Internet Appendix for:

The Federal Reserve's Balance Sheet as a Financial-Stability Tool[*]

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Appendix A: A Model of Government Crowding Out

In this Appendix we formalize crowding out logic using a simplified version of the model in GHS. We begin by considering the case where the government is the only actor that can issue short-term, money-like claims. In this case, the optimal mix of short- and long-term government debt hinges on a trade-off between capturing the money premium on short-term debt and limiting fiscal risk. We then extend the model to consider the case where financial intermediaries can also create short-term claims that are close substitutes for short-term government debt. When intermediaries do not fully internalized the financial stability costs associated with the private money they create, the government has an additional "crowding out" motive for issuing short-term money-like claims.

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A.1. A Trade-off Model of Government Debt Maturity

Consider a government with an initial accumulated debt (D), and no future expenditures, that must finance itself by issuing a combination of short-term bonds, long-term bonds, and taxes. Let S denote the fraction of the debt that is short-term and $1–S$ the fraction that is long-term.

Suppose that short-term debt offers many of the same services as base money, making it cheap to issue short-term debt in equilibrium. Let $v(M) > 0$ denote the value of these monetary services when the total amount of money-like claims is M. Thus, the money premium—i.e., the marginal benefit of additional money services and the corresponding reduction in equilibrium yields on money-like claims—is $v'(M) > 0$.[1] We assume that the demand for these monetary services is downward sloping—i.e., $v''(M) < 0$, so the equilibrium money premium is decreasing in the total amount of short-term debt.

All else equal, a higher value of the money premium makes the government want to issue more short-term debt. However, because the government must refinance this short-term debt at an uncertain future interest rate, issuing more short-term debt exposes taxpayers to refinancing risk and makes future taxes more volatile, which is costly. Specifically, a spike in interest rate would lead tax rates to jump. Formally, we assume that the deadweight costs of taxation are $(\lambda/2)\tau^2$, and that the variance of short-term interest rates is V_r.

To begin, we assume that the government is the only player that can issue short-term money-like claims. Thus, the total amount of money-like claims is simply $M = SD$. If the government finances itself by issuing fraction S of short-term debt, it generates total monetary services with value $v(SD)$. At the same time, this raises the volatility of taxes, which has expected cost $(\lambda/2)Var[\tau] = (\lambda/2)D^2V_r\ (S - S_0)^2$ where S_0 is a small number that reflects the maturity structure that minimizes fiscal risk in isolation. Thus, the optimal fraction of short-term government debt that maximizes total welfare, $U = v(SD) - (\lambda/2)Var[\tau]$, equates the marginal tax smoothing cost of issuing more short-term debt with the marginal benefit of money services:

$$\overbrace{\lambda V_r D(S - S_0)}^{\text{Marginal tax-smoothing cost}} = \overbrace{v'(SD)}^{\text{Marginal benefit of money services}} . \tag{A.1}$$

[1] Empirically, one can think of the money premium as being captured by our Z-spread measure.

Equation (A.1) captures the *traditional debt management tradeoff* that Treasury officials articulate between the expected interest savings of issuing more short-term debt (the right-hand side) versus the additional fiscal risk posed by issuing more short-term debt (the left-hand side).

Absent a money premium on short-term debt ($v'(SD) = 0$), the government immunizes itself against refinancing risk by opting for a long-term maturity structure, setting $S = S_0$. In contrast, if there is a money premium on short-term debt ($v'(SD) > 0$), the government issues more of it, exposing taxpayers to some fiscal risk in the process. The larger the money premium, the more aggressively the government relies on short-term debt.

When the volatility of short-term interest rates is lower (V_r is low), or when budget volatility is less costly (λ is low), the government issues more short-term debt. If there is no cost associated with budgetary volatility ($\lambda = 0$), the government will continue to shorten the maturity of the debt until the special demand for short-term debt is satiated—i.e., until $v'(SD)$ is driven to 0. This result generalizes the Friedman (1969) rule—which says that, absent any costs, the Fed Reserve should expand the monetary base until the demand for base money is satiated—from base money to money-like financial claims.

Equation (A.1) also suggests that for larger values of accumulated debt, the government should issue longer term, with both sides of key trade-off pointing in the same direction. First, as the overall debt burden grows, the fiscal costs associated with refinancing risk loom larger. Second, because the demand for liquidity services is downward sloping, the liquidity premium on short-term debt falls as D rises, further reducing the incentive to tilt towards short-term debt.

A.2. The Crowding Out Motive for Issuing Short-Term Government Debt

We started by assuming that the government was the only entity that could issue money-like claims: we now consider financial intermediaries as another party that can be also create money-like claims. For simplicity, we assume that money-like claims produced by intermediaries are perfect substitutes for those produced by the government. Thus, total money-like claims are $M = M_G + M_P$ where $M_G = SD$ denotes the government's supply of money-like claims and M_P denotes the supply of private money.

Unlike government-issued money-like claims, we assume that private intermediaries' issuance is associated with financial stability concerns. Specifically, we assume that the total financial stability costs of providing an amount M_P of private money is $c(M_P)$ where $c(\cdot)$ is an increasing, convex function of M_P—i.e., $c'(\cdot) > 0$ and $c''(\cdot) > 0$. However, we assume that private intermediaries only internalize a fraction $\phi < 1$ of these costs.[2] As a result, the equilibrium private supply of money-like claims satisfies

$$\overbrace{\phi}^{\text{Fraction internalzied}} \quad \times \quad \overbrace{c'(M_P^*)}^{\text{Marginal financial stability cost}} \quad = \quad \overbrace{v'(M_G + M_P^*)}^{\text{Marginal benefit of money services}} \quad . \tag{A.2}$$

Since $\phi < 1$, equation (A.2) means that private intermediaries will supply a quantity of money-like that exceeds the first-best quantity that fully internalizes the relevant financial stability costs.

We now return to the government's debt maturity choice problem, recognizing that the participation of financial intermediaries changes the government's optimization problem. Note that equation (A.2) implies that an expansion of government money $M_G = SD$, depresses the equilibrium money premium and crowds out private money creation:

$$\frac{\partial M_P^*}{\partial M_G} = \frac{v''(M_G + M_P^*)}{\phi c''(M_P^*) - v''(M_G + M_P^*)} < 0. \tag{A.3}$$

Now, when the government considers issuing more short-term debt it must take into account how its choices will impact the equilibrium quantity of private money creation as in equation (A.3). Specifically, the fraction of short-term government debt that maximizes total social welfare, $U = v(SD) - (\lambda/2)Var[\tau] - c(M_P^*)$, now satisfies:[3]

$$\overbrace{\lambda V_r(S - S_0)}^{\text{Marginal tax-smoothing cost}} \quad = \quad \overbrace{v'(SD + M_P^*)}^{\text{Marginal benefit of money services}} \quad \overbrace{+(\phi - 1)c'(M_P^*)\frac{\partial M_P^*}{\partial M_G}}^{\text{Crowding-out benefit}}. \tag{A.4}$$

Relative to equation (A.1), which summarized the traditional debt management tradeoff, equation (A.4) shows that, when private money creation creates uninternalized financial stability costs, there is an additional crowding-out benefit of issuing short-term government debt. (When $\phi < 1$, this crowding out term is positive since $\partial M_P^*/\partial M_G < 0$and $c'(M_P^*) > 0$.) In effective, there is a *regulatory motive* for issuing short-term debt, implying that *the government should issue more short-term debt than it would on the basis of traditional debt management*

[2] For instance, in GHS and Stein (2012), financial intermediaries create short-term safe claims that are backed by risky long-term assets. To always pay off short-term creditors in full, these long-term assets must be liquidated if a crisis arrives. However, these asset fire sales impose uninternalized costs on the broader economy because they raise the hurdle rate on investments in new real projects, thereby lowering total output.

[3] This is because the marginal benefit (per unit of D) of an increase in the fraction of short-term debt (S) is given by $v'(SD + M_P^*) \times (1 + \partial M_P^*/\partial M_G) - c'(M_P^*) \times \partial M_P^*/\partial M_G = v'(SD + M_P^*) + (\phi - 1) \times c'(M_P^*) \times \partial M_P^*/\partial M_G$, where the equality makes use of equation (A.2).

considerations alone. The more severe are the externalities associated with private money creation, the shorter is the optimal short-term share of government debt.

A.3. Crowding Out versus Liquidity Regulation

Thus far we have assumed that the government's only tool for reigning in excess private maturity transformation is by issuing more short-term debt itself. But suppose the government could also directly regulate private money creation. Does this then change our conclusion about how aggressively the government should crowd out private maturity transformation?

Concretely, suppose that the government levies on Pigouvian tax at rate θ_P on private money creation. With such a Pigouvian tax in place, the equilibrium supply of private money-like claims satisfies

$$\theta_P + \phi \times c'(M_P^*) = v'(M_G + M_P^*). \tag{A.5}$$

If such a regulation could be imposed on all private intermediaries, including both traditional banks and shadow bank, and this regulation did not create any unintended distortions, the government could achieve the first best amount of private money creation simply by using a regulatory tax set at $\theta_P = (1-\phi) \times c'(M_P^*) > 0$—i.e., by levying a tax that forces intermediaries to internalize all of the financial stability costs associated with private money creation. In this case, the optimal quantity of short-term debt would satisfy equation (A.1). Thus, in a world with perfectly effective and non-distortionary regulation, the crowding out motive vanishes: debt management and regulation would effectively decouple.

However, this separation principle no longer holds when regulation is only partially effective or is distortionary. For instance, suppose that a Pigouvian tax of θ_P itself creates deadweight costs equal to $(\Upsilon/2)\theta_P^2$, say because, in additional to crowding out private maturity transformation, this tax also discourages intermediaries from undertaking some socially valuable activities. The government's now chooses S and θ_P to maximize total welfare $U = v(SD) - (\lambda/2)Var[\tau] - c(M_P^*) - (\Upsilon/2)\theta_P^2$. Then letting $\partial M_P^*/\partial\theta_P = 1/[v''(M_G + M_P^*) + \phi c''(M_P^*)] < 0$ denote the impact of the regulatory tax on private money creation, GHS show that the optimal regulatory tax on private money creation is

$$\theta_P = \frac{\left|\partial M_P^* / \partial\theta_P\right|}{\left|\partial M_P^* / \partial\theta_P\right| + \Upsilon} \times (1-\phi)c'(M_P^*) < (1-\phi)c'(M_P^*), \tag{A.6}$$

and optimal short-term fraction of government debt satisfies

$$\overbrace{\lambda V_r(S-S_0)}^{\text{Marginal tax-smoothing cost}} = \overbrace{v'(SD+M_P^*)}^{\text{Marginal benefit of money services}} + \frac{\Upsilon}{\left|\partial M_P^* / \partial \theta_P\right| + \Upsilon} \times \overbrace{(\phi-1)c'(M_P^*)\frac{\partial M_P^*}{\partial M_G}}^{\text{Crowding-out benefit}}. \tag{A.7}$$

Relative to equation (A.4), equation (A.7) shows that, when the government can also use regulation to reign in excessive maturity transformation, the crowding out benefit is attenuated by a term that reflects the deadweight costs (Υ) and efficacy ($|\partial M_P^*/\partial \theta_P|$) of direct liquidity regulation. Indeed, GHS show that (under regulatory conditions) the government should rely more on crowding out and less on regulation when the deadweight costs of liquidity regulation are higher or the efficacy of regulation is low.[4] Thus, a key message of GHS is that, *in the realistic case where liquidity regulation is imperfect, the separation between debt management and regulation fails: the government should use both tools in combination to reign in excessive maturity transformation*.

[4] Similarly, the government should rely more on regulation and less on crowding out when the tax-smoothing costs associated with issuing more short-term debt are higher (i.e., when λ or V_r are higher).

Appendix B: Estimating Remittance Risk

We use the term structure model in Greenwood, Hanson, and Vayanos (2016) hereafter "GHV" to simulate how different configurations of the Fed's balance sheet might impact fiscal risk. We first describe our procedure for simulating (i) the consolidated Federal interest expense and (ii) the net interest income the Federal Reserve remits to the Treasury. We then briefly explain the GHV model.

B.1. Simulating Remittances

The consolidated federal interest expense is the interest that the government pays on the government liabilities that are held by the public (i.e., net of Fed holdings), including both publicly held Treasury debt and interest-bearing Fed liabilities. Thus, consolidated interest expense equals the interest on Treasury debt minus the net interest income the Federal Reserve remits to the Treasury, all as a fraction of Treasury debt:

$$INT_t^{TOT}(\tau_{UST},\tau_{FED}) = INT_t^{UST}(\tau_{UST}) - \frac{A}{D} REMIT_t^{FED}(\tau_{FED}, Z)\,, \tag{B.1}$$

where $INT_t^{UST}(\tau_{UST})$ is the interest expense on Treasury debt, *A*/*D* is the ratio of the Fed's assets to Treasury Debt, and $REMIT_t^{FED}(\tau_{FED}, Z)$ is the Fed's remittance to Treasury, financed by a fraction *Z* of interest-bearing reserves and fraction 1 – *Z* of non-interest bearing currency.

To simulate the Treasury's interest expense INT_t^{UST}, we assume a total Treasury debt of \$13 trillion with a weighted average maturity 5.75 years. For example, if the Treasury's average maturity was 5 years, we would assume that each month the Treasury reissues the principal from the 10-year bonds that it issues 10 years ago into new 10-year bonds. This implies that the interest expense on a 5-year WAM portfolio is just a 120-month moving average of 10-year yields, —i.e., $\sum_{j=1}^{120} y_{t-(j-1)}^{(10)}/120$ where $y_t^{(n)}$ denotes the *n*-year yield at time *t*. More generally, we assume that the Treasury follows a uniform issuance "ladder" with a weighted-average maturity of τ_{UST} years, which implies that:[5]

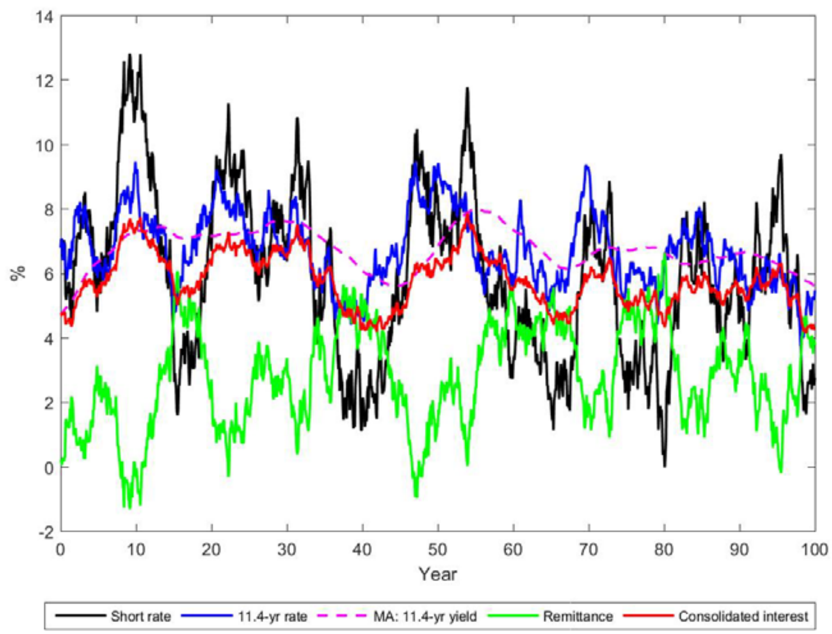
[5] One could easily adapt our methodology to allow for non-uniform maturity distributions that better mimic the Treasury debt distribution shown in Figure 8. We do not pursue this here for the sake of simplicity. For instance, one might model the interest expense as a weighted average of moving averages of the kind given in equation (B.1). For instance, letting $w_{(n)}$ denote the fraction of outstanding debt that had an initial maturity of *n*-years one would compute $INT_t^{UST} = \sum_{\tau\in \mathrm{T}} w_{(\tau)} INT_t^{UST}(\tau)$ where T is the set of maturities issued by the Treasury. For instance, we currently have $\mathrm{T} = \left\{\frac{1}{12}, \frac{1}{4}, \frac{1}{2}, 1, 2, 3, 5, 7, 10, 30\right\}$.

$$INT_t^{UST}(\tau_{UST}) = \frac{\sum_{j=1}^{12\times\tau_{UST}} y_{t-(j-1)}^{(2\times\tau_{UST})}}{12\times\tau_{UST}}, \tag{B.2}$$

We simulate Fed remittances similarly, assuming that the distribution of maturities in the Fed's portfolio is uniform—i.e., that the Fed follows a simple ladder investment strategy. Remittances expressed as a fraction of Fed assets, for a balance sheet that invests in Treasuries with a weighted average maturity of τ_{FED} years and is financed with fraction $Z \in [0,1]$ of interest-bearing reserves, are given by:

$$REMIT_t^{FED}(\tau_{FED}, Z) = \frac{\sum_{j=1}^{12\times\tau_{FED}} y_{t-(j-1)}^{(2\times\tau_{FED})}}{12\times\tau_{FED}} - Z\times r_t. \tag{B.3}$$

We simulate the term structure of interest rates using 100,000 years of monthly term structure data, compute the consolidated Federal interest expense and remittances using equations (B.1) and (B.3.), and then take the standard deviation of these series.



Simulated interest rates paths: The above figure shows a representative simulated path of interest rates, the net interest income the Fed remits to Treasury (as a % of Fed assets), and the consolidated Federal interest expense (as a % of Treasury debt). The figure assumes $13 trillion in Treasury debt with a weighted average maturity of 5.7 years; a $4.5 trillion Fed balance sheet that also holds a portfolio with weighted average maturity of 5.7 years and that is 1/3 financed with currency and 2/3 financed with interest-bearing reserves. In other words the figure assumes that $\tau_{UST} = \tau_{FED} = 5.7$ years, Z = 2/3, D = 13, and A = 4.5. We plot (i) the overnight short rate r_t, (i) the 11.4-year zero coupon yields $y_t^{(11.4)}$, (iii) an 11.4-year moving average of $y_t^{(11.4)}$—our proxy for Treasury's interest expense and the Fed's interest income, (iv) Fed remittances, and (v) the consolidated Federal interest expenses.

B.2. Understanding the Fiscal Risk of Different Fed Balance Sheet Scenarios

Subject to mild regularity conditions, the volatility of the consolidated Federal interest expenses, $Var[INT_t^{TOT}(\tau_{UST}, \tau_{FED})]$, will always be *decreasing* in τ_{UST} and *increasing* τ_{FED}.[6] The intuition is straightforward. The consolidated federal interest expense is the interest that the government pays on the liabilities that are held by the public (net of Fed holdings), including both publicly held Treasury debt and interest-bearing Fed liabilities. Holding the Fed's asset portfolio fixed, the shorter the weighted average maturity of Treasury debt, the greater is the quantity of short-term debt that the consolidated government needs to refinance each period, and therefore the more volatile is the government's consolidated interest expense. Similarly, holding the weighted average maturity of Treasuries fixed, the longer is the Fed's asset portfolio, the greater the amount of short-term debt held by the public, the greater is the quantity of short-term debt that the consolidated government needs to refinance each period, and therefore the more volatile is the government's consolidated interest expense. Thus, the volatility of the consolidated Federal interest expense is our preferred metric for assessing the Fed's contribution to fiscal risk.[7]

As noted in the main text, the volatility of the net interest income that the Fed remits to Treasury is not necessarily increasing in τ_{FED}. Thus, we would argue that looking at remittance volatility in isolation is an inferior measure of fiscal risk.

As noted in the text, looking at the volatility of the consolidated interest expense or the volatility of Fed remittances will yield directionally similar conclusions when the Fed is primarily financed with interest-bearing reserves. In other words, when Z is large, $Var[REMIT_t^{FED}(\tau_{FED}, Z)]$ will generally be an increasing function of τ_{FED}. To see this think about the limiting case where Z = 1 (i.e., the Fed is 100% financed with interest-bearing reserves) and where term premia for a given maturity are constant over time—i.e., where the expectations hypothesis holds. In this case, $Var[REMIT_t^{FED}(\tau_{FED}, 1)]$ will always be increasing in τ_{FED}. This is true because, in this expectations hypothesis world, the best past predictors of the

[6] Specifically, these statements may fail to hold is there is tremendous time-series variation in term premia.

[7] All else equal, the volatility of the consolidated interest expense is increasing in the fraction of the Fed's balance sheet that is financed with interest-bearing reserves, denoted *Z* above. If we hold non-interest bearing currency fixed at C dollars, we have $Z = 1 - C/A$ where A is the size the Fed's balance sheet. Thus, everything else constant, the volatility of the consolidated interest expense will rise with the size of the Fed's balance sheet. This can be seen by comparing across the three Panels in Table III.

current short rate that the Fed pays on its liabilities are the recent yields on short-dated Treasury bonds. Indeed, we have $\lim_{\tau \to 0} Var[REMIT_t^{FED}(\tau, 1)] = 0$. By contrast, the yields on long-term bonds in the distant past will not be highly correlated with the current short rate that the Fed pays on its liabilities. This case corresponds to the conventional intuition that the Fed is taking on greater remittances risk by buying longer term bonds.

By contrast, when most of the Fed balance sheet is financed with non-interest bearing currency (i.e., when Z is small), $Var[REMIT_t^{FED}(\tau_{FED}, Z)]$ will generally be a decreasing function of τ_{FED}. To see this think about the opposite limiting case where $Z = 0$ (i.e., the Fed is 100% financed with non-interest bearing currency) and where term premia for a given maturity are constant over time. In this case, $Var[REMIT_t^{FED}(\tau_{FED}, 0)]$ will always be decreasing in τ_{FED}. This is true because (i) in an expectations hypothesis world, long-term term yields are always less volatile that short-term yields and (ii) $Var[REMIT_t^{FED}(\tau_{FED}, 0)]$ for longer WAM strategy depend on the variance of a longer moving average of short rates.[8] Indeed, we have $\lim_{\tau_{FED} \to \infty} Var[REMIT_t^{FED}(\tau_{FED}, 0)] = 0$.

B.3. GHV Term Structure Model

The GHV model is set in continuous time and is meant to describe the term structure of interest rates when there is both volatility in short-term interest rates (set by the Fed) and the net supply of long-term government bonds (effectively jointly set by the Fed and the Treasury). We use the model here as an input to equations (B.1), (B.2), and (B.3). We use parameters from Table 1 of GHV and simulate 100,000 years of monthly data in order to compute time-series averages and standard deviations. Their key parameters are chosen to match the time-series volatility and persistence of nominal short rates from 1961 to 2015.

The short term nominal interest in the GHV model follows an exogenous "stochastic mean" process where there are both high-frequency and low-frequency shocks to short rates. Specifically, the current short rate, r_t, evolves according to

$$dr_t = \kappa_r(\bar{r}_t - r_t) + \sigma_r dB_{r,t} \tag{B.4}$$

and target short rate, $\bar{r}_t$, evolves according to

$$d\bar{r}_t = \kappa_{\bar{r}}(\bar{r} - \bar{r}_t) + \sigma_{\bar{r}} dB_{\bar{r},t}. \quad \text{(B.5).}$$

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[8] If there is tremendous time-series variation in term premia, $Var[REMIT_t^{FED}(\tau_{FED}, 0)]$ could be increasing in τ_{FED}.

where $B_{r,t}$ and $B_{\bar{r},t}$ are Brownian motions (assumed to be orthogonal for simplicity) and where κ_r, σ_r, $\kappa_{\bar{r}}, \sigma_{\bar{r}}$, and $\bar{r}$ are positive constants.

A set of fixed-income arbitrageurs must absorb the net supply of bonds from the consolidated government and preferred-habitat investors. Arbitrageurs have mean-variance preferences over instantaneous changes in their wealth and choose bond holding to maximize $E_t[dW_t]-(a/2)Var_t[dW_t]$ subject to the budget constraint

$$dW_t = \int_0^T x_t^{(\tau)} \frac{dP_t^{(\tau)}}{P_t^{(\tau)}} d\tau + \left(W_t - \int_0^T x_t^{(\tau)} d\tau \right) r_t \quad \text{(B.6).}$$

where $x_t^{(\tau)}$ denotes arbitrageurs holdings of τ-period bonds at time *t*.

The net supply of τ-period bonds that arbitrageurs must absorb from the consolidated government and other investors at time *t* is

$$s_t^{(\tau)} = \xi(\tau) + \theta(\tau)\beta_t \quad \text{(B.7).}$$

where $\xi(\tau)$ and $\theta(\tau)$ are deterministic functions of bond maturity, τ, and β_t is a stochastic bond supply factor. Specifically, the current supply factor, β_t, evolves according to

$$d\beta_t = \kappa_\beta(\bar{\beta}_t - \beta_t) + \sigma_\beta dB_{\beta,t} \quad \text{(B.8)}$$

and target supply $\bar{\beta}_t$, evolves according to

$$d\bar{\beta}_t = -\kappa_{\bar{\beta}}\bar{\beta}_t + \sigma_{\bar{\beta}} dB_{\bar{\beta},t}. \quad \text{(B.9).}$$

where $B_{\beta,t}$ and $B_{\bar{\beta},t}$ are Brownian motions (assumed to be orthogonal for simplicity) and where $\kappa_\beta, \sigma_\beta$, $\kappa_{\bar{\beta}}$, and $\sigma_{\bar{\beta}}$ are positive constants.

GHV conjecture that the equilibrium prices of τ-year zero coupon bonds takes the affine form

$$P_t^{(\tau)} = e^{-[A_r(\tau)r_t + A_{\bar{r}}(\tau)\bar{r}_t + A_\beta(\tau)\beta_t + A_{\bar{\beta}}(\tau)\bar{\beta}_t + C(\tau)]} \quad \text{(B.10)}$$

where $A_i(\tau)$ are factor sensitivities that measure the percentage decline in price of τ-year bonds per unit increase in factor i = r, $\bar{r}$, β, $\bar{\beta}$. Thus, the yield on τ-year zero coupon bonds is

$$y_t^{(\tau)} = \frac{A_r(\tau)r_t + A_{\bar{r}}(\tau)\bar{r}_t + A_\beta(\tau)\beta_t + A_{\bar{\beta}}(\tau)\bar{\beta}_t + C(\tau)}{\tau} \quad \text{(B.11)}$$

Arbitrageurs' first-order condition for holdings of τ-year bonds, $x_t^{(\tau)}$, is

$$E_t\left[\frac{dP_t^{(\tau)}}{P_t^{(\tau)}}\right]-r_t=-aCov\left[\frac{dP_t^{(\tau)}}{P_t^{(\tau)}},dW_t\right] \tag{B.12}$$
$$=A_r(\tau)\lambda_{r,t}+A_{\bar{r}}(\tau)\lambda_{\bar{r},t}+A_\beta(\tau)\lambda_{\beta,t}+A_{\bar{\beta}}(\tau)\lambda_{\bar{\beta},t}.$$

Where, by market clearing $x_t^{(\tau)}=s_t^{(\tau)}$, the prices of factor risk $\lambda_{i,t}$ for factors $i = r, \bar{r}, \beta, \bar{\beta}$ are

$$\lambda_{i,t}=\int_0^T A_i(\tau)s_t^{(\tau)}d\tau. \tag{B.13}$$

Combining our affine conjecture, arbitrageurs' first order condition, and market clearing delivers an affine equation in the four risk factors $(r, \bar{r}, \beta, \bar{\beta})$. Identifying terms yields ordinary differential equations (ODEs) for $A_r(\tau)$, $A_{\bar{r}}(\tau)$, $A_\beta(\tau)$, $A_{\bar{\beta}}(\tau)$, $C(\tau)$. There are closed form solutions for $A_r(\tau)$ and $A_{\bar{r}}(\tau)$. As GHV show, the ODEs for $A_\beta(\tau)$, $A_{\bar{\beta}}(\tau)$, and $C(\tau)$ can be solved in closed form, up to system of two scalar equations.