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Analyzing Scrip Systems

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Scrip systems provide a non-monetary trade economy for exchange of resources. We model a scrip system as a stochastic game and study system design issues on selection rules to match potential trade partners over time. We show the optimality of one particular rule in terms of maximizing social welfare for a given scrip system that guarantees players' incentives to participate. We also investigate the optimal number of scrips to issue under this rule. In particular, if the time discount factor is close enough to 1, or trade benefits one partner much more than it costs the other, the maximum social welfare is always achieved no matter how many scrips are in the system. When the benefit of trade and time discount are not sufficiently large, on the other hand, injecting more scrips in the system hurts most participants; as a result, there is an upper bound on the number of scrips allowed in the system, above which some players may default. We show that this upper bound increases with the discount factor as well as the ratio between the benefit and cost of service. Finally, we demonstrate similar properties for a different service provider selection rule that has been analyzed in previous literature.

Keywords: Repeated games, stochastic games, dynamic program.

1. Introduction

Scrips are coupons that are used in place of currency to exchange goods and services; typically, they cannot be exchanged for money, and therefore their sole use is for the good or service which they are intended for. In this paper we study scrip systems, which are markets that use scrips rather than money to exchange goods and services. Such markets are typically implemented when the use of governmental currency is impractical or undesirable.

One example of a scrip system is that of the Capitol Hill babysitting co-op, documented in Sweeney and Sweeney (1977). A group of about 150 married couples with children who lived in the Washington, D.C. area were tired of looking for and hiring babysitters to watch their children every time they wanted to enjoy a night out, so they decided to join together to form a babysitting co-op, managed by a scrip system. Every couple in the co-op was given an initial amount of coupons, or scrips, to pay for babysitting service by another couple in the co-op who was willing to provide the service. Free riding was mitigated in the system since a couple could only enjoy the service when they had coupons, and earning coupons required providing services.

It turns out that this babysitting co-op experienced market crashes similar to many other types of markets. Initially, they distributed too few coupons and trade rarely occurred. This was likely because either a couple ran out of coupons to pay for service or they hoarded the few coupons for later special situations. In order to solve this issue, the group collectively decided to give every couple additional coupons, to the point that each couple valued one additional coupon too little, and therefore was not willing to provide services to earn an additional coupon. This story was popularized by Krugman (1999), who related the scrip system's crashes to economic slumps and monetary policies. Since crashes occurred due to having the wrong number of scrips in the system, a natural and important question to ask is, therefore, what is the "right" number of scrips in a system?

There are many other examples of scrip systems that have been implemented in resource exchange environments, such as online peer-to-peer systems, to prevent free riding (see, e.g., Vishnumurthy et al. 2003, Gupta et al. 2003, Ioannidis et al. 2003, Sirivianos et al. 2007, Peterson and Sirer 2009, Belenkiy et al. 2007, and Satoshi Nakamoto 2009). The idea is similar to that of the babysitting co-op example, where scrips are credited to users who provide products/services (e.g., share files) and debited when users receive them (e.g., access other's files), so that in the long-run, the amount of products/services participants can receive matches what they provide.

Other common uses of scrip systems include online resource allocation environments; for example, grid computing networks (see, e.g., Brunelle et al. 2006), research testbeds (see, e.g., Chun et al. 2005, AuYoung et al. 2007 and (AuYoung et al. 2007)), distributed database systems (Stonebraker et al. 1996), and privacy-enhancing technologies, where a volunteer network of servers are needed to route Internet traffic in order to conceal the user's IP address (Humbert et al. 2011). Some scholars suggest that the academic journal refereeing process may also be managed by a scrip system!

Clearly scrip systems are important in a variety of settings, yet there has been relatively little work done to analyze their behavior. In all of these examples, the number of scrips injected into the system is a determinant of system performance. As seen from the babysitting co-op example, having too few or too many scrips in the system can cause market crashes. Another important question arises in the online applications of scrip systems – how should the service provider be chosen? In this paper, we analyze a class of scrip systems and provide insights regarding system design: the way service providers should be selected as well as the optimal number of scrips that should be used in the system.

We show the optimality of one particular service provider selection rule for a given scrip system in terms of maximizing social welfare, i.e., total utility of all players in the system over time, while making sure that players have the incentive to follow the rules of the scrip system. For scrip systems where the time discount factor is close enough to 1, or trade benefits one partner much more than it costs the other, the maximum social welfare is always achieved no matter how many scrips are in the system. As a result, system optimal performance can be achieved under individual incentive constraints. When the benefit of trade and time discount are not sufficiently large, on the other hand, injecting more scrips in the system hurts most participants; in this case, there is an upper bound on the number of scrips allowed in the system, above which some players may default. We show that this upper bound increases with the discount factor as well as the ratio between the benefit and cost of service.

In the remainder of this section, we provide a literature review on the modeling and analysis of scrip systems. The basics of our model, as well as the optimal centralized control policy, are introduced in Section 2. We then study the stochastic game played in the absence of a central planner in Section 3 and demonstrate that the central optimal solution can be achieved in the game when the discount factor is sufficiently large. Section 4 investigates the impact of the number of scrips in the system. Finally, Section 5 concludes the paper with a summary of our results and potential areas for future work.

1.1. Literature Review

As mentioned before, in the computer science literature, several papers have been written regarding the application of scrip systems and their implementation. There have been, however, only a few papers that formally model and analyze scrip systems. Aperjis and Johari (2006) is one of the earliest papers that studies a peer-to-peer file sharing system as an exchange economy. They propose a static game where users decide uploading/downloading rates, and they study the market clearing prices in equilibrium. The papers that are most closely related to ours are Friedman et al. (2006), Kash et al. (2007), Kash et al. (2012a) and Kash et al. (2012b). In fact, this stream of papers motivated our study.

Friedman et al. (2006) is one of the first papers that analyzed players' strategies as well as design characteristics of scrip systems in a stochastic setting. Their model considers a homogeneous population of players in a scrip system with a finite number of scrips. Services are provided at a fixed price (in terms of the number of scrips) and incur a fixed and identical utility gain (loss) to each player receiving (providing) the service. They consider an infinite time horizon game with discounting. In each period, a player is chosen uniformly at random to request service, while all other players have the option to volunteer as a service provider, one of whom is selected randomly. Our model, described in Section 2, is similar to their model with one major departure. Instead of having a player chosen randomly to provide service, we allow the system designer to choose a service provider selection rule for all players to follow; then we determine the number of scrips that should be injected in the system accordingly. At the end of our analysis, we return to their assumption of randomly selecting a service provider, and we show similar results regarding the number of scrips that should be injected into such a scrip system.

A key assumption adopted in Friedman et al. (2006) is that each player chooses to volunteer to provide service following a threshold strategy. That is, a player is willing to volunteer to provide service only if his scrip stock is lower than a threshold number of scrips. The paper shows that when the discount factor is close enough to one, there exists an ϵ-Nash Equilibrium in which each player follows such a threshold strategy. One implication of the threshold strategy is that there exists a total threshold number of scrips in the system, above which no trade occurs and therefore the system will experience a market crash. Our model, on the other hand, does not restrict players' strategies to be of threshold type; rather, we show the existence of a total threshold number of scrips as a result. Follow-on work in Kash et al. (2012b) shows that social welfare increases as the number of scrips in the system increases up to this threshold of total scrips, after which the social welfare drops to zero due to the market crash. Kash et al. (2012a), where the model is further generalized to include multiple player types, characterizes each player's threshold that achieves the optimal social welfare. In Kash et al. (2012b), the authors further analyze the impact of altruists and hoarders in the scrip system.

Motivated by the application of scrip systems to privacy-enhancing technologies, Humbert et al. (2011) analyzes scrip systems where each service request requires n providers to satisfy. This model directly extends the model in Kash et al. (2007) to require n service providers instead of one. The authors show similar results to those in Kash et al. (2007) including the existence of an ϵ-Nash Equilibrium where all players act according to a threshold policy.

Finally, recent literature in economics also studies scrip systems, motivated by the babysitting co-op, as the micro-foundation of monetary policy. Hens et al. (2007) provides an overview of this line of work. There are quite a few differences between their model and ours. The main difference is that they assume no cost to provide service, while we assume providing service incurs a negative utility, and the ratio between the benefit and cost of service plays an important role in our model. Similar to Friedman et al. (2006), Hens et al. (2007) also focuses on the random service provider selection rule, which is, one may argue, simple and realistic in many economic settings.

2. Basic Model Description

Consider an economy with a non-empty set N of players. Each player $i \in N$ has r_i scrips, where we abuse notation to use N to represent the number of players as well. In each time period, one player at random will be the "service requester" with probability $1/N$, and all other players' types are 0. In any given period, the service requester is able to obtain utility u if the player chosen as a service provider (one of the type 0 players) is willing to sacrifice utility $c < u$ in exchange for 1 scrip and if the service requester has a scrip and is willing to pay 1 scrip for service. Assume a time discount factor γ for the system.

Denote the "state" of the system to be $s = (r, j)$, where r is the vector of scrip stocks $(r_i)_{i \in N}$ and j is the service requester. Thus, the total number of scrips in the system is $R = \sum_i r_i$, which does not change over time. "State space" S is the collection of all possible states s. We denote a stationary policy π to map the state (r, j) into a probability distribution on the remaining players other than j. The purpose of π is to select a service provider for any possible state s. Denote set Π to represent the set of admissible policies.

At this point it is worth introducing a particular service provider selection rule in Π, the "minimum scrip selection rule" $\bar{\pi}$, where in each round, the type 0 player with the least number of scrips is selected as the service provider. If more than one type 0 player has the fewest scrips, one player is chosen randomly from them with equal likelihood to be the service provider. Another example of a selection rule in Π is the "random provider selection rule", a common selection rule considered in previous literature, where a player is selected as the service provider uniformly at random, independent of her scrip stock.

We first demonstrate that the minimum scrip selection rule, $\bar{\pi}$, maximizes social welfare among all policies in Π in a central planner setting, where we assume that a hypothetical central planner not only chooses the service provider, but also decides whether trade should occur.

2.1. Central Planner Setting

Now we consider a hypothetical central planner who tries to maximize the total social welfare over an infinite time horizon with discount factor $\gamma \in (0,1)$. In each period, given state (r, j), the central planner decides whether trade should occur when player j has at least one scrip, and if so, chooses a player i to be the service provider. Let e_k be a vector in $\Re^N$ with every component equal to zero except the kth component equal to one. $J(r, j)$ is the system social welfare from state (r, j), and $\mathcal{J}(r) = \sum_{j'=1}^{N} J(r, j')$ is the total system social welfare across all states with the same vector of scrip stocks. The corresponding Bellman equation is:

$$J(r,j) = \begin{cases} \max\{\max_{i\neq j}(u-c) + \frac{\gamma}{N}\mathcal{J}(r+e_i-e_j),\ \frac{\gamma}{N}\mathcal{J}(r)\} \ , & r_j > 0 \\ \frac{\gamma}{N}\mathcal{J}(r) \ , & r_j = 0 \end{cases} , \tag{1}$$

$$\text{where} \quad \mathcal{J}(r) = \sum_{j'=1}^{N} J(r, j') \ .$$

In the Bellman equation (1), the outer maximization decides whether to trade a scrip for service, while the inner maximization selects the trading partner.

Equivalently, we can express the Bellman equation in terms of $\mathcal{J}$ as $\mathcal{J} = \Gamma\mathcal{J}$, where

$$(\Gamma\mathcal{J})(r) = \frac{\gamma(N-N_r)}{N}\mathcal{J}(r) + \sum_{j:\ r_j\geq 1} \max\left\{(u-c) + \frac{\gamma}{N}\max_{i:i\neq j}\mathcal{J}(r+e_i-e_j),\ \frac{\gamma}{N}\mathcal{J}(r)\right\} , \tag{2}$$

in which N_r is the number of players with positive scrips in r.

Next we define the following properties for a generic function $\mathcal{J}$ defined on the integer simplex $\{r \in \mathbb{Z}_+^N : \sum_i r_i = R\}$, and show that the optimal system social welfare function $\mathcal{J}^*$ satisfies them, which further implies that the minimum scrip selection rule is optimal in the central planner setting.

(**C1**) *Symmetry:* For any r and r' with $r_k = r'_k$ for all k except i, j, where $r_i = r'_j$ and $r_j = r'_i$,

$$\mathcal{J}(r) = \mathcal{J}(r') \ . \tag{3}$$

(**C2**) For any r and players i and j such that $r_i > r_j$,

$$\mathcal{J}(r - e_i + e_j) \geq \mathcal{J}(r) \ . \tag{4}$$

(**C3**) For any scrip distribution r and player j with $r_j > 0$,

$$\mathcal{J}(r) - \max_{i:i\neq j} \mathcal{J}(r + e_i - e_j) \leq \frac{N}{\gamma}(u - c) \ . \tag{5}$$

Furthermore, for a player i, denote set I_{r_i} to contain all players with at most r_i scrips. If

$$\sum_{j \in I_{r_i}} r_j \geq r_i(|I_{r_i}| - 1) + 1 \ , \tag{6}$$

then for any player $j \in I_{r_i}$ we have

$$\mathcal{J}(r) - \mathcal{J}(r + e_i - e_j) \leq \frac{N}{\gamma}(u - c) \ . \tag{7}$$

Proposition 1. *The solution $\mathcal{J}^*$ to the Bellman equation $\mathcal{J} = \Gamma \mathcal{J}$ satisfies conditions (**C1**), (**C2**), and (**C3**).*

The proof is based on showing that for any function $\mathcal{J}$ that satisfies these properties, so does $\Gamma\mathcal{J}$. The detailed proof is presented in the Appendix. Proposition 1 implies the following characterization of the central planner's optimal policy.

Theorem 1. *In the central planner setting, trade always occurs, and the minimum scrip selection rule $\bar{\pi}$ is optimal.*

Proof: Condition (5) for $\mathcal{J}^*$ suggests

$$(u - c) + \frac{\gamma}{N} \max_{m:m\neq l} \mathcal{J}^*(r + e_m - e_l) \geq \frac{\gamma}{N}\mathcal{J}^*(r) \ ,$$

for $r_l > 0$, which implies that trade always occurs if the service requester has at least one scrip to pay for service. Therefore, for a vector r with $r_j > 0$, Bellman equation (1) becomes

$$J^*(r, j) = (u - c) + \frac{\gamma}{N} \max_{i:i\neq j} \mathcal{J}^*(r + e_i - e_j) \ . \tag{8}$$

Condition (**C2**) further implies that the optimal i that solves the maximization in (8) must be the one with the least number of scrips. Q.E.D.

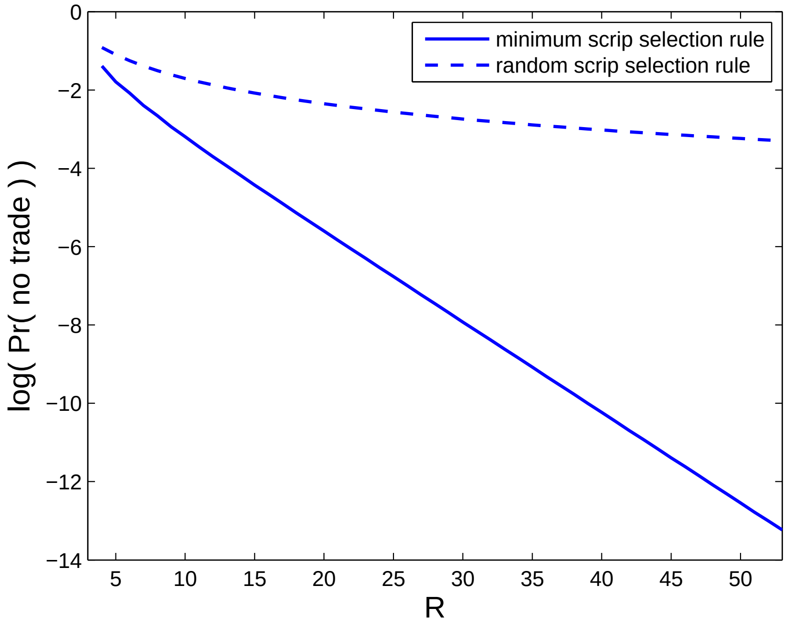

Figure 1 Probability of no trade with $N=3$ players.

The intuition behind Theorem 1 is clear. In order to maximize social welfare, the central planner always prefers trading, which generates $u-c>0$, over not, whenever possible. Trade cannot occur when the service requester has no scrips. The minimum scrip selection rule tries to balance the scrip holdings among players, therefore minimizing the chance that a player runs out of scrips. An important observation is that with more scrips in the system, there is a lower probability that a service requester will have no scrips, thus resulting in a higher probability of social welfare increasing by $u-c$ in each round; this implies that in the central planner setting, social welfare increases with the number of scrips in the system.

In particular, the solid curve in Figure 1 illustrates that the probability that trade does not occur, due to lack of scrips by the service requester, appears log linearly decreasing with the number of scrips R in the system when R is sufficiently large; each point on this curve is obtained through standard numerical iteration approaches to compute steady state probabilities for a 3 player Markov Chain. In comparison, the dotted curve represents the same probability following the random provider selection rule. It is clear that the chance of no trade is much lower with the minimum scrip selection rule than the random provider selection rule. More generally, we are able to obtain a closed form expression of the probability of no trade following the random provider selection rule (see Proposition 7 in the Appendix, and Kash et al. (2012a) Lemmas A.3 and A.4 for steady state probabilities in more general settings).

3. Stochastic Game and Optimality of Minimum Scrip Selection Rule

In this section, we remove the existence of a central planner, and we formally define the game in which the pair of players selected to be potential trade partners can decide whether to exchange a scrip for service. We then demonstrate that even in the absence of a central planner, the minimum scrip selection rule, $\bar{\pi}$, achieves the maximum social welfare obtained in the central planner setting under certain conditions, which corresponds to the Folk Theorem for stochastic games (see, e.g., Fudenberg and Tirole 1991). Furthermore, we show that in the game setting there exist thresholds on the discount factor as well as the relative benefit of receiving a service above which all players have the incentive to trade a scrip for service whenever the service requester has at least one scrip.

3.1. Stochastic Game

Now we consider a game in which all players follow a particular service provider selection policy $\pi \in \Pi$; we assume that no players collude. We focus on a stochastic game setting where the planning horizon is infinite, due to the well known distinction between finite horizon stochastic games and infinite horizon stochastic games (see, e.g., Fudenberg and Tirole 1991). In our setting, if the planning horizon was finite, given that scrips have no salvage value at the end of the horizon, no type 0 player would offer service and suffer a negative utility $-c$ in the last period. Using backward induction and following the same logic throughout the time horizon, no trade ever occurs in any finite horizon game.

In the infinite horizon setting, each player's strategy may depend on the entire history of the game. In particular, we denote vector $\theta = (r, j, i, \Omega)$ to represent the "state of game" at each period, in which r is the scrip distribution vector, j is the service requester, i is the player selected to be the service provider according to selection rule $\pi(r, j)$, and set Ω contains all players who *i*) have never refused to trade a scrip for service with anyone within set Ω, and *ii*) have never traded with anyone not in set Ω at the time of trade[1]. Obviously, in the beginning of the time horizon, Ω contains all the N players in the game. Further denote set $D_k(\theta)$ to represent player $k \in N$'s action space at state of game θ. In particular, if player j has positive scrips ($r_j > 0$), she can choose whether or not to give one scrip to player i in exchange for service ($d_j = 1$), or not to spend the scrip for service ($d_j = 0$); therefore, $D_j(r, j, i, \Omega) = \{1, 0\}$. Player i, on the other hand, can choose whether to accept the scrip and serve player j ($d_i = 1$) or not ($d_i = 0$); therefore, $D_i(r, j, i, \Omega) = \{1, 0\}$. Any player other than i and j must take no action, so $D_k(r, j, i, \Omega) = \{0\}$ for $k \neq i, j$. Note that at state of game $\theta = (r, j, i, \Omega)$, trade of a scrip for service occurs only if $r_j d_i d_j > 0$. Denote the action profile $D(\theta) = \times_{k \in N} D_k(\theta)$, with element $d(\theta) = (d_k(\theta))_{k \in N} \in D(\theta)$.

Given state of game $\theta = (r, j, i, \Omega)$ and action profile d, single period utilities for players i and j depend on whether or not trade occurred; $u_i(\theta, d) = -c$ and $u_j(\theta, d) = u$ if $r_j d_i d_j > 0$ (trade occurred), and $u_i(\theta, d) = u_j(\theta, d) = 0$ if $r_j d_i d_j = 0$ (trade did not occur). In either case, for any player $k \neq i, j$, the utility $u_k(\theta, d) = 0$. Given service provider selection rule π, each player tries to maximize her own total utility over an infinite horizon with discount factor $\gamma \in (0, 1)$.

Following Myerson (1997), it is sufficient to consider stationary strategy profiles, i.e., strategies that only depend on state of the game rather than the entire history. Specifically, consider stationary strategy τ_k for player k that maps state of the game θ to a particular action $d_k \in D_k(\theta)$ and the corresponding policy profile for all players, denoted as $\tau = (\tau_k)$. Let $v_k(\tau, \theta)$ denote player k's expected γ-discounted average payoff if players commit to stationary policy profile τ and the current state of game is θ. Further denote $Y_k(\tau, d_k, v_k, \theta)$ to represent player k's discounted average payoff starting from state θ if all players commit to stationary policy τ except player k, who deviates in the first round with action d_k,

$$Y_k(\tau, d_k, v_k, \theta) = (1-\gamma) u_k\big(\theta, (d_k, \tau_{-k}(\theta))\big) + \frac{\gamma}{N} \sum_{j' \in N} v_k\Big(\tau, \big(r', j', \pi(r', j'), \omega(\theta, d_k)\big)\Big) \ ,$$

$$\text{where} \quad \omega(\theta, d_k) = \Omega \quad \text{if } d_k = 1 \text{ or } k \neq i, j \ , \text{ and}$$

$$\omega(\theta, d_k) = \Omega \setminus k \quad \text{if } d_k = 0 \text{ and } k = i \text{ or } j \ .$$

Theorem 7.1 of Myerson (1997) states that the stationary strategy τ is an equilibrium strategy profile of the stochastic game if for every player k we have

$$v_k(\tau,\theta) = \max_{d_k \in D_k(\theta)} Y_k(\tau, d_k, v_k, \theta) \ . \tag{9}$$

In other words, if each player's optimal strategy is to not deviate from τ in a single period, then τ is an equilibrium strategy profile. Using these notations, it is straightforward to verify the following result, which we will use to prove Lemma 3 to support the Folk Theorem result.

LEMMA 1. *Following any service provider selection rule $\pi \in \Pi$, it is an equilibrium for every player $i = \pi(r,j)$ to always refuse providing service. The corresponding equilibrium discounted average payoff for each player is* 0.

In order to demonstrate the next result, we define the *always trade* strategy profile $\bar{\tau}$ to be such that in each time period the service requester always chooses to pay for service whenever she has positive scrips, and the selected type 0 player always chooses to provide service as long as the service requester belongs to set Ω and refuses to provide service if the service requester does not belong to set Ω. We also define *unichain selection rules* to be the set of service provider selection rules under which if players follow the *always trade* strategy profile, the resulting Markov chain on the state space S has a single recurrent class. It is easy to verify that both the minimum and random provider selection rules mentioned earlier in the paper are examples of unichain selection rules, along with many others.

LEMMA 2. *Following any unichain service provider selection rule π, if players follow the* always trade *strategy $\bar{\tau}$, the total discounted payoff is positive for all players at any state s for γ close enough to* 1.

Proof: Since policy π is unichain, the long run average payoff is independent of the initial state (r,j) (Bertsekas 2007). In any time period, the chance that a random service requester has a positive number of scrips is at least $1/N$. As a result, by following the *always trade* strategy profile, the expected social welfare gain per time period is at least $(u-c)/N$, which lower bounds the long run average social welfare gain. Since the N players are indistinguishable, the per player long run average payoff is lower bounded by $(u-c)/N^2 > 0$.

Following Proposition 4.1.2 of Bertsekas (2007), the total average discounted payoff $v_k(\bar{\tau}, \theta)$ converges to the long run average payoff as discount γ approaches 1, and therefore is also positive. Q.E.D.

Lemma 2 essentially states that when the discount factor is close enough to 1, the value function is positive at all states under any unichain service provider selection rule and the *always trade* strategy. Following the idea behind the Folk Theorem, if a player wants to refuse requesting or providing service in exchange for a scrip, and therefore deviate from the *always trade* strategy, the entire group of players can punish this player by refusing to provide service in the future. This results in an inferior, zero, total future utility for the focal player. This threat prevents a player from deviating from the *always trade* strategy. The following result summarizes this idea.

LEMMA 3. *Under any unichain service provider selection rule π, there exists a $\underline{\gamma}$ such that for any $\gamma \in [\underline{\gamma}, 1]$, the* always trade *strategy profile $\bar{\tau}$ is an equilibrium.*

This result follows from the Folk Theorem for stochastic games (Theorem 9, Dutta (1995)). The complete proof in the Appendix verifies the conditions of Theorem 9 in Dutta (1995) based on Lemmas 1 and 2.

Lemma 3 implies that as the discount factor is getting close to 1, the centralized optimal solution can be achieved in the stochastic game. This is by no means surprising, in light of the Folk Theorem. The above sequence of lemmas, however, motivates our next analysis of the *always trade* strategy when either the policy π is not unichain or the discount factor is not sufficiently close to 1.[2]

3.2. *Always Trade* Strategy

In this section we show that under certain conditions the central planner's optimal social welfare obtained in Section 2.1 can be achieved in the stochastic game. Motivated by Lemmas 1 - 3, we next focus on the case in which each player follows the *always trade* strategy.

Without loss of generality, denote $V(s)$ to represent the total discounted value function of player 1 at state of the system $s=(r,j)$, under service provider selection rule π and the *always trade* strategy $\bar{\tau}$. Therefore, function V satisfies the following recursive equation,

$$V = TV \ , \tag{10}$$

in which

$$(TV)(r,1) = \begin{cases} (\gamma/N)\sum_{j'} V(r,j') \ , & r_1 = 0 \\ u + \gamma/(N|\Upsilon^\pi(r,1)|)\sum_{j'}\sum_{i\in\Upsilon^\pi(r,1)} V(r-e_1+e_i,j') \ , & r_1>0 \end{cases} , \tag{11}$$

and

$$(TV)(r,j) = \begin{cases} (\gamma/N)\sum_{j'} V(r,j') \ , & r_j = 0 \\ \gamma/(N|\Upsilon^\pi(r,j)|)\sum_{j'}\sum_{i\in\Upsilon^\pi(r,j)} V(r-e_j+e_i,j') \ , & r_j>0, \ 1\notin\Upsilon^\pi(r,j) \\ \left(-c+\gamma/N\sum_{j'}\sum_{i\in\Upsilon^\pi(r,j)} V(r-e_j+e_i,j')\right)/|\Upsilon^\pi(r,j)| \ , & r_j>0, \ 1\in\Upsilon^\pi(r,j) \end{cases} . \tag{12}$$

Here the set $\Upsilon^\pi(r,j)$ represents the set of players eligible to be selected as the service provider, according to selection policy π; we assume ties are broken randomly. For example, under the minimum scrip selection rule $\bar{\pi}$, set $\Upsilon^{\bar{\pi}}(r,1)$ includes all players who hold the smallest number of scrips, excluding player 1. If the cardinality of the set $|\Upsilon^\pi(r,j)|>1$, each player in the set has the same chance of being chosen to be the service provider. When the service provider selection rule is clear in the context, we remove the superscript in Υ^π for simplicity.

Using the recursive expression of value function V, we show the following result.

PROPOSITION 2. *For any given service provider selection rule $\pi\in\Pi$ and model parameters u, c, N and R, there is a unique threshold $\bar{\gamma}\in(0,1)$, such that $V(r,j)\geq 0$ for all r and j if and only if $\gamma\geq\bar{\gamma}$.*

Proof: Denote V^γ to be the solution to recursive equations (10)-(12). That is, $V^\gamma = TV^\gamma$. Now consider a slightly revised value iteration,

$$(T^\gamma V)(r,1) = \begin{cases} (\gamma/N)\sum_{j'} V(r,j') \ , & r_1 = 0 \\ \gamma u + \gamma/(N|\Upsilon(r,1)|)\sum_{j'}\sum_{i\in\Upsilon(r,1)} V(r-e_1+e_i,j') \ , & r_1>0 \end{cases} ,$$

$$(T^{\gamma}V)(r,j)=\begin{cases}(\gamma/N)\sum_{j'}V(r,j')\ , & r_j=0\\ \gamma/(N|\Upsilon(r,j)|)\sum_{j'}\sum_{i\in\Upsilon(r,j)}V(r-e_j+e_i,j')\ , & r_j>0,\ 1\notin\Upsilon(r,j)\\ \left(-\gamma c+\gamma/N\sum_{j'}\sum_{i\in\Upsilon(r,j)}V(r-e_j+e_i,j')\right)/|\Upsilon(r,j)|\ , & r_j>0,\ 1\in\Upsilon(r,j)\end{cases}.$$

Denote $\hat{V}^{\gamma}$ to be the solution to $\hat{V}^{\gamma}=T^{\gamma}\hat{V}^{\gamma}$, which is also the total discounted value function of player 1 with γu and γc as the benefit and cost of trade instead of u and c. Therefore, $\hat{V}^{\gamma}=\gamma V^{\gamma}$.

Now consider a discount factor $\hat{\gamma}$ such that $V^{\hat{\gamma}}\geq 0$, which implies $\hat{V}^{\hat{\gamma}}\geq 0$. Consider any discount factor γ' such that $\gamma'>\hat{\gamma}$. We have $T^{\gamma'}\hat{V}^{\hat{\gamma}}\geq T^{\hat{\gamma}}\hat{V}^{\hat{\gamma}}=\hat{V}^{\hat{\gamma}}\geq 0$. Following the convergence of the value iteration algorithm and monotonicity of the operator T^{γ} (Corollary 1.2.1.1 and Lemma 1.1.1 in Bertsekas (2007)),

$$\hat{V}^{\gamma'}=\lim_{t\to\infty}(T^{\gamma'})^t\hat{V}^{\hat{\gamma}}\geq\lim_{t\to\infty}(T^{\hat{\gamma}})^t\hat{V}^{\hat{\gamma}}=\hat{V}^{\hat{\gamma}}\geq 0\ ,$$

which implies $V^{\gamma'}=\hat{V}^{\gamma'}/\gamma'\geq 0$. Q.E.D.

Note that this result is stronger than Lemma 2 because it holds for policies π that are not unichain and shows a unique threshold $\bar{\gamma}$. Parallel to Proposition 2, we have the following intuitive result.

PROPOSITION 3. *For any given service provider selection rule $\pi\in\Pi$ and model parameters γ, N and R, there is a unique threshold on u/c, such that $V(r,j)\geq 0$ for all r and j if and only if u/c is larger than this threshold.*

The proof is very similar to the proof of Proposition 2 and thus is omitted here.

Propositions 1, 2 and 3 imply the following main result of this section, which is stronger than Lemma 3.

THEOREM 2. *For any model parameters u, c, N and R, there is a unique threshold of the discount factor, $\bar{\gamma}$, such that when $\gamma>\bar{\gamma}$, the centralized optimal social welfare is achieved in equilibrium. That is, under the minimum scrip selection rule $\bar{\pi}$, in equilibrium all players follow the* always trade *strategy.*

Similarly, for any given model parameters γ, N and R, there is a unique threshold on u/c, above which the centralized optimal social welfare is achieved in equilibrium by all players following the always trade *strategy under the minimum scrip selection rule.*

The equilibrium result is proved by applying the definition of the *always trade* strategy to equilibrium condition (9).

4. Number of Scrips in the System

In the previous section we demonstrated the optimality of the minimum scrip selection rule in the stochastic game setting with a fixed number of scrips and when the discount factor is large enough. In this section we investigate the appropriate number of scrips to ensure that *always trade* is an equilibrium strategy under the minimum scrip selection rule. In particular, we show that under fairly general conditions, there is a unique threshold of the number of scrips in the system, below which *always trade* is an equilibrium. Furthermore, the threshold increases with the discount factor γ and the benefit of receiving service, u/c.

First we present a condition under which no matter how many scrips are in the system, the value function for any state is non-negative, implying that *always trade* is an equilibrium.

THEOREM 3. *Under any service provider selection rule $\pi \in \Pi$, if*

$$\frac{u}{c} \geq \frac{N}{\gamma} \ , \tag{13}$$

no matter how many scrips are in the system, the value function V that solves the recursive equations (10)-(12) is non-negative; that is, the always trade *strategy is an equilibrium.*

The proof is presented in the Appendix.

The condition (13), rewritten as $c \leq \gamma u/N$, reflects the trade-off between the cost of serving today versus the expected benefit of receiving service tomorrow. It is intuitive that if the cost of earning a scrip today is less than the expected benefit of spending it in the next period, providing service never generates a negative net expected profit. Interestingly, the condition does not depend on the service provider selection rule.

Since the condition is rather restrictive, we next analyze what happens under the minimum scrip selection rule when the condition $c \leq \gamma u/N$ is violated. First, we present a technical characterization of recurrent states, which is somewhat interesting in its own right and useful for proving our main result.

LEMMA 4. *Consider the case when $R \geq N$, i.e., the number of scrips in the system is no less than the number of players. Under the minimum scrip selection rule and* always trade *strategy, any state with more than one player having 0 scrips is transient.*

The proof is based on induction on the number of players with 0 scrips. The detailed proof is presented in the Appendix. Lemma 4 allows us to restrict attention to only those states where no more than 1 player has 0 scrips, which will be useful to prove Proposition 4, constituting the foundation of our main result, Theorem 4.

Analogous to the symmetry condition (C1) in the central planner setting, Lemma 5 below provides a symmetry argument needed for the proofs of Propositions 4 and 5.

LEMMA 5. *Assume value function V satisfies recursive equations (10)-(12) for a system with $R \geq N$ scrips with the minimum scrip selection rule. For any nonnegative integer vectors r and r' such that $\sum_j r_j = R$ and $\sum_j r'_j = R$ with $r_k = r'_k$ for all k except $l \neq 1$, $m \neq 1$, where $r_l = r'_m$ and $r_m = r'_l$,*

$$\sum_j V(r,j) = \sum_j V(r',j) \ . \tag{14}$$

The proof is presented in the Appendix.

PROPOSITION 4. *Assume value function V satisfies recursive equations (10)-(12) for a system with $R \geq N$ scrips with the minimum scrip selection rule, and value function $\bar{V}$ satisfies recursive equations (10)-(12) for a system with the same parameters except with $R+1$ scrips. Further, assume that*

$$\frac{u}{c} \leq \frac{N}{\gamma}\left[\frac{N}{\gamma} - (N-1)\right] - (N-1) \ . \tag{15}$$

For any nonnegative integer vector r in the recurrent class such that $\sum_j r_j = R$, and any player index j and $k \neq 1$, we have

$$\bar{V}(r+e_k,j) \leq V(r,j) \ , \tag{16}$$

$$\sum_j \bar{V}(r+e_1,j) - \sum_j V(r,j) \leq \frac{cN}{\gamma} \ , \quad \forall r: r_1 > 0 \ , \quad \textit{and} \tag{17}$$

$$\sum_j \bar{V}(r+e_1,j) - \sum_j V(r,j) \leq \frac{N}{\gamma}\left[\frac{N}{\gamma} - (N-1)\right] c \ , \ \forall r: r_1 = 0 \ . \tag{18}$$

Property (16) is a monotonicity property for value functions across different state spaces, and it states that if we inject one more scrip in the system, every player, other than the one who receives the scrip, is worse off (measured by the value function). The basic intuition behind it is that giving more scrips to others makes a player more likely to become the minimum scrip holder, and therefore work sooner than otherwise. For the one who does receive the additional scrip, while it is intuitive that the person is better off, properties (17) and (18) show that the benefit is, in fact, upper bounded. The intuition is that even though the player who receives the additional scrip is better off by, at some point, spending it for service, such a trade is followed by a state where the additional scrip belongs to someone else, therefore the player will be worse off afterwards, following (16). The basic logic of the proof for Proposition 4 is that these properties are preserved through value iteration. The complete proof, however, needs to verify the properties under all possible scenarios of scrip distribution among players in both the R scrip system as well as the $R+1$ scrip system. Furthermore, in order to establish each one of properties (16)-(18), we need all the properties to hold to begin the value iteration, as well as condition (15). As a result, the proof is rather involved and is presented in the Appendix.

Property (16) is the key property that we focus on. It implies that there is a threshold (possibly infinity) on the number of scrips, above which the value function at some state may become negative. When the value function does become negative, the threat of zero utility does not work anymore, and the corresponding player at this state is better off claiming bankruptcy by leaving the group. In order to prevent such an undesirable outcome, the number of scrips issued in the system must be lower than this threshold. As discussed in Section 2.1, the social welfare of the system increases with the number of scrips. Therefore, assuming players follow the minimum scrip selection rule, the system designer will choose the number of scrips in the system to be just below this threshold in order to achieve the greatest possible social welfare while making it in each player's best interest to not leave the group.

The monotonicity property (16), however, holds only under condition (15). Figure 2 depicts the condition. That is, in the area below the solid curve, due to monotonicity there is a threshold on

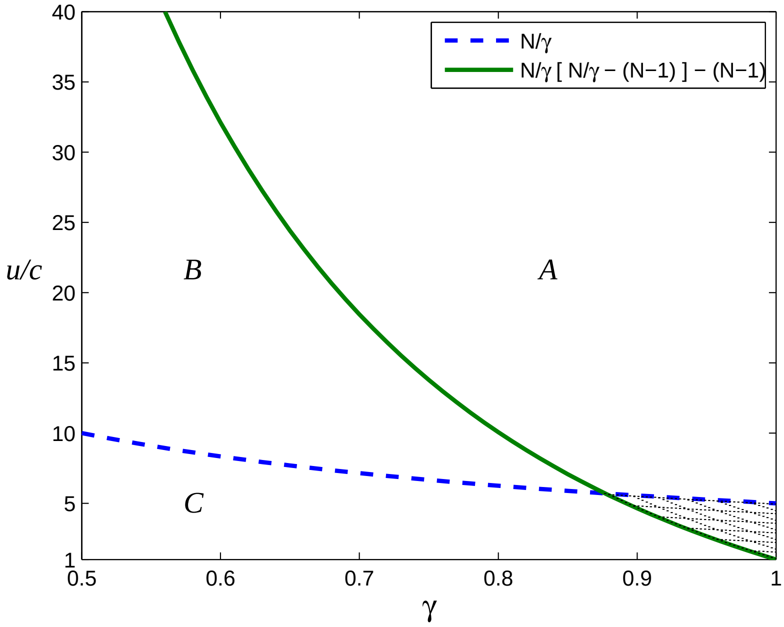

Figure 2 **Conditions (13) and (15), with** $N = 5$**.**

the number of scrips (possibly infinity), below which *always trade* is an equilibrium strategy. The dashed curve in Figure 2 corresponds to condition (13) in Theorem 3. In the area above the dashed curve, no matter how many scrips are in the system, *always trade* is an equilibrium strategy. These curves partition Figure 2 into four ares. In area A, no matter how many scrips are in the system, *always trade* is an equilibrium strategy. In area B, adding a scrip to the system decreases every player's total discounted value except that of the player with the additional scrip; however, we know that each player's total discounted value remains positive and thus no matter how many scrips are in the system, *always trade* is an equilibrium. In area C, adding a scrip to the system decreases every player's total discounted value except that of the player with the additional scrip; in this case, there is a threshold (possibly infinity) on the number of scrips above which at least one of the player's total discounted value is negative. This leaves the shaded area depicted in the figure not covered by theoretical results. Later in Section 4.1, we conduct a numerical study on the shaded area, which indicates that although the monotonicity property (16) does not hold, it is very likely that there still exists a unique upper bound on the number of scrips.

Theorem 2 in the previous section states that for any given number of scrips R, there is a threshold on γ above which the value function V is always positive. Proposition 4 further states that under condition (15), for a given discount γ, there is a threshold R below which the value function V is always positive. The combination of the two results implies that in the $R-\gamma$ space, the region that guarantees that value function V is always positive is characterized by a threshold in R that is monotone in γ. The result is formally stated in the following Theorem. (The result on u/c follows the exact same argument.)

THEOREM 4. *In a scrip system with N players and at least N scrips, for any given set of model parameters such that*

$$\frac{u}{c} \geq \frac{2(N-1)}{\sqrt{N^2+4N-4}-N} \ , \quad or \quad \gamma \leq \frac{N(\sqrt{N^2+4N-4}-N)}{2(N-1)} \ , \tag{19}$$

there is an upper bound $\bar{R}$ (possibly infinity) on the number of scrips, below which always trade *is an equilibrium strategy under the minimum scrip selection rule, and the system optimal social welfare is achieved in the game. Furthermore, the upper threshold $\bar{R}$ increases with γ and u/c.*

Sufficient condition (19) is obtained by equating conditions (13) and (15). Illustrated in Figure 2, condition (19) covers the area to the left and above of the intersection between the solid and dashed curves. As we will demonstrate in numerical studies in Section 4.1, the monotone threshold structure presented in Theorem 4 likely holds even without these conditions being met.

4.1. Shaded Area

We do not have theoretical results when conditions (13) and (15) are both violated, depicted by the shaded area in Figure 2. Therefore, we conducted numerical studies to check the structure of the value function in its minimum recurrent state. In particular, we take a grid of values for u/c and γ in the shaded area when $N=3$,[3] and we see how the minimum value function's value (over recurrent states) changes with increasing R. We observe that in every case, the value function is unimodal and therefore monotonically decreases as R increases to be large enough. Figure 3 depicts one such example.

The findings indicate that when condition (15) in Proposition 4 is violated, a player's value function does not always decrease monotonically with more scrips given to others. On the other

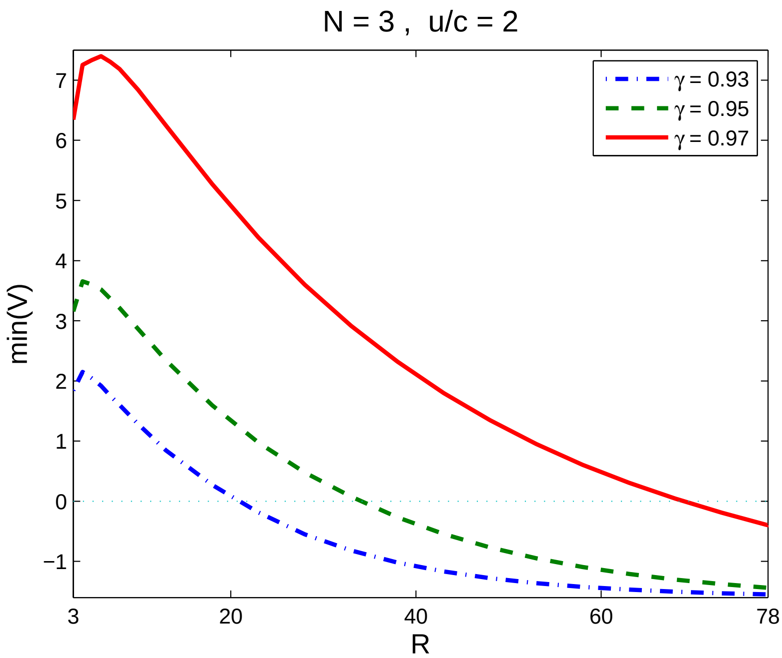

Figure 3 **The non-monotone, unimodal structure of the value function in its minimum recurrent state.**

hand, in our numerical examples, it always first increases when the number of scrips R is small, and then decreases. Therefore, as long as the minimum value function over recurrent states is positive at the smallest scrip number $R = N$, there still is a unique upper bound (possibly infinity) on the number of scrips, below which the value function is positive in all states. If so, the threshold on the number of scrips in the system increases with γ and u/c, even without the necessity of condition (19).

The following result indicates that when $R = N$ the value function is indeed positive in all recurrent states.

PROPOSITION 5. *Assume $R = N$ and*

$$\frac{u}{c} \geq \frac{N}{\gamma}\left[\frac{N}{\gamma} - (N-1)\right] - (N-1) \ . \tag{20}$$

The value function V that solves the recursive equations (10)-(12) under the minimum scrip selection rule is positive in all recurrent states.

The proof is included in the Appendix.

4.2. Random provider selection rule

It is important to note that there is a possibility that a different service provider selection rule may permit a greater number of scrips than the minimum scrip selection rule. We have shown that for a given number of scrips in the system, the minimum scrip selection rule achieves the maximum social welfare. It could be possible, however, that a different service provider selection rule outperforms the minimum scrip selection rule because it allows more scrips in the system without a player defaulting compared to the minimum scrip selection rule. Here we analyze the random provider selection rule, a common selection rule considered in previous literature. More precisely, we consider a generalization of the minimum scrip selection rule that also includes the random provider selection rule.

In particular, consider the following service provider selection rule. First, K players out of the $N-1$ players are randomly selected as "potential providers". Then the potential provider who has the least number of scrips is selected as the service provider. Such a setting covers the possibility that not every player can provide service in each period, which, to a certain extent, relates to settings studied in Kash et al. (2007), Kash et al. (2012a), and Kash et al. (2012b). Obviously, when $K=1$, we have the random provider selection rule, while the case $K=N-1$ is the minimum scrip selection rule. The recursive equations (10)-(12) are customized to $V=TV$, where

$$(TV)(r,j) = \mathrm{E}_{\kappa_j}[(\Xi_{\kappa_j}V)(r,j)] \ , \tag{21}$$

in which κ_j represents the set of K randomly selected potential providers among the $N-1$ players who are not j, and $\Xi_{\kappa_j}V$ follows

$$(\Xi_{\kappa_1}V)(r,1) = \begin{cases} (\gamma/N)\sum_{j'} V(r,j') \ , & r_1 = 0 \\ u + \gamma/(N|\Upsilon_{\kappa_1}(r,1)|)\sum_{j'}\sum_{i\in\Upsilon_{\kappa_1}(r,1)} V(r-e_1+e_i,j') \ , & r_1 > 0 \end{cases} , \tag{22}$$

and

$$(\Xi_{\kappa_j}V)(r,j) = \begin{cases} (\gamma/N)\sum_{j'} V(r,j') \ , & r_j = 0 \\ \gamma/(N|\Upsilon_{\kappa_j}(r,j)|)\sum_{j'}\sum_{i\in\Upsilon_{\kappa_j}(r,j)} V(r-e_j+e_i,j') \ , & r_j > 0, \ 1 \notin \Upsilon_{\kappa_j}(r,j) \\ \Big(-c + \gamma/N\sum_{j'}\sum_{i\in\Upsilon_{\kappa_j}(r,j)} V(r-e_j+e_i,j')\Big)/|\Upsilon_{\kappa_j}(r,j)| \ , & r_j > 0, \ 1 \in \Upsilon_{\kappa_j}(r,j) \end{cases} . \tag{23}$$

Here $\Upsilon_{\kappa_j}(r, j)$ represents the set of minimum scrip holders among players in the set κ_j.

Similar to Proposition 4 for the minimum scrip selection rule, the following result holds for the general service provider selection rule described above, including the random provider selection rule.

Proposition 6. *Assume value function V satisfies $V = TV$, where operator T is defined in equations (21)-(23), and value function $\bar{V}$ satisfies the same recursive equation in a system with the same parameters except with $R+1$ scrips. Further, assume that*

$$\frac{u}{c} \leq \frac{N}{\gamma} - (N-1) \ . \tag{24}$$

For any nonnegative integer vector r such that $\sum_j r_j = R$ and any player index j and $k \neq 1$, we have

$$\bar{V}(r + e_k, j) \leq V(r, j) \ , \quad \textit{and} \tag{25}$$

$$\sum_j \bar{V}(r + e_1, j) - \sum_j V(r, j) \leq \frac{cN}{\gamma} \ . \tag{26}$$

The proof is similar to that of Proposition 4 and is presented in the Appendix.

For similar reasons as discussed above for the minimum scrip selection rule, property (25) implies that there is a threshold (possibly infinity) on the number of scrips in the system, above which the value function at some state may become negative. Numerical studies similar to those described in Section 4.1 suggest that conditions (13) and (24) are sufficient, but not necessary, for the threshold structure to hold.

Interestingly, therefore, both the minimum scrip selection rule and the random provider selection rule permit a threshold number of scrips in the system (possibly infinity), above which the system crashes. Depending upon the system parameters u, c, N, and γ, numerical results show that sometimes the minimum scrip selection rule permits at least as many scrips as the random provider selection rule; in this case, the minimum scrip selection rule is the preferable service provider selection rule for the scrip system because it provides a greater social welfare. For other parameters, though, the random provider selection rule permits more scrips than the minimum scrip selection rule, and in some cases the difference is enough to cause the random provider selection rule to outperform the minimum scrip selection rule in terms of social welfare. Depending on the system parameters and application, the system designer may choose to compare the performance of the minimum scrip selection rule and the random provider selection rule before creating the scrip system. Since both exhibit a threshold property on the number of permissable scrips in the system, this should be relatively simple to do.

5. Conclusion

In this paper we study design issues for managing a scrip system in a stochastic setting. In particular, in each period one player becomes a service requester and receives positive utility if another player is willing to provide the service in exchange for a scrip. We first show that a central planner would always prefer a trade of scrip for service to occur and would select the player who has the least number of scrips to be the service provider.

In a stochastic game setting with the absence of a central planner, such a system optimal solution can be achieved in equilibrium when the time discount factor is high enough or when the benefit of service is high enough compared with the cost to the service provider. When the time discount factor, or ratio between the benefit and cost of service, is not that high, we show that when using the minimum scrip selection rule or random provider selection rule there is an upper bound on the number of scrips that are allowed in the system, above which some players may decide to default and exit the game when their scrip stock becomes low. Furthermore, this upper bound increases with the time discount factor as well as the ratio between the benefit and cost of service.

From a system design point of view, our results demonstrate that, assuming players follow the minimum scrip selection rule, the number of scrips in the system should be at the upper bound, and all players have the incentive to trade a scrip for service whenever the service requester has at least one scrip. We also analyzed a commonly used service provider selection rule, the random provider selection rule, and showed similar threshold results as the minimum scrip selection rule. This makes it simple for the system designer to compare the performance of the minimum scrip selection rule and the random provider selection rule and choose the rule that results in the greatest social welfare.

One inherent assumption of our work is that each player is able to provide service when asked, and, furthermore, the detection and punishment of players who do not contribute is possible. In practice for large scrip systems, this could be difficult, especially if the system relies on the service requester to report whether trade occurred, which could open the possibility of malicious players getting others kicked out of the system. Previous work by Kash et al. (2007), Kash et al. (2012a) and Kash et al. (2012b) has not required the ability of each player to provide service nor the detection and punishment of players who do not contribute. With a few additional assumptions and model differences as described in Section 1.1, they show the existence of an equilibrium in which players follow a threshold policy under the random provider selection rule. One useful extension of our work would be to possibly combine these two streams of work by using the minimum scrip rule without the severe punishment that the Folk Theorem equilibrium relies upon.

There are a number of other possible extensions to our paper that deserve further exploration. The most notable one is that preferences are not homogeneous among players. For example, some players may value the service more than others, or providing service may cost some players more than others. Similarly, the benefit and cost of service may change over time or be stochastic. We suspect that the reason why our system does not experience a market crash when there are too few scrips, as observed in some applications, is a result of our current assumptions on the homogeneity of utilities across players and over time. Future work will hopefully provide us with more insights on this type of market crash. Another extension that is worth studying is that the price of service may not be fixed at one scrip, but is instead determined according to the scrip distribution among players.

Endnotes

1. In the stochastic game, generally speaking, the service provider selection rule may allow selection of another player in case a first selected player rejects to serve. Such a generalized selection rule

π maps the distribution of scrips to a sequence of provider selections, contingent upon acceptance. All equilibrium results hold with this generalization.

2. When the policy π is not unichain, Lemma 2 does not hold. Therefore Condition 1 of Dutta's theorem may be violated. Furthermore, Dutta's Folk Theorem for stochastic games requires the discount factor to be sufficiently close to 1.

3. Since the state space grows exponentially with N, which poses significant computational challenges, we did not check for higher values of N.

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Online Appendix

Appendix

Proof of Proposition 1

In this proof we show that starting from a function $\mathcal{J}$ that satisfies conditions (**C1**), (**C2**) and (**C3**), $\Gamma\mathcal{J}$ also satisfies them. First note that function $\mathcal{J}_0 = 0$ for all r (obtained from $J_0 = 0$ for all (r, j)) satisfies all these conditions.

Condition (5) for $\mathcal{J}$ implies

$$(u-c) + \frac{\gamma}{N} \max_{m:m\neq l} \mathcal{J}(r + e_m - e_l) \geq \frac{\gamma}{N}\mathcal{J}(r) \ ,$$

which further implies

$$(\Gamma\mathcal{J})(r) = N_r(u-c) + \frac{\gamma}{N}\left[(N-N_r)\mathcal{J}(r) + \sum_{l:r_l\geq 1} \max_{m:m\neq l} \mathcal{J}(r+e_m-e_l)\right] \ . \tag{27}$$

1. Condition (**C1**)

We first show that condition (3) holds for $\Gamma\mathcal{J}$ in (27). That is, $(\Gamma\mathcal{J})(r) = (\Gamma\mathcal{J})(r')$. Note that

$$(\Gamma\mathcal{J})(r') = N_{r'}(u-c) + \frac{\gamma}{N}\left[(N-N_{r'})\mathcal{J}(r') + \sum_{l:r'_l\geq 1} \max_{m:m\neq l} \mathcal{J}\Big(r'+e_m-e_l\Big)\right] \ .$$

Obviously, $N_r = N_{r'}$, and $\mathcal{J}(r) = \mathcal{J}(r')$ due to (3). We only have the following cases.

- $l \neq i, j$. For $m \neq i, j$, we have $\mathcal{J}(r+e_m-e_l) = \mathcal{J}(r'+e_m-e_l)$ since $(r+e_m-e_l)_i = (r'+e_m-e_l)_j$ and $(r+e_m-e_l)_j = (r'+e_m-e_l)_i$. Furthermore, we have $\mathcal{J}(r+e_i-e_l) = \mathcal{J}(r'+e_j-e_l)$ and $\mathcal{J}(r+e_j-e_l) = \mathcal{J}(r'+e_i-e_l)$. Therefore, $\max_{m:m\neq l}\mathcal{J}(r+e_m-e_l) = \max_{m:m\neq l}\mathcal{J}\Big(r'+e_m-e_l\Big)$.
- $r_i \geq 1$ and $l = i$. For $m \neq j$, we have $\mathcal{J}(r+e_m-e_i) = \mathcal{J}(r'+e_m-e_j)$. Furthermore, we have $\mathcal{J}(r+e_j-e_i) = \mathcal{J}(r'+e_i-e_j)$. Therefore, $\max_{m:m\neq l}\mathcal{J}(r+e_m-e_l) = \max_{m:m\neq l}\mathcal{J}\Big(r'+e_m-e_l\Big)$.
- $r_j \geq 1$ and $l = j$. The same logic as in the previous case reveals $\max_{m:m\neq l}\mathcal{J}(r+e_m-e_l) = \max_{m:m\neq l}\mathcal{J}\Big(r'+e_m-e_l\Big)$.

2. Condition (**C2**)

First of all, when $r_i = r_j + 1$, then symmetry condition **C1** implies $\mathcal{J}(r - e_i + e_j) = \mathcal{J}(r)$ so the condition (4) holds. Therefore we only need to show the result carries through value iteration when $r_i \geq r_j + 2$. Denote $m_l^* = \arg\max_{m:m\neq l}\mathcal{J}(r+e_m-e_l)$, which has to be a minimum scrip holder other than l. That is,

$$(\Gamma J)(r) = N_r(u-c) + \frac{\gamma}{N}\left[(N-N_r)\mathcal{J}(r) + \sum_{l:r_l\geq 1}\mathcal{J}(r+e_{m_l^*}-e_l)\right]$$

Consider the following cases.

1. $r_j > 0$. In this case, since $r_i \geq r_j + 2$, we know that $N_r = N_{r-e_i+e_j}$, and for any l with $r_l > 0$, we also have $(r - e_i + e_j)_l > 0$, and vice versa.

1.1 $r_i \geq r_j + 3$. Note that for any l such that $r_l \geq 1$, we have $(r + e_{m_l^*} - e_l)_i > (r + e_{m_l^*} - e_l)_j$. Therefore

$$(\Gamma J)(r) \leq N_r(u-c) + \frac{\gamma}{N}\left[(N - N_r)\mathcal{J}(r - e_i + e_j) + \sum_{l:r_l\geq 1} \mathcal{J}(r + e_{m_l^*} - e_l - e_i + e_j)\right] \leq (\Gamma J)(r - e_i + e_j) \ .$$

1.2 $r_i = r_j + 2$ and $m_i^* \neq j$. In this case we also have that for any l such that $r_l \geq 1$, $(r + e_{m_l^*} - e_l)_i > (r + e_{m_l^*} - e_l)_j$. Therefore the situation is the same as the previous case.

1.3 $r_i = r_j + 2$ and $m_i^* = j$. In this case we only have that for any $l \neq i$ such that $r_l \geq 1$, $(r + e_{m_l^*} - e_l)_i > (r + e_{m_l^*} - e_l)_j$. Therefore

$$(\Gamma J)(r) = N_r(u-c) + \frac{\gamma}{N}\left[(N - N_r)\mathcal{J}(r) + \mathcal{J}(r - e_i + e_j) + \sum_{l\neq i:r_l\geq 1} \mathcal{J}(r + e_{m_l^*} - e_l)\right]$$

$$\leq N_r(u-c) + \frac{\gamma}{N}\left[\mathcal{J}(r) + (N - N_r)\mathcal{J}(r - e_i + e_j) + \sum_{l\neq i:r_l\geq 1} \mathcal{J}(r + e_{m_l^*} - e_l - e_i + e_j)\right]$$

Note that if $m_i^* = j$, then j is a minimum scrip holder in r other than i. Following symmetry, we have either $\max_{m:m\neq i} \mathcal{J}(r - 2e_i + e_j + e_m) = \mathcal{J}(r)$ (when j is the unique minimum scrip holder in r other than i) or $\max_{m:m\neq i} \mathcal{J}(r - 2e_i + e_j + e_m) = \mathcal{J}(r - e_i + e_j)$ (when j is not the unique minimizer in r other than i). In either case,

$$(\Gamma J)(r) \leq (\Gamma J)(r - e_i + e_j) \ .$$

2. $r_j = 0$. In this case $N_r = N_{r-e_i+e_j} - 1$.

$$(\Gamma J)(r - e_i + e_j) \geq (N_r + 1)(u-c) + \frac{\gamma}{N}\Big[(N - N_r - 1)\mathcal{J}(r - e_i + e_j) + \max_{m:m\neq j} \mathcal{J}(r - e_i + e_j + e_m - e_j)$$

$$+ \sum_{l:r_l\geq 1} \mathcal{J}(r + e_{m_l^*} - e_l - e_i + e_j)\Big] \geq N_r(u-c) + \frac{\gamma}{N}\Big[(N - N_r)\mathcal{J}(r - e_i + e_j) + \sum_{l:r_l\geq 1} \mathcal{J}(r + e_{m_l^*} - e_i + e_j - e_l)\Big] \geq \Gamma\mathcal{J}(r) \ ,$$

where the second inequality follows Condition (**C3**), that is,

$$\frac{N}{\gamma}(u-c) + \max_{m:m\neq j} \mathcal{J}(r - e_i + e_j + e_m - e_j) \geq \mathcal{J}(r - e_i + e_j) \ ;$$

and the last inequality follows the same logic as in the previous cases.

3. Condition (**C3**)

First, we show condition (5). From the previous proof, we know that for $\mathcal{J}$ satisfying (C1) and (C2), when $r_j > r_i$, we have $\mathcal{J}(r) \leq \mathcal{J}(r + e_i - e_j)$, which implies the condition automatically. Therefore we focus on cases when $r_i \geq r_j$. Furthermore, player i must be the minimum scrip holder in r except j.

Denote m_l^* to represent the player that achieves $\max_{m:m\neq l} \mathcal{J}(r + e_m - e_l)$.

- $r_j \geq 2$. In this case $N_r = N_{r+e_i-e_j} = N$. Therefore,

$$(\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r+e_i-e_j) = \frac{\gamma}{N}\sum_l \left(\max_{m:m\neq l}\mathcal{J}(r+e_m-e_l) - \max_{m:m\neq l}\mathcal{J}(r+e_i-e_j+e_m-e_l)\right)$$
$$\leq \frac{\gamma}{N}\sum_l \Big(\mathcal{J}(r+e_{m_l^*}-e_l) - \mathcal{J}\big((r+e_{m_l^*}-e_l)+e_i-e_j\big)\Big)$$

—$|I_{r_i}| = 2$. This means that $I_{r_i} = \{i,j\}$ and i is the unique minimum scrip holder in r except j. Thus for each possible value of l, i is also the minimum scrip holder in $r+e_{m_l^*}-e_l$; note that if $r_i = r_j$ and $l = k \neq i,j$, then by (C1) we have $\mathcal{J}(r+e_i-e_k) = \mathcal{J}(r+e_j-e_k)$ and we can choose $m_k^* = j$ to make this statement true. Therefore, condition (5) implies the following result:

$$(\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r+e_i-e_j) \leq \frac{\gamma}{N}\left[N\frac{N}{\gamma}(u-c)\right] < \frac{N}{\gamma}(u-c)\ .$$

—$|I_{r_i}| > 2$. This means that player i is not a unique minimum scrip holder in r except j; in other words, another player(s) has r_i scrips. For $l = j$, we know from (C1) that $\mathcal{J}(r+e_{m_j^*}-e_j) = \mathcal{J}(r+e_k-e_j)$ for any player $k \neq i$ with $r_k = r_i$. Thus, i is a minimum scrip holder in $r+e_k-e_j$ and we can apply condition (5).

For $l = i$, i is the minimum scrip holder in $r+e_j-e_i$ except j, so again we can apply condition (5). For $l = k \neq i,j$ with $r_k > r_i$, i is the minimum scrip holder in $r+e_j-e_k$ except j, so again we can apply condition (5).

Finally, for $l = k \neq i,j$ with $r_k = r_i$, i is no longer a minimum scrip holder in $r+e_j-e_k$. In this case, if $r_i = r_j$, we know from (C1) that $\mathcal{J}(r+e_{m_k^*}-e_k) = \mathcal{J}(r+e_j-e_k)$ and by choosing $m_k^* = j$, we see that $(r+e_j-e_k)_i < (r+e_j-e_k)_j$ and from (C1) and (C2) we have $\mathcal{J}(r+e_j-e_k) \leq \mathcal{J}(r+e_j-e_k+e_i-e_j)$. On the other hand, if $r_i > r_j$, we have $|I_{(r+e_j-e_k)_i}| \geq 3$ and includes i, j, and k; since $(r+e_j-e_k)_j \geq 2$ we know condition (6) holds so we can apply condition (7). Putting all these cases together, we again have

$$(\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r+e_i-e_j) \leq \frac{\gamma}{N}\left[N\frac{N}{\gamma}(u-c)\right] < \frac{N}{\gamma}(u-c)\ .$$

- $r_j = 1$, $r_i > 1$. In this case $N = N_r = N_{r+e_i-e_j} + 1$.

$$(\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r+e_i-e_j) = (u-c) + \frac{\gamma}{N}\sum_{l:l\neq j}\left(\max_{m:m\neq l}\mathcal{J}(r+e_m-e_l) - \max_{m:m\neq l}\mathcal{J}(r+e_i-e_j+e_m-e_l)\right)$$

Following the exact same line of argument as above, we have

$$(\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r+e_i-e_j) \leq (u-c) + \frac{\gamma}{N}\left((N-1)\frac{N}{\gamma}(u-c)\right) < \frac{N}{\gamma}(u-c)\ .$$

- $r_j = 1$, $r_i = 1$. In this case we must have $\max_{m:m\neq l}\mathcal{J}(r+e_m-e_l) = \max_{m:m\neq l}\mathcal{J}(r+e_i-e_j+e_m-e_l) = \mathcal{J}(r+e_i-e_l)$, when $l \neq i,j$. And $\max_{m:m\neq i}\mathcal{J}(r+e_m-e_i) = \mathcal{J}(r+e_j-e_i) \leq \mathcal{J}(r) = \max_{m:m\neq i}\mathcal{J}(r+e_i-e_j+e_m-e_i)$. As a result, $(\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r+e_i-e_j) \leq (u-c)$.

Now we focus on showing condition (7). Again, from the previous proof, we know that for $\mathcal{J}$ satisfying (C1) and (C2), when $r_j > r_i$, we have $\mathcal{J}(r) \leq \mathcal{J}(r + e_i - e_j)$, which implies the condition automatically. Therefore we focus on cases when $r_i \geq r_j$. Also, note that in order for condition (6) to hold, we know that no players can have 0 coupons.

- $r_j \geq 2$. In this case $N_r = N_{r+e_i-e_j} = N$.

$$(\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r+e_i-e_j) = \frac{\gamma}{N}\sum_l \left(\max_{m:m\neq l}\mathcal{J}(r+e_m-e_l) - \max_{m:m\neq l}\mathcal{J}(r+e_i-e_j+e_m-e_l)\right)$$
$$\leq \frac{\gamma}{N}\sum_l \left(\mathcal{J}(r+e_{m_l^*}-e_l) - \mathcal{J}\big((r+e_{m_l^*}-e_l)+e_i-e_j\big)\right)$$

(1) There exists $r_k = 1$. According to condition (6), all other players in I_{r_i} must have exactly r_i scrip. For any player $l \neq k$, $r + e_{m_l^*} - e_l$ satisfies (6). Note that $\mathcal{J}(r+e_j-e_k) = \mathcal{J}(r+e_i-e_k)$ since $r_j = r_i$. Thus by choosing $m_k^* = j$, we have

$$(\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r+e_i-e_j) \leq \frac{\gamma}{N}\left((N-1)\frac{N}{\gamma}(u-c)\right) < \frac{N}{\gamma}(u-c)\ .$$

(2) $r_l \geq 2$ for all $l \in I_{r_i}$. By choosing $m_l^* \neq i$ when possible, $r + e_{m_l^*} - e_l$ always satisfies condition (6), which implies our result.

- $r_j = 1$. Condition (6) implies that for all players $l \in I_{r_i} \setminus j$, $r_l = r_i$. Therefore, i achieves $\max_{i:i\neq j}\mathcal{J}(r + e_i - e_j)$, which has been shown earlier in the proof.

We have shown that starting from a function $\mathcal{J}$ that satisfies conditions (C1), (C2), and (C3), $\Gamma\mathcal{J}$ also satisfies them. Therefore, the conditions hold for any function $\mathcal{J}$ through value iteration. The set of functions that satisfy the three conditions is closed (and convex), which implies that the limit $\mathcal{J}^*$ also satisfy (C1)-(C3). See, for example, Proposition 1 of **?**. (Strictly speaking, the proposition in **?** directly implies that (C1) and (C2) are closed, convex cone properties. Its proof, on the other hand, clearly indicates that (C3) is a closed, convex property.)

Q.E.D.

Steady state probabilities for random provider selection rule: Here we provide closed form expressions on the steady state probabilities and the probability of no trade under the random provider selection rule.

PROPOSITION 7. *In a system with N players and R scrips following the random provider selection rule, the steady state probability of any possible scrip distribution among players is $\rho = 1/\binom{R+N-1}{N-1}$. Furthermore,* *the probability that trade would not occur due to service requester's lack of scrip is* $\rho \sum_{k=1}^{N-1} \binom{N-1}{k-1} \cdot \binom{R-1}{N-k-1}$.

Proof: Under the random provider selection rule, it is obvious that the transition probability matrix is doubly stochastic, and therefore the steady state probability is the same for all states. The number of states for a system with N players and R scrips is the number of combinations of distributing R scrips to N players, which is $\binom{R+N-1}{N-1}$. This implies the expression for ρ.

The event of no trade occurs when the service requester has zero scrips. Divide the event into sub-events of exactly $k \le N-1$ players having zero scrips. The possible ways of having exactly k zero scrip players is $\binom{N}{k}\binom{R-1}{N-k-1}$, where the second term is the number of ways of distributing the R scrips among the remaining $N-k$ players, which is the same as distributing $R-N+k$ scrips among $N-k$ players without no-zero-scrip restrictions. As a result, the probability of no trade is

$$\sum_{k=1}^{N-1} \rho \frac{k}{N} \binom{N}{k} \binom{R-1}{N-k-1} = \rho \sum_{k=1}^{N-1} \binom{N-1}{k-1} \binom{R-1}{N-k-1} .$$

Q.E.D.

Proof of Lemma 3: This result follows directly from the Folk Theorem for stochastic games (Theorem 9, Dutta (1995)). In particular, we need to verify the following three conditions to invoke Theorem 9 in Dutta (1995):

1. Unichain selection rule guarantees that the set of feasible long-run average payoffs is independent of the starting state.

2. Each player's long-run average min-max payoff needs to be independent of the starting state. Instead of considering the min-max payoff, we consider the "always refuse providing service" equilibrium strategy, which generates 0 long run average payoff, and therefore independent of the starting state.

3. The dimension of the set of feasible payoffs should be N. This condition holds because any single player i can be ruled out from service, which creates a payoff vector that has a constant positive value for all players except player i, who has value zero. The resulting N vectors, each corresponding to one player being ruled out, span $\Re^N$.

Q.E.D.

Proof of Theorem 3: Start the value iteration algorithm (10)-(12) from $\bar{V}(r,j)$ defined as,

$$\bar{V}(r,1) = u \sum_{k=0}^{r_1-1} \left(\frac{\gamma}{N}\right)^k \ , \quad \text{if } r_1 \ge 1 \ ; \quad \text{and } \bar{V}(r,j) = 0 \ , \quad \text{otherwise.}$$

Note that $\bar{V} \geq 0$ in general, and, in particular, $\bar{V}(r,1) \geq u$ if $r_1 \geq 1$.

Next we verify that $T\bar{V} \geq \bar{V} \geq 0$.

1. When $j=1$ and $r_1 = 0$ we have,

$$(T\bar{V})(r,1) = \frac{\gamma}{N} \sum_{j'} \bar{V}(r,j') = 0 = \bar{V}(r,1) \ .$$

2. When $j=1$ and $r_1 > 0$ we have,

$$(T\bar{V})(r,1) = u + \frac{\gamma}{N|\Upsilon(r,1)|} \sum_{j'} \sum_{i \in \Upsilon(r,1)} \bar{V}(r - e_1 + e_i, j') = u \sum_{k=0}^{r_1 - 1} \left(\frac{\gamma}{N}\right)^k = \bar{V}(r,1) \ . \tag{28}$$

3. When $j \neq 1$ and $r_j = 0$ we have,

$$(T\bar{V})(r,j) = \frac{\gamma}{N} \sum_{j'} \bar{V}(r,j') \geq 0 = \bar{V}(r,j) \ .$$

4. When $j \neq 1$, $r_j > 0$, and $1 \notin \Upsilon(r,j)$ we have,

$$(T\bar{V})(r,j) = \frac{\gamma}{N|\Upsilon(r,j)|} \sum_{j'} \sum_{i \in \Upsilon(r,j)} \bar{V}(r - e_j + e_i, j') \geq 0 = \bar{V}(r,j) \ .$$

5. When $j \neq 1$, $r_j > 0$, and $1 \in \Upsilon(r,j)$, following (28) we have, $(T\bar{V})(r - e_j + e_1, 1) \geq u$. Therefore,

$$\begin{aligned}
(T\bar{V})(r,j) &= \Big[-c + \gamma/N \sum_{i \in \Upsilon(r,j)} \Big(\bar{V}(r - e_j + e_i, 1) + \sum_{j' \neq 1} \bar{V}(r - e_j + e_i, j') \Big] / |\Upsilon(r,j)| \\
&= \Big[-c + \gamma/N \sum_{i \in \Upsilon(r,j)} \bar{V}(r - e_j + e_i, 1) \Big] / |\Upsilon(r,j)| \\
&\geq \Big[-c + (\gamma/N) \bar{V}(r - e_j + e_1, 1) \Big] / |\Upsilon(r,j)| \\
&\geq \Big[-c + u\gamma/N \Big] / |\Upsilon(r,j)| \geq 0 = \bar{V}(r,j) \ ,
\end{aligned}$$

in which the first inequality is due to $\bar{V} \geq 0$, the second inequality due to $(r - e_j + e_1)_1 \geq 1$ and therefore $\bar{V}(r - e_j + e_1, 1) \geq u$, and the last inequality from $u/c \geq N/\gamma$.

Monotonicity and convergence of operator T imply that $V^\gamma = \lim_{t \to \infty} T^t \bar{V} \geq \bar{V} \geq 0$. Q.E.D.

Proof of Lemma 4: Consider a scrip distribution vector $r^{n_0}_{n_1 n_2}$, where exactly n_0 players have zero scrip, n_1 players have 1 scrip, and n_2 players have more than 1 scrip; therefore $n_0 + n_1 + n_2 = N$. For $n_0 \geq 1$, under the minimum scrip selection rule, from state $(r^{n_0}_{n_1 n_2}, j)$, the probability of transitioning into another state also with n_0 zero scrip players (including itself) is $(n_0 + n_1)/N$ (this happens when either a zero scrip player or a one scrip player becomes the service requester, j). Otherwise, with probability n_2/N, the system transitions into a state with exactly $n_0 - 1$ zero scrip players. For $n_0 = 0$, the probability of transitioning into a state with one zero scrip player is n_1/N, and the probability of transitioning into a state where there are no zero scrip players is n_2/N. This implies that when $n_0 \geq 2$ (and $N \geq 3$), the system never transitions into state $(r^{n_0}_{n_1 n_2}, j)$ from another state with fewer number of zero scrip players. In addition, since $R \geq N$, we must have $n_2 > 0$ when $n_0 \geq 1$. That is, there is a positive probability when $n_0 \geq 1$ that the system transitions out of state $(r^{n_0}_{n_1 n_2}, j)$ and into another state with one fewer zero scrip players. Starting from (r^{N-1}_{01}, j), where the R scrips concentrate in one player, leaving the other $N-1$ players with 0 scrips, the above analysis implies that state (r^{N-1}_{01}, j) must be transient. Decrease n_0 one by one from $N-2$ to 2, and we have that state $(r^{n_0}_{n_1 n_2}, j)$ is transient as long as $n_0 \geq 2$.

Note that for the special case when $N = 2$, it is impossible to have a state where more than one player has 0 scrips since the $R \geq 2$ scrips must be held by at least one player. Q.E.D.

Proof of Lemma 5: We need to show that starting from any function V that satisfies (14), the condition still holds after one step value iteration defined by the right hand side of recursive equations (10)-(12), denoted as TV. For simplification of notation, we denote $\nu(r) = \sum_j V(r, j)$. We will show that every term $(TV)(r, j)$ corresponds to an equivalent term $(TV)(r', \hat{j})$ for some $\hat{j}$ and vice versa, thus implying the result.

(1) $j = 1$. We show the equivalence of $(TV)(r, 1)$ with $(TV)(r', 1)$ under the following two possibilities. Note that if $r_1 = 0$, then $r'_1 = 0$.

(a) $r_1 = 0$. By directly applying the induction hypothesis (14),

$$(TV)(r,1) = \frac{\gamma}{N}\nu(r) = \frac{\gamma}{N}\nu(r') = (TV)(r',1)$$

(b) $r_1 > 0$.

$$(TV)(r,1) = u + \frac{\gamma}{N|\Upsilon(r,1)|}\sum_{i \in \Upsilon(r,1)} \nu(r - e_1 + e_i)$$
$$(TV)(r',1) = u + \frac{\gamma}{N|\Upsilon(r',1)|}\sum_{i \in \Upsilon(r',1)} \nu(r' - e_1 + e_i)$$

For $i \in \Upsilon(r,1) \neq l, m$, note that $i \in \Upsilon(r',1)$ and $\nu(r - e_1 + e_i) = \nu(r' - e_1 + e_i)$ by the induction hypothesis. If $l \in \Upsilon(r,1)$, then $m \in \Upsilon(r',1)$ and $\nu(r - e_1 + e_l) = \nu(r' - e_1 + e_m)$ by the induction hypothesis. Similarly, if $m \in \Upsilon(r,1)$, then $l \in \Upsilon(r',1)$ and $\nu(r - e_1 + e_m) = \nu(r' - e_1 + e_l)$. Therefore, $(TV)(r,1) = (TV)(r',1)$ in this case.

(2) $j \neq l, m$. We show the equivalence of $(TV)(r, j)$ with $(TV)(r', j)$ under the following possibilities. Note that if $r_j = 0$, then $r'_j = 0$. Also, if $1 \notin \Upsilon(r, j)$, then $1 \notin \Upsilon(r', j)$.

(a) $r_j = 0$. By directly applying the induction hypothesis (14),

$$(TV)(r,j) = \frac{\gamma}{N}\nu(r) = \frac{\gamma}{N}\nu(r') = (TV)(r',j)$$

(b) $r_j > 0$ and $1 \notin \Upsilon(r,j)$.

$$(TV)(r,j) = \frac{\gamma}{N|\Upsilon(r,j)|} \sum_{i\in\Upsilon(r,j)} \nu(r - e_j + e_i)$$

$$(TV)(r',j) = \frac{\gamma}{N|\Upsilon(r',j)|} \sum_{i\in\Upsilon(r',j)} \nu(r' - e_j + e_i)$$

For $i \in \Upsilon(r,j) \neq l,m$, note that $i \in \Upsilon(r',j)$ and $\nu(r - e_j + e_i) = \nu(r' - e_j + e_i)$ by the induction hypothesis. If $l \in \Upsilon(r,j)$, then $m \in \Upsilon(r',j)$ and $\nu(r - e_j + e_l) = \nu(r' - e_j + e_m)$ by the induction hypothesis. Similarly, if $m \in \Upsilon(r,j)$, then $l \in \Upsilon(r',j)$ and $\nu(r - e_j + e_m) = \nu(r' - e_j + e_l)$. Therefore, $(TV)(r,j) = (TV)(r',j)$ in this case.

(c) $r_j > 0$ and $1 \in \Upsilon(r,j)$.

$$(TV)(r,j) = \frac{-c}{|\Upsilon(r,j)|} + \frac{\gamma}{N|\Upsilon(r,j)|} \sum_{i\in\Upsilon(r,j)} \nu(r - e_j + e_i)$$

$$(TV)(r',j) = \frac{-c}{|\Upsilon(r',j)|} + \frac{\gamma}{N|\Upsilon(r',j)|} \sum_{i\in\Upsilon(r',j)} \nu(r' - e_j + e_i)$$

Similarly to the previous case, for $i \in \Upsilon(r,j) \neq l,m$, note that $i \in \Upsilon(r',j)$ and $\nu(r - e_j + e_i) = \nu(r' - e_j + e_i)$ by the induction hypothesis. If $l \in \Upsilon(r,j)$, then $m \in \Upsilon(r',j)$ and $\nu(r - e_j + e_l) = \nu(r' - e_j + e_m)$ by the induction hypothesis. Similarly, if $m \in \Upsilon(r,j)$, then $l \in \Upsilon(r',j)$ and $\nu(r - e_j + e_m) = \nu(r' - e_j + e_l)$. Therefore, $(TV)(r,j) = (TV)(r',j)$ in this case.

(3) $j = l$. We show the equivalence of $(TV)(r,l)$ with $(TV)(r',m)$ under the following possibilities. Note that if $r_l = 0$, then $r'_m = 0$. Also, if $1 \notin \Upsilon(r,l)$, then $1 \notin \Upsilon(r',m)$.

(a) $r_l = 0$. By directly applying the induction hypothesis (14),

$$(TV)(r,l) = \frac{\gamma}{N}\nu(r) = \frac{\gamma}{N}\nu(r') = (TV)(r',m)$$

(b) $r_l > 0$ and $1 \notin \Upsilon(r,l)$.

$$(TV)(r,l) = \frac{\gamma}{N|\Upsilon(r,l)|} \sum_{i\in\Upsilon(r,l)} \nu(r - e_l + e_i)$$

$$(TV)(r',m) = \frac{\gamma}{N|\Upsilon(r',m)|} \sum_{i\in\Upsilon(r',m)} \nu(r' - e_m + e_i)$$

For $i \in \Upsilon(r,l) \neq m$, note that $i \in \Upsilon(r',m)$ and $\nu(r - e_l + e_i) = \nu(r' - e_m + e_i)$ by the induction hypothesis. If $m \in \Upsilon(r,l)$, then $l \in \Upsilon(r',m)$ and $\nu(r - e_l + e_m) = \nu(r' - e_m + e_l)$ by the induction hypothesis. Therefore, $(TV)(r,l) = (TV)(r',m)$ in this case.

(c) $r_l > 0$ and $1 \in \Upsilon(r,l)$.

$$(TV)(r,l) = \frac{-c}{|\Upsilon(r,l)|} + \frac{\gamma}{N|\Upsilon(r,l)|} \sum_{i\in\Upsilon(r,l)} \nu(r - e_l + e_i)$$

$$(TV)(r',m) = \frac{-c}{|\Upsilon(r',m)|} + \frac{\gamma}{N|\Upsilon(r',m)|} \sum_{i\in\Upsilon(r',m)} \nu(r' - e_m + e_i)$$

Similarly to the previous case, for $i \in \Upsilon(r,l) \neq m$, note that $i \in \Upsilon(r',m)$ and $\nu(r - e_l + e_i) = \nu(r' - e_m + e_i)$ by the induction hypothesis. If $m \in \Upsilon(r,l)$, then $l \in \Upsilon(r',m)$ and $\nu(r - e_l + e_m) = \nu(r' - e_m + e_l)$ by the induction hypothesis. Therefore, $(TV)(r,l) = (TV)(r',m)$ in this case.

(4) $j = m$. This case is identical to case 3 above. Following the same logic with l and m interchanged, one can show that $(TV)(r,m) = (TV)(r',l)$.

Q.E.D.

Proof of Proposition 4

Obviously, constant 0 functions of the corresponding dimensions satisfy conditions (16)-(18). We just need to show that starting from any functions $\bar{V}$ and V that satisfy (16)-(18), the condition still holds after one step value iteration defined by the right hand side of equations (10)-(12), denoted as $T\bar{V}$ and TV, respectively.

First consider condition (16). That is, we show that if conditions (16), (17), and (18) hold for some functions V and $\bar{V}$, condition (16) also holds for TV and $T\bar{V}$. For simplification of notation, we denote $\nu(r) = \sum_j V(r,j)$, and $\bar{\nu}$ similarly, which will be used in the proof.

(1.1) $j = 1$, $r_1 = 0$.

$$(TV)(r,1) = (\gamma/N)\nu(r) \geq (\gamma/N)\bar{\nu}(r + e_k) = (T\bar{V})(r + e_k, 1) \ .$$

(1.2) $j = 1$, $r_1 > 0$,

$$(TV)(r,1) = u + \gamma/(N|\Upsilon(r,1)|) \sum_{i\in\Upsilon(r,1)} \nu(r - e_1 + e_i)$$

$$(T\bar{V})(r + e_k, 1) = u + \gamma/(N|\Upsilon(r + e_k,1)|) \sum_{i\in\Upsilon(r+e_k,1)} \bar{\nu}(r - e_1 + e_i + e_k) \ .$$

Consider the following 3 possibilities.

(i) $\Upsilon(r+e_k,1) = \Upsilon(r,1)$. In this case, $\bar{V}(r - e_1 + e_i + e_k, j') \leq V(r - e_1 + e_i, j')$ from induction hypothesis. Therefore we have $(T\bar{V})(r + e_k, 1) \leq (TV)(r,1)$.

(ii) $\Upsilon(r+e_k,1)\cup\{k\}=\Upsilon(r,1)$. In this case, $\bar{\nu}(r-e_1+e_i+e_k)$ is the same for all $i\in\Upsilon(r+e_k,1)$. Therefore $\sum_{i\in\Upsilon(r+e_k,1)}\bar{\nu}(r-e_1+e_i+e_k)/|\Upsilon(r+e_k,1)|$ equals $\bar{\nu}(r-e_1+e_i+e_k)$ for any particular i. The same can be said for $\nu(r-e_1+e_i)$. The result then follows the comparison between $\nu(r-e_1+e_i)$ and $\bar{\nu}(r-e_1+e_i+e_k)$ according to the induction hypothesis.

(iii) $\Upsilon(r,1)=\{k\}$. We have $V(r-e_1+e_k,j')\geq \bar{V}(r-e_1+e_i+e_k,j')$ for any $i\neq 1$. Therefore,

$$(TV)(r,1)=u+\gamma/N\nu(r-e_1+e_k)$$

$$\geq u+\gamma/(N|\Upsilon(r+e_k,1)|)\sum_{i\in\Upsilon(r+e_k,1)}\bar{\nu}(r-e_1+e_i+e_k)=(T\bar{V})(r+e_k,1)\ .$$

(2.1.1) $j\neq 1$, $r_j=0$, $k\neq j$,

$$(TV)(r,j)=(\gamma/N)\nu(r)\geq(\gamma/N)\bar{\nu}(r+e_k)=(T\bar{V})(r+e_k,j)\ .$$

(2.1.2.1) $j\neq 1$, $r_j=0$, $k=j$, $1\notin\Upsilon(r+e_j,j)$

$$(T\bar{V})(r+e_j,j)=\gamma/(N|\Upsilon(r+e_j,j)|)\sum_{i\in\Upsilon(r+e_j,j)}\bar{\nu}(r+e_i)$$

$$\leq\gamma/(N|\Upsilon(r+e_j,j)|)\sum_{i\in\Upsilon(r+e_j,j)}\nu(r)=(\gamma/N)\nu(r)=(TV)(r,j)\ .$$

(2.1.2.2) $j\neq 1$, $r_j=0$, $k=j$, $1\in\Upsilon(r+e_j,j)$. Lemma 4 implies that no more than one player has 0 scrips in any state in the recurrent class. Therefore, we must have $r_1>0$ and hence,

$$(T\bar{V})(r+e_j,j)=\left(-c+\gamma/N\sum_{i\in\Upsilon(r+e_j,j)}\bar{\nu}(r+e_i)\right)/|\Upsilon(r+e_j,j)|$$

$$=\frac{-c}{|\Upsilon(r+e_j,j)|}+\frac{\gamma}{N|\Upsilon(r+e_j,j)|}\left(\sum_{i\in\Upsilon(r+e_j,j)\setminus 1}\bar{\nu}(r+e_i)+\bar{\nu}(r+e_1)\right)$$

$$\leq\frac{-c}{|\Upsilon(r+e_j,j)|}+\frac{\gamma}{N|\Upsilon(r+e_j,j)|}\left(\sum_{i\in\Upsilon(r+e_j,j)\setminus 1}\bar{\nu}(r+e_i)+\nu(r)+\frac{cN}{\gamma}\right)$$

$$\leq\frac{-c}{|\Upsilon(r+e_j,j)|}+\frac{\gamma}{N|\Upsilon(r+e_j,j)|}\left(\sum_{i\in\Upsilon(r+e_j,j)\setminus 1}\nu(r)+\nu(r)+\frac{cN}{\gamma}\right)$$

$$=\frac{-c}{|\Upsilon(r+e_j,j)|}+\frac{\gamma}{N|\Upsilon(r+e_j,j)|}\left(|\Upsilon(r+e_j,j)|\nu(r)+\frac{cN}{\gamma}\right)$$

$$=\frac{\gamma}{N}\nu(r)=(TV)(r,j)$$

where the first inequality follows from (17), and the second inequality follows from (16).

(2.2.1) $j \neq 1$, $r_j > 0$, $1 \notin \Upsilon(r,j)$, $1 \notin \Upsilon(r+e_k,j)$.

$$(TV)(r,j) = \gamma/(N|\Upsilon(r,j)|) \sum_{i \in \Upsilon(r,j)} \nu(r - e_j + e_i)$$

$$(T\bar{V})(r+e_k,j) = \gamma/(N|\Upsilon(r+e_k,j)|) \sum_{i \in \Upsilon(r+e_k,j)} \bar{\nu}(r - e_j + e_k + e_i) \ .$$

Following the 3 cases as in (1.2), we have the result.

(2.2.2) $j \neq 1$, $r_j > 0$, $1 \notin \Upsilon(r,j)$, $1 \in \Upsilon(r+e_k,j)$. This means that we must have $\Upsilon(r,j) = \{k\}$, and $r_1 = r_k + 1$ (therefore $r_1 > 0$).

$$\begin{aligned}
(T\bar{V})(r+e_k,j) &= \left(-c + \gamma/N \sum_{i \in \Upsilon(r+e_k,j)} \bar{\nu}(r - e_j + e_k + e_i) \right) / |\Upsilon(r+e_k,j)| \\
&= \frac{-c}{|\Upsilon(r+e_k,j)|} \\
&\quad + \frac{\gamma}{N|\Upsilon(r+e_k,j)|} \left(\sum_{i \in \Upsilon(r+e_k,j) \setminus 1} \bar{\nu}(r - e_j + e_k + e_i) + \bar{\nu}(r - e_j + e_k + e_1) \right) \\
&\leq \frac{-c}{|\Upsilon(r+e_k,j)|} \\
&\quad + \frac{\gamma}{N|\Upsilon(r+e_k,j)|} \left(\sum_{i \in \Upsilon(r+e_k,j) \setminus 1} \bar{\nu}(r - e_j + e_k + e_i) + \nu(r - e_j + e_k) + \frac{cN}{\gamma} \right) \\
&\leq \frac{-c}{|\Upsilon(r+e_k,j)|} \\
&\quad + \frac{\gamma}{N|\Upsilon(r+e_k,j)|} \left(\sum_{i \in \Upsilon(r+e_k,j) \setminus 1} \nu(r - e_j + e_k) + \nu(r - e_j + e_k) + \frac{cN}{\gamma} \right) \\
&= \frac{-c}{|\Upsilon(r+e_k,j)|} + \frac{\gamma}{N|\Upsilon(r+e_k,j)|} \left(|\Upsilon(r+e_k,j)| \nu(r - e_j + e_k) + \frac{cN}{\gamma} \right) \\
&= \frac{\gamma}{N} \nu(r - e_j + e_k) = (TV)(r,j) \ .
\end{aligned}$$

Here, the first inequality follows from (17), the second inequality follows from (16), and the last equation follows from the fact that $\Upsilon(r,j)$ is a singleton with only one element, k.

(2.3.1) $j \neq 1$, $r_j > 0$, $1 \in \Upsilon(r,j)$, $1 \in \Upsilon(r+e_k,j)$.

$$(TV)(r,j) = \left(-c + \gamma/N \sum_{i \in \Upsilon(r,j)} \nu(r - e_j + e_i) \right) / |\Upsilon(r,j)|$$

$$(T\bar{V})(r+e_k,j) = \left(-c + \gamma/N \sum_{i \in \Upsilon(r+e_k,j)} \bar{\nu}(r - e_j + e_k + e_i) \right) / |\Upsilon(r+e_k,j)|$$

Similar to case (1.2), we consider the three cases. When $\Upsilon(r+e_k,j) = \Upsilon(r,j)$, we have the result directly from condition (16). Since here we assume that $1 \in \Upsilon(r,j)$, it is impossible that $\Upsilon(r,j) = \{k\}$.

Now we focus on the case where $|\Upsilon(r+e_k,j)|+1=|\Upsilon(r,j)|=L>1$, that is, $\Upsilon(r+e_k,j)\cup\{k\}=\Upsilon(r,j)$. Since, by Lemma 4, no more than one player can have 0 scrips in any state in the recurrent class, we must have $r_1>0$.

$$\begin{aligned}
&L(L-1)\left(T\bar{V}(r+e_k,j)-T\bar{V}(r,j)\right)\\
&=-c+\frac{\gamma}{N}\left[L\left(\sum_{i\in\Upsilon(r+e_k,j)\setminus 1}\bar{\nu}(r-e_j+e_k+e_i)+\bar{\nu}(r-e_j+e_k+e_1)\right)\right.\\
&\left.-(L-1)\left(\sum_{i\in\Upsilon(r+e_k,j)\setminus 1}\nu(r-e_j+e_i)+\nu(r-e_j+e_k)+\nu(r-e_j+e_1)\right)\right]\\
&=-c+\frac{\gamma}{N}\Big[(L-1)\sum_{i\in\Upsilon(r+e_k,j)\setminus 1}\Big(\bar{\nu}(r-e_j+e_i+e_k)-\nu(r-e_j+e_i)\Big)\\
&+\sum_{i\in\Upsilon(r+e_k,j)\setminus 1}\Big(\bar{\nu}(r-e_j+e_i+e_k)-\nu(r-e_j+e_k)\Big)-\nu(r-e_j+e_k)\\
&+(L-1)\Big(\bar{\nu}(r-e_j+e_k+e_1)-\nu(r-e_j+e_1)\Big)+\bar{\nu}(r-e_j+e_k+e_1)\Big]\\
&\le -c+\frac{\gamma}{N}\Big[\bar{\nu}(r-e_j+e_k+e_1)-\nu(r-e_j+e_k)\Big]\le -c+\frac{\gamma}{N}\frac{cN}{\gamma}=0\ ,
\end{aligned}$$

where the first inequality follows (16), and the second inequality follows (17).

(2.3.2) $j\neq 1$, $r_j>0$, $1\in\Upsilon(r,j)$, $1\notin\Upsilon(r+e_k,j)$. This is not possible when we assume $k\neq 1$.

Next, we consider condition (18) when $r_1=0$. That is, we show that if (16) and (17) hold for some V and $\bar{V}$, condition (18) also holds for TV and $T\bar{V}$. Since, by Lemma 4, no more than one player can have 0 scrips in any state in the recurrent class, we know that player 1 is the only one with 0 scrips and hence,

$$\sum_j TV(r,j)=-c(N-1)+\frac{\gamma}{N}\left(\sum_{j\neq 1}\nu(r+e_1-e_j)+\nu(r)\right)$$

In this case $\sum_j T\bar{V}(r+e_1,j)$ has the following two possibilities.

R0.1 Player 1 remains the only minimum scrip holder in $r+e_1$. There is a set L with $|L|=l$ of other players having the second least number of scrips.

$$\sum_j T\bar{V}(r+e_1,j)=u-c(N-1)+\frac{\gamma}{N}\left(\sum_{j\neq 1}\bar{\nu}(r+2e_1-e_j)+\frac{1}{l}\sum_{i\in L}\bar{\nu}(r+e_i)\right)$$

Therefore,

$$\begin{aligned}
\sum_j T\bar{V}(r+e_1,j)-\sum_j TV(r,j)=u+\frac{\gamma}{N}\Bigg(&\frac{1}{l}\sum_{i\in L}\Big(\bar{\nu}(r+e_i)-\nu(r)\Big)\\
&+\sum_{j\neq 1}\Big(\bar{\nu}(r+2e_1-e_j)-\nu(r+e_1-e_j)\Big)\Bigg)
\end{aligned}$$

$$\leq u + \frac{\gamma}{N}\left(\sum_{j\neq 1}\Big(\bar{\nu}(r+2e_1-e_j) - \nu(r+e_1-e_j)\Big)\right)$$
$$\leq u + \frac{\gamma}{N}\frac{cN}{\gamma}(N-1) = u + c(N-1) \leq \frac{N}{\gamma}\left[\frac{N}{\gamma} - (N-1)\right]c\ ,$$

where the first inequality follows induction hypothesis (16), the second inequality follows (17), and the last inequality follows condition (15).

R0.2 Player 1 becomes a member of the set L, $|L| = l$, of minimum scrip holders in $r+e_1$.

$$\sum_j T\bar{V}(r+e_1,j) = u - c - \frac{N-l}{l}c$$
$$+\frac{\gamma}{N}\left(\sum_{j\notin l}\frac{1}{l}\Big(\sum_{i\in L\setminus 1}\bar{\nu}(r+e_1+e_i-e_j) + \bar{\nu}(r+2e_1-e_j)\Big)\right.$$
$$\left.+\sum_{j\in L\setminus 1}\frac{1}{l-1}\Big(\sum_{i\in L\setminus j,1}\bar{\nu}(r+e_1+e_i-e_j) + \bar{\nu}(r+2e_1-e_j)\Big) + \frac{1}{l-1}\sum_{i\in L\setminus 1}\bar{\nu}(r+e_i)\right)$$

Therefore

$$\sum_j T\bar{V}(r+e_1,j) - \sum_j TV(r,j) \leq u + c\left((N-2) - \frac{N-l}{l}\right)$$
$$+\frac{\gamma}{N}\left(\frac{1}{l}\sum_{j\notin L}\Big(\bar{\nu}(r+2e_1-e_j) - \nu(r+e_1-e_j)\Big)\right.$$
$$\left.+\frac{1}{l-1}\sum_{j\in L\setminus 1}\Big(\bar{\nu}(r+2e_1-e_j) - \nu(r+e_1-e_j)\Big)\right)$$
$$\leq u + c\left((N-2) - \frac{N-l}{l}\right) + \frac{\gamma}{N}\left(\frac{N-l}{l}+1\right)\frac{cN}{\gamma} \leq \frac{N}{\gamma}\left[\frac{N}{\gamma} - (N-1)\right]c\ ,$$

where the first inequality follows induction hypothesis (16), the second inequality follows (17), and the last inequality follows condition (15).

Now we consider condition (17) when $r_1 > 0$. Following the definition of ν, we have the following cases for $\sum_j TV(r,j)$.

1. Player 1 is the only minimum scrip holder in both r and $r+e_1$. Let L, $|L| = l$, be the set of the second least number of scrip holders.

$$\sum_j T\bar{V}(r+e_1,j) - \sum_j TV(r,j) = \frac{\gamma}{N}\left(\frac{1}{l}\sum_{i\in L}\Big(\bar{\nu}(r+e_i) - \nu(r-e_1+e_i)\Big)\right.$$
$$\left.+\sum_{j\neq 1}\Big(\bar{\nu}(r+2e_1-e_j) - \nu(r+e_1-e_j)\Big)\right)$$
$$\leq \frac{\gamma}{N}\left\{\frac{N}{\gamma}\left[\frac{N}{\gamma} - (N-1)\right] + (N-1)\frac{N}{\gamma}\right\}c = \frac{cN}{\gamma}\ ,$$

where the inequality follows both (17) and (18). Note that if $r_1 > 1$, using only (17) leads to the same inequality, although not tight.

2. Player 1 is the only minimum scrip holder in r. Let L, $|L|=l$, be the set that includes player one plus the $l-1$ others with exactly one more scrip ($l>1$).

$$\sum_j TV(r,j) = u - c(N-1) + \frac{\gamma}{N}\left(\sum_{j\neq 1}\nu(r+e_1-e_j) + \frac{1}{l-1}\sum_{i\in L\setminus 1}\nu(r+e_i-e_1)\right)$$

As a result, player 1 is one of l minimum scrip holders in $r+e_1$.

$$\sum_j T\bar{V}(r+e_1,j) = u - c - \frac{N-l}{l}c$$
$$+\frac{\gamma}{N}\left(\sum_{j\notin L}\frac{1}{l}\Big(\sum_{i\in L\setminus 1}\bar{\nu}(r+e_1+e_i-e_j)+\bar{\nu}(r+2e_1-e_j)\Big)\right.$$
$$\left.+\sum_{j\in L\setminus 1}\frac{1}{l-1}\sum_{i\in L\setminus j}\bar{\nu}(r+e_1+e_i-e_j)+\frac{1}{l-1}\sum_{i\in L\setminus 1}\bar{\nu}(r+e_i)\right)$$

Therefore

$$\sum_j T\bar{V}(r+e_1,j) - \sum_j TV(r,j) \le c\left((N-2)-\frac{N-l}{l}\right)$$
$$+\frac{\gamma}{N}\left(\sum_{j\notin L}\frac{1}{l}\Big(\bar{\nu}(r+2e_1-e_j)-\nu(r+e_1-e_j)\Big)+\frac{1}{l-1}\sum_{i\in L\setminus 1}\Big(\bar{\nu}(r+e_i)-\nu(r+e_i-e_1)\Big)\right.$$
$$\left.+\frac{1}{l-1}\sum_{j\in L\setminus 1}\Big(\bar{\nu}(r+2e_1-e_j)-\nu(r+e_1-e_j)\Big)\right)$$
$$\le c\left((N-2)-\frac{N-l}{l}\right)+\frac{\gamma}{N}\left(\frac{N-l}{l}+1\right)\frac{cN}{\gamma}+\frac{\gamma}{N}\frac{N}{\gamma}\left[\frac{N}{\gamma}-(N-1)\right]c=\frac{cN}{\gamma}\ .$$

where the first inequality follows from (16), and the second inequality follows both (17) and (18). Similar to before, if $r_1>1$, we only need (17), which leads to the same inequality, although not tight.

3. Player 1 is in the set L, $|L|=l\ge 2$ of players with minimum scrips.

$$\sum_j TV(r,j) = u - c - \frac{N-l}{l}c + \frac{\gamma}{N}\left(\sum_{j\notin L}\frac{1}{l}\Big(\sum_{i\in L\setminus 1}\nu(r+e_i-e_j)+\nu(r+e_1-e_j)\Big)\right.$$
$$\left.+\sum_{j\in L\setminus 1}\frac{1}{l-1}\Big(\sum_{i\in L\setminus\{j,1\}}\nu(r+e_i-e_j)+\nu(r+e_1-e_j)\Big)+\frac{1}{l-1}\sum_{i\in L\setminus 1}\nu(r+e_i-e_1)\right)$$

(1) $l=2$. Therefore in $r+e_1$ there is a unique minimum scrip holder, say player k. That is, $L=\{1,k\}$ and player 1 is among the set M, $|M|=m$, of second minimum scrip holders in $r+e_1$.

$$\sum_j T\bar{V}(r+e_1,j) = u - \frac{c}{m} + \frac{\gamma}{N}\left(\sum_{j\notin L}\bar{\nu}(r+e_1+e_k-e_j)+\bar{\nu}(r+e_k)\right.$$
$$\left.+\frac{1}{m}\Big(\sum_{i\in M\setminus 1}\bar{\nu}(r+e_1+e_i-e_k)+\bar{\nu}(r+2e_1-e_k)\Big)\right)$$

$$\sum_j T\bar{V}(r+e_1,j) - \sum_j TV(r,j) \leq \left(1+\frac{N-2}{2}-\frac{1}{m}\right)c + \frac{\gamma}{N}\frac{N}{\gamma}\left[\frac{N}{\gamma}-(N-1)\right]c$$
$$+\frac{\gamma}{N}\frac{cN}{\gamma}\left(\frac{N-2}{2}+\frac{1}{m}\right) = \frac{cN}{\gamma} \ ,$$

where the inequality follows both (17) and (18). Again, if $r_1 > 1$, it is fine using (17) only, which leads to a more relaxed version of the same inequality.

(2) $l > 2$. Therefore there are $l-1$ minimum scrip holders in $r+e_1$.

$$\sum_j T\bar{V}(r+e_1,j) = u + \frac{\gamma}{N}\left(\sum_{j\notin L}\frac{1}{l-1}\sum_{i\in L\setminus 1}\bar{\nu}(r+e_1+e_i-e_j)\right.$$
$$\left.+\frac{1}{l-1}\sum_{i\in L\setminus 1}\bar{\nu}(r+e_i) + \sum_{j\in L\setminus 1}\frac{1}{l-2}\sum_{i\in L\setminus\{1,j\}}\bar{\nu}(r+e_i+e_1-e_j)\right)$$
$$\sum_j T\bar{V}(r+e_1,j) - \sum_j TV(r,j) \leq \left(1+\frac{N-l}{l}\right)c$$
$$+\frac{\gamma}{N}\frac{cN}{\gamma}\left(\frac{N-l}{l}(l-1)+l-2\right) + \frac{\gamma}{N}\frac{N}{\gamma}\left[\frac{N}{\gamma}-(N-1)\right]c = \frac{cN}{\gamma}$$

4. There is a unique minimum scrip holder $k \neq 1$ with $r_k > 0$, and player 1 is in the set $L, |L| = l$, of players who have the second minimum scrips.

$$\sum_j TV(r,j) = u - \frac{c}{l} + \frac{\gamma}{N}\left(\sum_{j\neq k,1}\nu(r+e_k-e_j) + \nu(r+e_k-e_1)\right.$$
$$\left.+\frac{1}{l}\Big(\sum_{i\in L\setminus 1}\nu(r+e_i-e_k) + \nu(r+e_1-e_k)\Big)\right)$$
$$\sum_j T\bar{V}(r+e_1,j) = u + \frac{\gamma}{N}\left(\sum_{j\neq k,1}\bar{\nu}(r+e_1+e_k-e_j) + \bar{\nu}(r+e_k)\right.$$
$$\left.+\frac{1}{l-1}\sum_{i\in L\setminus 1}\bar{\nu}(r+e_1+e_i-e_k)\right)$$
$$\sum_j T\bar{V}(r+e_1,j) - \sum_j TV(r,j) \leq \left(\frac{1}{l}+(N-2)+\frac{l-1}{l}\right)c + \frac{\gamma}{N}\frac{N}{\gamma}\left[\frac{N}{\gamma}-(N-1)\right]c = \frac{cN}{\gamma}$$

5. There is a unique minimum scrip holder $k \neq 1$ with $r_k > 0$, and there are l players who have the second minimum scrips, not including player 1. Following the same logic as before, we obtain

$$\sum_j T\bar{V}(r+e_1,j) - \sum_j TV(r,j) \leq \frac{cN}{\gamma}$$

6. There is a unique minimum scrip holder $k \neq 1$ with $r_k = 0$.

$$\sum_j T\bar{V}(r+e_1,j) - \sum_j TV(r,j) \leq \frac{cN}{\gamma}$$

7. There are $l \geq 2$ minimum scrip holders, not including Player 1.

$$\sum_j T\bar{V}(r+e_1,j) - \sum_j TV(r,j) \leq \frac{cN}{\gamma}$$

Q.E.D.

Proof of Proposition 5

First we show the result for the general case where $N \geq 3$. Denote $\nu_1 = \sum_j V(e,j)$, where e is the vector of all ones; $\nu_2 = \sum_j V(e+e_1-e_2,j)$, that is, player 1 has two scrips while another player has zero; $\nu_3 = \sum_j V(e-e_1+e_2,j)$, that is, player 1 has zero scrips while another player has two; $\nu_4 = \sum_j V(e+e_2-e_3,j)$. Due to symmetry (see Lemma 5), (e,j), $(e+e_1-e_2,j)$, $(e-e_1+e_2,j)$, and $(e+e_2-e_3,j)$ for any j are the only types of recurrent states in the system with $R=N$ scrips. The corresponding recursive equations (10) - (12) give us

$$\begin{aligned} \nu_1 &= (u-c)+(\gamma/N)\big(\nu_2+\nu_3+(N-2)\nu_4\big) , & \nu_2 &= u+(\gamma/N)\big(\nu_1+(N-1)\nu_2\big) , \\ \nu_3 &= -(N-1)c+(\gamma/N)\big(\nu_1+\nu_3+(N-2)\nu_4\big) , & \nu_4 &= u+(\gamma/N)\big(\nu_1+\nu_3+(N-2)\nu_4\big) \end{aligned}$$

The solution is

$$\nu_1 = \frac{(u-c)N}{(1-\gamma)(N+\gamma)} \geq 0 , \quad \nu_2 = \Big(1-\frac{\gamma}{N}(N-1)\Big)^{-1}\Big(\frac{\gamma}{N}\nu_1+u\Big) \geq 0 ,$$

$$\nu_3 = \Big(1-\frac{\gamma}{N}(N-1)\Big)^{-1}\left[\frac{\gamma}{N}\nu_1+(N-1)c\left(\frac{\gamma(N-2)}{N}-1\right)+\frac{\gamma(N-2)}{N}u\right] ,$$

$$\nu_4 = \Big(1-\frac{\gamma}{N}(N-1)\Big)^{-1}\left[\frac{\gamma}{N}\nu_1+\frac{1}{N}\Big((N-\gamma)u-(N-1)\gamma c\Big)\right] \geq 0 .$$

$\nu_3 \geq 0$ iff

$$\frac{u}{c} \geq \frac{\Big(\gamma N+(N-1)(N-\gamma(N-2))(1-\gamma)(N+\gamma)\Big)}{\gamma\Big(N+(N-2)(1-\gamma)(N+\gamma)\Big)} = 1+\frac{N}{\gamma}\frac{\Big((N-1)-(N-2)\gamma\Big)(1-\gamma)(N+\gamma)}{\Big(N+(N-2)(1-\gamma)(N+\gamma)\Big)}$$

$$= \frac{N}{\gamma}\left(\frac{\Big((N-1)(1-\gamma)(N+\gamma)+N\gamma\Big)}{\Big(N+(N-2)(1-\gamma)(N+\gamma)\Big)}\right)-(N-1) = \frac{N}{\gamma}\left(1+\frac{(1-\gamma)\gamma}{N+(N-2)(1-\gamma)(N+\gamma)}\right)-(N-1)$$

Under condition (20), the above inequality holds.

For the special case where $N=2$ and following the same notation as above, the only types of recurrent states are (e,j), $(e+e_1-e_2,j)$, and $(e-e_1+e_2,j)$ for $j=1,2$. Thus we keep our definitions $\nu_1 = \sum_j V(e,j)$, $\nu_2 = \sum_j V(e+e_1-e_2,j)$, and $\nu_3 = \sum_j V(e-e_1+e_2,j)$, and we no longer have $\nu_4 = \sum_j V(e+e_2-e_3,j)$ since there are only two players. The corresponding recursive equations (10) - (12) give us

$$\nu_1 = u-c+\frac{\gamma}{2}(\nu_2+\nu_3) ; \quad \nu_2 = u+\frac{\gamma}{2}(\nu_1+\nu_2) ; \quad \nu_3 = -c+\frac{\gamma}{2}(\nu_1+\nu_3)$$

The solution is

$$\nu_1 = \frac{-2(u-c)}{(\gamma+2)(\gamma-1)} \geq 0 \ ; \quad \nu_2 = \frac{-2(u\gamma^2+c\gamma-2u)}{(\gamma+2)(\gamma-1)(\gamma-2)} \ ; \quad \nu_3 = \frac{2(c\gamma^2+u\gamma-2c)}{(\gamma+2)(\gamma-1)(\gamma-2)}$$

$\nu_2 \geq 0$ iff $\frac{u}{c} \geq \frac{-\gamma}{\gamma^2-2}$, which is true under condition (20); $\nu_3 \geq 0$ iff $\frac{u}{c} \geq \frac{2-\gamma^2}{\gamma}$, which is also true under condition (20).

Q.E.D.

Proof of Proposition 6

Similar to the proof of Proposition 4, we first show that for any functions $\bar{V}$ and V that satisfy (25)-(26), condition (25) holds with $\Xi_\kappa \bar{V}$ and $\Xi_\kappa V$, with any given κ.

First consider condition (25). Still denote $\nu(r) = \sum_j V(r,j)$, and $\bar{\nu}$ similarly.

(1.1) $j=1$, $r_1=0$.

$$(\Xi_\kappa V)(r,1) = (\gamma/N)\nu(r) \geq (\gamma/N)\bar{\nu}(r+e_k) = (\Xi_\kappa \bar{V})(r+e_k,1) \ .$$

(1.2) $j=1$, $r_1>0$,

$$(\Xi_\kappa V)(r,1) = u + \gamma/(N|\Upsilon_\kappa(r,1)|) \sum_{i\in\Upsilon_\kappa(r,1)} \nu(r-e_1+e_i)$$

$$(\Xi_\kappa \bar{V})(r+e_k,1) = u + \gamma/(N|\Upsilon_\kappa(r+e_k,1)|) \sum_{i\in\Upsilon_\kappa(r+e_k,1)} \bar{\nu}(r-e_1+e_i+e_k) \ .$$

Consider the following 3 possibilities.

(i) $\Upsilon_\kappa(r+e_k,1) = \Upsilon_\kappa(r,1)$. In this case, $\bar{V}(r-e_1+e_i+e_k,j') \leq V(r-e_1+e_i,j')$ from induction hypothesis. Therefore we have $(\Xi_\kappa \bar{V})(r+e_k,1) \leq (\Xi_\kappa V)(r,1)$.

(ii) $\Upsilon_\kappa(r+e_k,1) \cup \{k\} = \Upsilon_\kappa(r,1)$. In this case, $\bar{\nu}(r-e_1+e_i+e_k)$ is the same for all $i \in \Upsilon_\kappa(r+e_k,1)$. Therefore $\sum_{i\in\Upsilon_\kappa(r+e_k,1)} \bar{\nu}(r-e_1+e_i+e_k)/|\Upsilon_\kappa(r+e_k,1)|$ equals $\bar{\nu}(r-e_1+e_i+e_k)$ for any particular i. The same can be said for $\nu(r-e_1+e_i)$. The result then follows the comparison between $\nu(r-e_1+e_i)$ and $\bar{\nu}(r-e_1+e_i+e_k)$ according to the induction hypothesis.

(iii) $\Upsilon_\kappa(r,1) = \{k\}$. We have $V(r-e_1+e_k,j') \geq \bar{V}(r-e_1+e_i+e_k,j')$ for any $i \neq 1$. Therefore,

$$(\Xi_\kappa V)(r,1) = u + (\gamma/N)\nu(r-e_1+e_k)$$

$$\geq u + \gamma/(N|\Upsilon_\kappa(r+e_k,1)|) \sum_{i\in\Upsilon_\kappa(r+e_k,1)} \bar{\nu}(r-e_1+e_i+e_k) = (\Xi_\kappa \bar{V})(r+e_k,1) \ .$$

(2.1.1) $j \neq 1$, $r_j = 0$, $k \neq j$,

$$(\Xi_\kappa V)(r,j) = (\gamma/N)\nu(r) \geq (\gamma/N)\bar{\nu}(r+e_k) = (\Xi_\kappa \bar{V})(r+e_k,j) \ .$$

(2.1.2.1) $j \neq 1,\ r_j = 0,\ k = j,\ 1 \notin \Upsilon_\kappa(r+e_j, j)$

$$(\Xi_\kappa \bar{V})(r+e_j, j) = \gamma/(N|\Upsilon_\kappa(r+e_j,j)|) \sum_{i \in \Upsilon_\kappa(r+e_j,j)} \bar{\nu}(r+e_i)$$

$$\leq \gamma/(N|\Upsilon_\kappa(r+e_j,j)|) \sum_{i \in \Upsilon_\kappa(r+e_j,j)} \nu(r) = (\gamma/N)\nu(r) = (\Xi_\kappa V)(r,j)\ .$$

(2.1.2.2) $j \neq 1,\ r_j = 0,\ k = j,\ 1 \in \Upsilon_\kappa(r+e_j, j)$.

$$(\Xi_\kappa \bar{V})(r+e_j, j) = \left(-c + \gamma/N \sum_{i \in \Upsilon_\kappa(r+e_j,j)} \bar{\nu}(r+e_i)\right) / |\Upsilon_\kappa(r+e_j,j)|$$

$$= \frac{-c}{|\Upsilon_\kappa(r+e_j,j)|} + \frac{\gamma}{N|\Upsilon_\kappa(r+e_j,j)|}\left(\sum_{i \in \Upsilon_\kappa(r+e_j,j)\setminus 1} \bar{\nu}(r+e_i) + \bar{\nu}(r+e_1)\right)$$

$$\leq \frac{-c}{|\Upsilon_\kappa(r+e_j,j)|} + \frac{\gamma}{N|\Upsilon_\kappa(r+e_j,j)|}\left(\sum_{i \in \Upsilon_\kappa(r+e_j,j)\setminus 1} \bar{\nu}(r+e_i) + \nu(r) + \frac{cN}{\gamma}\right)$$

$$\leq \frac{-c}{|\Upsilon_\kappa(r+e_j,j)|} + \frac{\gamma}{N|\Upsilon_\kappa(r+e_j,j)|}\left(\sum_{i \in \Upsilon_\kappa(r+e_j,j)\setminus 1} \nu(r) + \nu(r) + \frac{cN}{\gamma}\right)$$

$$= \frac{-c}{|\Upsilon_\kappa(r+e_j,j)|} + \frac{\gamma}{N|\Upsilon_\kappa(r+e_j,j)|}\left(|\Upsilon_\kappa(r+e_j,j)|\nu(r) + \frac{cN}{\gamma}\right)$$

$$= \frac{\gamma}{N}\nu(r) = (\Xi_\kappa V)(r,j)$$

where the first inequality follows from (26), and the second inequality follows from (25).

(2.2.1) $j \neq 1,\ r_j > 0,\ 1 \notin \Upsilon_\kappa(r,j),\ 1 \notin \Upsilon_\kappa(r+e_k, j)$.

$$(\Xi_\kappa V)(r,j) = \gamma/(N|\Upsilon_\kappa(r,j)|) \sum_{i \in \Upsilon_\kappa(r,j)} \nu(r - e_j + e_i)$$

$$(\Xi_\kappa \bar{V})(r+e_k, j) = \gamma/(N|\Upsilon_\kappa(r+e_k,j)|) \sum_{i \in \Upsilon_\kappa(r+e_k,j)} \bar{\nu}(r - e_j + e_k + e_i)\ .$$

Following the 3 cases as in (1.2), we have the result.

(2.2.2) $j \neq 1,\ r_j > 0,\ 1 \notin \Upsilon_\kappa(r,j),\ 1 \in \Upsilon_\kappa(r+e_k, j)$. This means that we must have $\Upsilon_\kappa(r,j) = \{k\}$, and $r_1 = r_k + 1$.

$$(\Xi_\kappa \bar{V})(r+e_k, j) = \left(-c + \gamma/N \sum_{i \in \Upsilon_\kappa(r+e_k,j)} \bar{\nu}(r - e_j + e_k + e_i)\right) / |\Upsilon_\kappa(r+e_k,j)|$$

$$= \frac{-c}{|\Upsilon_\kappa(r+e_k,j)|} + \frac{\gamma}{N|\Upsilon_\kappa(r+e_k,j)|}\left(\sum_{i \in \Upsilon_\kappa(r+e_k,j)\setminus 1} \bar{\nu}(r - e_j + e_k + e_i) + \bar{\nu}(r - e_j + e_k + e_1)\right)$$

$$\leq \frac{-c}{|\Upsilon_\kappa(r+e_k,j)|} + \frac{\gamma}{N|\Upsilon_\kappa(r+e_k,j)|}\left(\sum_{i \in \Upsilon_\kappa(r+e_k,j)\setminus 1} \bar{\nu}(r - e_j + e_k + e_i) + \nu(r - e_j + e_k) + \frac{cN}{\gamma}\right)$$

$$\leq \frac{-c}{|\Upsilon_\kappa(r+e_k,j)|} + \frac{\gamma}{N|\Upsilon_\kappa(r+e_k,j)|}\left(\sum_{i\in\Upsilon_\kappa(r+e_k,j)\setminus 1} \nu(r-e_j+e_k)+\nu(r-e_j+e_k)+\frac{cN}{\gamma}\right)$$
$$= \frac{-c}{|\Upsilon_\kappa(r+e_k,j)|} + \frac{\gamma}{N|\Upsilon_\kappa(r+e_k,j)|}\left(|\Upsilon_\kappa(r+e_k,j)|\nu(r-e_j+e_k)+\frac{cN}{\gamma}\right)$$
$$= \frac{\gamma}{N}\nu(r-e_j+e_k) = (\Xi_\kappa V)(r,j) \ .$$

Here, the first inequality follows from (26), the second inequality follows from (25), and the last equation follows from the fact that $\Upsilon_\kappa(r,j)$ is a singleton with only one element, k.

(2.3.1) $j\neq 1$, $r_j>0$, $1\in\Upsilon_\kappa(r,j)$, $1\in\Upsilon_\kappa(r+e_k,j)$.

$$(\Xi_\kappa V)(r,j) = \left(-c+\gamma/N\sum_{i\in\Upsilon_\kappa(r,j)}\nu(r-e_j+e_i)\right)/|\Upsilon_\kappa(r,j)|$$

$$(\Xi_\kappa\bar V)(r+e_k,j) = \left(-c+\gamma/N\sum_{i\in\Upsilon_\kappa(r+e_k,j)}\bar\nu(r-e_j+e_k+e_i)\right)/|\Upsilon_\kappa(r+e_k,j)|$$

Similar to case (1.2), we consider the three cases. When $\Upsilon_\kappa(r+e_k,j)=\Upsilon_\kappa(r,j)$, we have the result directly from condition (25). Since here we assume that $1\in\Upsilon_\kappa(r,j)$, it is impossible that $\Upsilon_\kappa(r,j)=\{k\}$.

Now we focus on the case where $|\Upsilon_\kappa(r+e_k,j)|+1=|\Upsilon_\kappa(r,j)|=L>1$, that is, $\Upsilon_\kappa(r+e_k,j)\cup\{k\}=\Upsilon_\kappa(r,j)$.

$$L(L-1)\left(\Xi_\kappa\bar V(r+e_k,j)-\Xi_\kappa\bar V(r,j)\right)$$
$$=-c+\frac{\gamma}{N}\left[L\left(\sum_{i\in\Upsilon_\kappa(r+e_k,j)\setminus 1}\bar\nu(r-e_j+e_k+e_i)+\bar\nu(r-e_j+e_k+e_1)\right)\right.$$
$$\left.-(L-1)\left(\sum_{i\in\Upsilon_\kappa(r+e_k,j)\setminus 1}\nu(r-e_j+e_i)+\nu(r-e_j+e_k)+\nu(r-e_j+e_1)\right)\right]$$
$$=-c+\frac{\gamma}{N}\Big[(L-1)\sum_{i\in\Upsilon_\kappa(r+e_k,j)\setminus 1}\Big(\bar\nu(r-e_j+e_i+e_k)-\nu(r-e_j+e_i)\Big)$$
$$+\sum_{i\in\Upsilon_\kappa(r+e_k,j)\setminus 1}\Big(\bar\nu(r-e_j+e_i+e_k)-\nu(r-e_j+e_k)\Big)-\nu(r-e_j+e_k)$$
$$+(L-1)\Big(\bar\nu(r-e_j+e_k+e_1)-\nu(r-e_j+e_1)\Big)+\bar\nu(r-e_j+e_k+e_1)\Big]$$
$$\leq -c+\frac{\gamma}{N}\Big[\bar\nu(r-e_j+e_k+e_1)-\nu(r-e_j+e_k)\Big]\leq -c+\frac{\gamma}{N}\frac{cN}{\gamma}=0\ ,$$

where the first inequality follows (25), and the second inequality follows (26).

(2.3.2) $j\neq 1$, $r_j>0$, $1\in\Upsilon_\kappa(r,j)$, $1\notin\Upsilon_\kappa(r+e_k,j)$. This is not possible when we assume $k\neq 1$.

Next, we consider condition (26). First, consider the case when $r_1=0$, and. Denote C_n^k to be n choose k. We have the following possibilities.

R0.1 Player 1 is the only one with zero scrip in r and remains the only minimum scrip holder in $r+e_1$. Denote M to be the product of the probability that player 1 provides service when another (random) player is the service requester times $N-1$.

$$\sum_j (TV)(r,j) = \sum_j \sum_{\kappa_j} (\Xi_{\kappa_j} V)(r,j)/C_{N-1}^K = -cM + \frac{\gamma}{N}\left[\nu(r) + \sum_{j\neq 1}\left(\sum_{\kappa_j:1\in\kappa_j} \nu(r+e_1-e_j) + \sum_{\kappa_j:1\notin\kappa_j} \nu(r'_{\kappa_j})\right)/C_{N-1}^K\right]$$

$$\sum_j (T\bar{V})(r+e_1,j) = \sum_j \sum_{\kappa_j} (\Xi_{\kappa_j}\bar{V})(r+e_1,j)/C_{N-1}^K$$

$$= u - cM + \frac{\gamma}{N}\left[\sum_{\kappa_1}\bar{\nu}(r+e_{(\kappa_1)}) + \sum_{j\neq 1}\left(\sum_{\kappa_j:1\in\kappa_j}\bar{\nu}(r+2e_1-e_j) + \sum_{\kappa_j:1\notin\kappa_j}\bar{\nu}(r'_{\kappa_j}+e_1)\right)\right]/C_{N-1}^K$$

Therefore,

$$\sum_j (T\bar{V})(r+e_1,j) - (TV)(r,j) \leq u + \frac{\gamma}{N}\frac{cN}{\gamma}(N-1) \leq \frac{N}{\gamma}c - c(N-1) + c(N-1) = \frac{cN}{\gamma}\ ,$$

where the first inequality follows induction hypothesis (25) as well as (25), and the second inequality follows condition (24).

R0.2 Player 1 is NOT the only one with zero scrip in r. Similar to the previous case, the difference between $\sum_j (T\bar{V})(r+e_1,j)$ and $\sum_j (TV)(r,j)$ is upper bounded by u coming from the case $j=1$, as well as a collection of c terms. In the summations of all the cases over j, κ_j and service providers, either player 1 is not the service provider, in which case $\bar{\nu}$ and ν differ by e_1, contributing a term cN/γ in the difference following (25), which needs to be multiplied by the γ/N factor; or player 1 is selected as the service provider, resulting in direct contribution of c in the difference, while the $\bar{\nu}$ and ν terms differ with e_i with $i\neq 1$. Collecting the c terms in all cases, the total difference is, therefore, exactly $c(N-1)$. Following the logic exactly as case R0.1, we have, again,

$$\sum_j (T\bar{V})(r+e_1,j) - (TV)(r,j) \leq u + c(N-1) \leq \frac{cN}{\gamma}\ .$$

Now consider $r_1 > 0$. There is no difference of u between $\sum_j (T\bar{V})(r+e_1,j)$ and $\sum_j (TV)(r,j)$ anymore. Instead, when $j=1$, the difference between the $\bar{\nu}$ and ν terms are always e_1. Otherwise all other arguments in R0.2 follows. Consequently, the overall difference is bounded by $cN < cN/\gamma$.

Q.E.D.