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Iavor Bojinov
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Jinglong Zhao

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Iavor Bojinov
Harvard Business School

David Simchi-Levi
Massachusetts Institute of Technology

Jinglong Zhao
Massachusetts Institute of Technology

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Iavor Bojinov
Technology and Operations Management Unit, Harvard Business School, Boston, MA 02163, ibojinov@hbs.edu

David Simchi-Levi
Institute for Data, Systems, and Society, Department of Civil and Environmental Engineering, and Operations Research Center, Massachusetts Institute of Technology, Cambridge, MA 02139, dslevi@mit.edu

Jinglong Zhao
Institute for Data, Systems, and Society, Massachusetts Institute of Technology, Cambridge, MA 02139, jinglong@mit.edu

In switchback experiments, a firm sequentially exposes an experimental unit to a random treatment, measures its response, and repeats the procedure for several periods to determine which treatment leads to the best outcome. Although practitioners have widely adopted this experimental design technique, the development of its theoretical properties and the derivation of optimal design procedures have been, to the best of our knowledge, elusive. In this paper, we address these limitations by establishing the necessary results to ensure that practitioners can apply this powerful class of experiments with minimal assumptions. Our main result is the derivation of the optimal design of switchback experiments under a range of different assumptions on the order of carryover effect — that is, the length of time a treatment persists in impacting the outcome. We cast the experimental design problem as a minimax discrete robust optimization problem, identify the worst-case adversarial strategy, establish structural results for the optimal design, and finally solve the problem via a continuous relaxation. For the optimal design, we derive two approaches for performing inference after running the experiment. The first provides exact randomization based p-values and the second uses a finite population central limit theorem to conduct conservative hypothesis tests and build confidence intervals. We further provide theoretical results for our inferential procedures when the order of the carryover effect is misspecified. For firms that possess the capability to run multiple switchback experiments, we also provide a data-driven strategy to identify the likely order of carryover effect. To study the empirical properties of our results, we conduct extensive simulations. We conclude the paper by providing some practical suggestions.

1. Introduction

Academic scholars have appreciated the benefits that experimentation brings to firms for many decades (March 1991, Sitkin 1992, Sarasvathy 2001, Thomke 2001, Kohavi and Thomke 2017). However, widespread adoption of the practice has only taken off in the last decade, partly fueled by the rapid cost reductions achieved by firms in the technology sector (Kohavi et al. 2007, 2009, Azevedo et al. 2019, Kohavi et al. 2020). Most large firms now possess internal tools for experimentation, and a growing number of smaller and more conventional companies are purchasing the capabilities from third-party sellers that offer full-stack integration (Thomke 2020). These tools typically allow simple "A/B" tests that compare the standard offering "A" to a new or improved version "B". The comparisons are made across a range of different business outcomes, and the tests are usually conducted for at least a week. This simple practice has provided tremendous value to firms (Koning et al. 2019). Some firms and authors, however, have recognized the limitations of these simple A/B tests (Bojinov et al. 2020b). Principle amongst these is adequately handling interference[1] (the scenario where the assignment of one subject impacts another) or estimating heterogeneous (or personalized) effects[2].

In this paper, we simultaneously tackle both of these challenges by developing a theoretical framework for the optimal design and analysis of switchback (or time series) experiments. In switchback experiments, we sequentially expose a unit to a random treatment, measure its response, and repeat the procedure for a fixed period of time (Robins 1986, Bojinov and Shephard 2019). By administering alternate treatments to the same unit, we can directly estimate an individual level causal effect and alleviate the challenges posed by interference.

There are two classes of applications where switchback experiments are widely used in practice. The first arises when units interfere with each other either through a network or some more complicated unknown structure. For example, consider a ride-hailing platform that wants to test a new fare pricing algorithm's effectiveness in a large city (Farronato et al. 2018). Administering the test version to a subset of drivers can impact their behavior, which, in turn, could change the behavior of drivers that are receiving the old version. Directly comparing the revenue generated by the drivers across the two groups will likely provide a biased estimate of what would happen if everyone were assigned to the new version compared to the old. Instead, practitioners treat the city as a single aggregated unit and use a switchback experiment to estimate the intervention's effectiveness, thereby alleviating the problem caused by interference. A similar issue often arises in revenue management when, for example, a retailer wants to test the effectiveness of a new promotion planning algorithm (Ferreira et al. 2016). Administering the new version to a subset of stock keeping units (SKU's) cannibalizes the sales from the other SKU's. Again comparing the generated revenue across the two groups is unlikely to provide an accurate measure of the promotion's effectiveness. Instead, practitioners can treat all the SKU's as a single aggregated unit and use a switchback experiment to obtain accurate estimates of the promotion's effectiveness. The

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[1] Many online platforms and retail marketplaces have observed different levels of interference when the assignment of one subject impacts another. See Kastelman and Ramesh (2018), Farronato et al. (2018), Glynn et al. (2020) for online platforms (*e.g.*, DoorDash, Lyft, Uber), and Caro and Gallien (2012), Ferreira et al. (2016), Ma et al. (2020) for retail markets (*e.g.,* AB InBev, Rue la la, Zara).

[2] See Nie et al. (2018), Deshpande et al. (2018), Hadad et al. (2019) for estimating heterogeneous effects.

second application arises when we have a limited number of experimental units, and we believe the effects are likely to be heterogeneous. For example, Bojinov and Shephard (2019) used switchback experiments to make causal claims about the relative effectiveness of algorithms compared with humans at executing large financial trades across a range of financial markets. More generally, psychologists and biostatisticians regularly use switchback experiments whenever studying the effectiveness of an intervention on a single unit (*e.g.*, Lillie et al. (2011) and Boruvka et al. (2018)).

There are three significant challenges to using switchback experiments. The first is that causal estimators from switchback experiments have large variances as the precision is a function of the total number of assignments. The second is that past interventions are likely to impact future outcomes; this is often referred to as a carryover effect. Typically, many authors assume that there are no carryover effects, *e.g.*, Chamberlain (1982), Athey and Imbens (2018), and Imai and Kim (2019), although some recent work has relaxed this assumption (Sobel 2012, Bojinov et al. 2020a). The third is that standard super population inference - where people either assume a model for the outcome[3], or that the units are sampled from an infinitely large population - requires unrealistic assumptions that fail to capture the problem's personalized nature (Bojinov and Shephard 2019).

This paper's main contributions are to address these three challenges and present a framework that allows firms and researchers to run reliable switchback experiments. First, we derive optimal designs for switchback experiments, ensuring that we can select a design that leads to the lowest variance among the most popular class assignment mechanisms. Second, we assume the presence of a carryover effect and show that our estimation and inference are valid both when the order of carryover effect is correctly specified and misspecified, the later leading to a minor increase in the variance. For practitioners and managers, we also propose a method to identify the order of carryover effect by running a series of carefully designed switchback experiments. Finally, we take a purely design-based perspective on uncertainty; that is, we treat the outcomes as unknown but fixed (or equivalently, we condition on the set of potential outcomes) and assume that the assignment mechanism is the only source of randomness (Abadie et al. 2020). The main benefit of a design-based perspective is that the inference, and in turn the causal conclusions, do not depend on our ability to correctly specify a model describing the phenomena we are studying, ensuring that our findings are wholly non-parametric and robust to model misspecification (Imbens and Rubin 2015, Chapter 5).

The paper is structured as follows. In Section 2 we define the notations, the assumptions, and the assignment mechanism that we focus on, which is referred to as the *regular switchback experiments*.

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[3] For example, Wager and Xu (2019), Johari et al. (2020) assume that the market they are experimenting on is in an equilibrium state, Glynn et al. (2020) assumes an underlying Markovian model for the outcomes, and Athey et al. (2018), Eckles et al. (2016), Sussman and Airoldi (2017), Puelz et al. (2019) make assumptions on the structure of the interference.

In Section 3, we discuss how to design an effective regular switchback experiment under the minimax rule. We cast the design problem as a minimax robust optimization problem. We identify the worst-case adversarial strategy, establish structural results, and then explicitly find the optimal design. In Section 4, we discuss how to perform inference and conduct statistical testing based on the results obtained from an optimally designed switchback experiments. We propose an exact test for sharp null hypotheses, and an asymptotic test for testing the average treatment effect. We provide p-values and construct confidence intervals based on such two hypotheses tests. In Section 5, we discuss cases when carryover effects are misspecified. We show that our estimation and inference still remain valid, with only a little more variance. In Section 6, we run simulations to test the correctness and effectiveness of our proposed experiments under various simulation setups. In Section 7, we discuss how to conduct hypothesis testing to identify the true order of carryover effects. We give empirical illustrations on how to conduct a switchback experiment in practice. All the proofs can be found in the Appendix.

2. Notations, Assumptions, and Regular Switchback Experiments

2.1. Assignment Paths and Potential Outcomes

We focus our discussion on a single experimental unit. For example, this unit could be a ride-hailing platform testing the effectiveness of a new fare pricing algorithm in a large city, or a retailer testing the effectiveness of a new promotion planning algorithm over all its SKU's. At each time point $t \in [T] = \{1, 2, ..., T\}$, we assign the unit to receive an intervention $W_t \in \{0, 1\}$. For example, one experimental period could be several minutes to one hour for a ride-hailing platform or one to two days for a retailer; the intervention could be a new pricing or promotion planning algorithm. In some applications, external factors determine the time horizon, T; however, when T is not fixed, Section 7.2, provides details for how the manager can determine an appropriate T.

Following convention, we say that the unit is assigned to treatment if $W_t = 1$ and control when $W_t = 0$; in A/B testing terminology, "A" is control and "B" is treatment. The assignment path is then the collection of assignments and is denoted using a vector notation whose dimensions are specified in the subscript, $\boldsymbol{W}_{1:T} = (W_1, W_2, ..., W_T) \in \{0, 1\}^T$. We adopt the convention that $\boldsymbol{W}_{1:t}$ stands for a random assignment path, while $\boldsymbol{w}_{1:t}$ stands for one realization of the random assignment path. Though we focus on binary assignments, our results easily extend to more complex settings.

After administering the assigned intervention, we observed a corresponding outcome. For example, this could be total traffic or total revenue generated during each experimental period. Let $\{t : t'\} = \{t, t+1, ..., t'\}$. Following the extended potential outcomes framework, at time $t \in [T]$, we posit that for each possible assignment path $\boldsymbol{w}_{1:T}$ there exists a corresponding potential outcome denoted by $Y_t(\boldsymbol{w}_{1:T})$. The set of all potential outcomes will then be written as $\mathbb{Y} = \{Y_i(\boldsymbol{w}_{1:T})\}_{t \in [T], \boldsymbol{w}_{1:T} \in \{0,1\}^T}$.

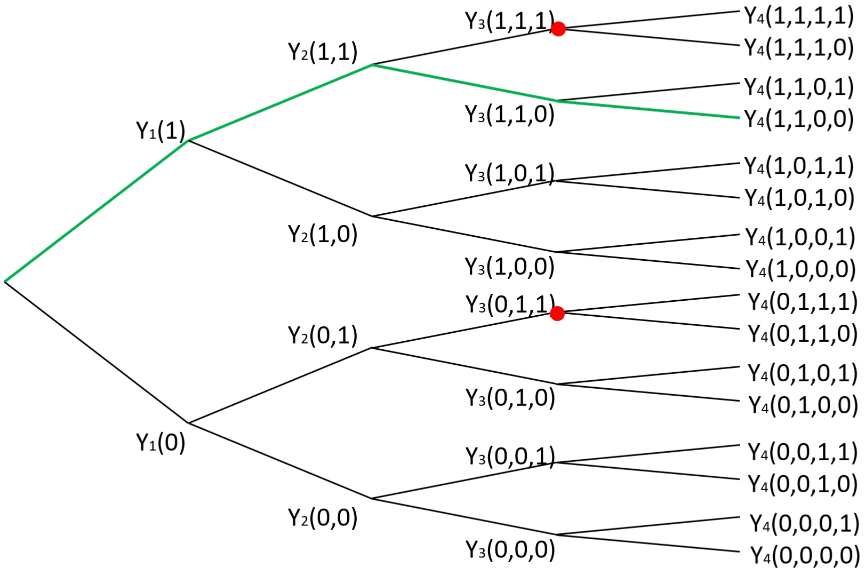

Figure 1 Illustrator of assignment paths and potential outcomes when $T = 4$. The green path stands for one assignment path $w_{1:4} = (1,1,0,0)$. The two red dots stand for two potential outcomes that are equal in Example 3.

EXAMPLE 1. When $T = 4$, there are 16 assignment paths as shown in Figure 1. Associated with each assignment path $\boldsymbol{w}_{1:4}$ are potential outcomes $Y_1(\boldsymbol{w}_{1:4}), Y_2(\boldsymbol{w}_{1:4}), Y_3(\boldsymbol{w}_{1:4}), Y_4(\boldsymbol{w}_{1:4})$. □

Throughout this paper, we do not directly model the potential outcomes or impose a parametric relationship with the assignment path; instead, we treat them as unknown but fixed quantities, or, equivalently, we will implicitly condition on $\mathbb{Y}$ (Imbens and Rubin 2015, Chapter 5). The benefit of this approach is that we can be completely agnostic to the outcome process, allowing us to make nonparametric causal claims. To make inference possible, we rely on the variation introduced by the random assignment path; this is commonly referred to as finite-sample or design-based perspective. Unlike traditional sampling-based inference, our approach does not require a hypothetical population from which we sampled our experimental units (Abadie et al. 2020).

Since the potential outcomes are fixed but unknown, we can assume that their absolute values are bounded from above (Robins et al. 1999, Bai 2019, Li et al. 2020). Assumption 1 is almost always satisfied, since it only assumes that the potential outcomes are bounded by the same constant B, *e.g.*, the total traffic or revenue generated from each experimental period is some finite amount.

ASSUMPTION 1 **(Bounded Potential Outcomes)**. *Assume that the potential outcomes are bounded by some constant, i.e.,* $\exists B > 0, s.t. \forall t \in [T], \forall \boldsymbol{w} \in \{0,1\}^T$,

$$|Y_t(\boldsymbol{w})| \leq B$$

and denote $\mathbb{Y} \in [-B, B]^T := \mathcal{Y}$. *In particular, knowledge about the magnitude of B is not required.*

We further make the following two assumptions that limit the dependence of the potential outcomes on assignment paths.

Assumption 2 **(Non-anticipating Potential Outcomes)**. *Assume for any* $t \in [T]$, $\boldsymbol{w}_{1:t} \in \{0,1\}^T$, *and for any* $\boldsymbol{w}'_{t+1:T}, \boldsymbol{w}''_{t+1:T} \in \{0,1\}^{T-t}$,

$$Y_t(\boldsymbol{w}_{1:t}, \boldsymbol{w}'_{t+1:T}) = Y_t(\boldsymbol{w}_{1:t}, \boldsymbol{w}''_{t+1:T}).$$

Assumption 2 states that the potential outcomes at time t do not depend on future treatments (Bojinov and Shephard 2019, Basse et al. 2019, Rambachan and Shephard 2019). Since we control the assignment mechanism, the design ensures that this assumption is satisfied.

Example 2 (Example 1 Continued). Under Assumption 2, $Y_3(1,1,1,1) = Y_3(1,1,1,0)$. So we use the red dot at $Y_3(1,1,1)$ to stand for both $Y_3(1,1,1,1)$ and $Y_3(1,1,1,0)$. ☐

Assumption 3 **(No m-Carryover Effects)**. *Assume there exists a fixed and given* m, *such that for any* $t \in \{m+1, m+2, ..., T\}, \boldsymbol{w}_{t-m:T} \in \{0,1\}^{T-t+m+1}$, *and for any* $\boldsymbol{w}'_{1:t-m-1}, \boldsymbol{w}''_{1:t-m-1} \in \{0,1\}^{t-m-1}$,

$$Y_t(\boldsymbol{w}'_{1:t-m-1}, \boldsymbol{w}_{t-m:T}) = Y_t(\boldsymbol{w}''_{1:t-m-1}, \boldsymbol{w}_{t-m:T}).$$

Assumption 3 restricts the order of carryover effect (Laird et al. 1992, Senn and Lambrou 1998, Bojinov and Shephard 2019, Basse et al. 2019). In many applications, Assumption 3 is satisfied; however, practitioners must rely on their domain knowledge to choose an appropriate m. For example, surge pricing on a ride-hailing platform is typically known not to carry over for more than $1 \sim 2$ hours, depending on the city size (Garg and Nazerzadeh 2019). Stockpiling of beverage drinks induced by a promotion is typically consumed within a few days, due to the "pantry effect" (Ailawadi and Neslin 1998, Bell et al. 1999). Assuming an incorrect m will not invalidate the subsequent inference, but will lead to a little increase in variance. See Section 5 and 6.5 for discussions. Moreover, we can always correctly identify m with a little more experimental budget. See Section 7.1 for discussions.

Under both Assumptions 2 and 3, we simplify notations as follows. For any $t \in \{m+1, ..., T\}$, $\boldsymbol{w}_{t-m:t} \in \{0,1\}^{m+1}$, and for any $\boldsymbol{w}'_{1:t-m-1}, \boldsymbol{w}''_{1:t-m-1} \in \{0,1\}^{t-m-1}$, any $\boldsymbol{w}'_{t+1:T}, \boldsymbol{w}''_{t+1:T} \in \{0,1\}^{T-t}$,

$$Y_t(\boldsymbol{w}'_{1:t-m-1}, \boldsymbol{w}_{t-m:t}, \boldsymbol{w}'_{t+1:T}) = Y_t(\boldsymbol{w}''_{1:t-m-1}, \boldsymbol{w}_{t-m:t}, \boldsymbol{w}''_{t+1:T}).$$

In the remainder of this paper, we will write $Y_t(\boldsymbol{w}_{1:t-m-1}, \boldsymbol{w}_{t-m:t}, \boldsymbol{w}_{t+1:T}) = Y_t(\boldsymbol{w}_{t-m:t})$.

Example 3 (Example 2 Continued). Suppose $m = 1$. Under Assumptions 2 and 3, the potential outcomes at the 2 red dots in Figure 1 are equal, i.e., $Y_3(1,1,1) = Y_3(0,1,1)$. ☐

2.2. Causal Effects

In the potential outcomes approach to causal inference, any comparison of potential outcomes has a causal interpretation. For any $p \in \mathbb{N}$, let $\mathbf{1}_{p+1} = (1,1,...,1)$ be a vector of $p+1$ ones; let $\mathbf{0}_{p+1} = (0,0,...,0)$ be a vector of $p+1$ zeros. In this paper, we focus on the average lag-p causal effect of consecutive treatments on the outcome, defined for any $p \in [T-1]$,

$$\tau_p(\mathbb{Y}) = \frac{1}{T-p} \sum_{t=p+1}^{T} [Y_t(\mathbf{1}_{p+1}) - Y_t(\mathbf{0}_{p+1})]. \tag{1}$$

This estimand captures the effects of permanently deploying a new policy[4], and has been widely studied in the longitudinal experiments since the early work of Robins (1986).

Although we focus on an average causal effect, all of our results and analysis trivially extend to the total causal effect, which does not divide the sum by the number of estimands, i.e., $(T-p)\tau_p(\mathbb{Y})$. The optimal design as we will show in Section 3 will not be changed. We mainly focus on the $p = m$ case in this section. We refer to Section 5 when m is misspecified, and Section 7.1 to identify m.

The challenge of causal inference on switchback experiments is that we only observe one assignment path. So in each period t, we observe at most either $Y_t(\mathbf{1}_{p+1})$ or $Y_t(\mathbf{0}_{p+1})$ (and sometimes neither). To link the observed and potential outcomes, we assume there is only one version of the treatment, and there is no non-compliance. Let $\boldsymbol{w}_{1:T}^{\mathsf{obs}}$ be the realized assignment path. Let $Y_t^{\mathsf{obs}} = Y_t(\boldsymbol{w}_{1:T}^{\mathsf{obs}})$ be the observed outcome at time t, under the realized assignment path $\boldsymbol{w}_{1:T}^{\mathsf{obs}}$.

2.3. Regular Switchback Experiments

It is the manager's decision to decide the design of switchback experiments. In this paper, we narrow our scope to the family of regular switchback experiments. This family of experiments are parameterized by

$$\mathbb{T} = \{t_0 = 1 < t_1 < t_2 < ... < t_K\} \subseteq [T],$$

where $K \in \mathbb{N}$ belongs to the set of all positive integers, and $\mathbb{T}$ contains a total of $K+1$ integers. For the ease of notations also let $t_{K+1} = T+1$.

Definition 1 (Regular Switchback Experiments). For any $K \in \mathbb{N}$ and any $\mathbb{T} = \{t_0 = 1 < t_1 < t_2 < ... < t_K\} \subseteq [T]$, a regular switchback experiment $\mathbb{T}$ administers a probabilistic treatment at any time t, given by:

$$\Pr(W_t = 1) = \begin{cases} 1/2, & \text{if } t \in \mathbb{T} \\ \mathbb{1}\{W_{t-1} = 1\}, & \text{if } t \notin \mathbb{T} \end{cases} \tag{2}$$

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[4] What is typical in companies is that after the switchback experiment is finished, managers decide to permanently deploy (or not to deploy) a new policy. So the unit is always exposed to treatment (or control) hereon.

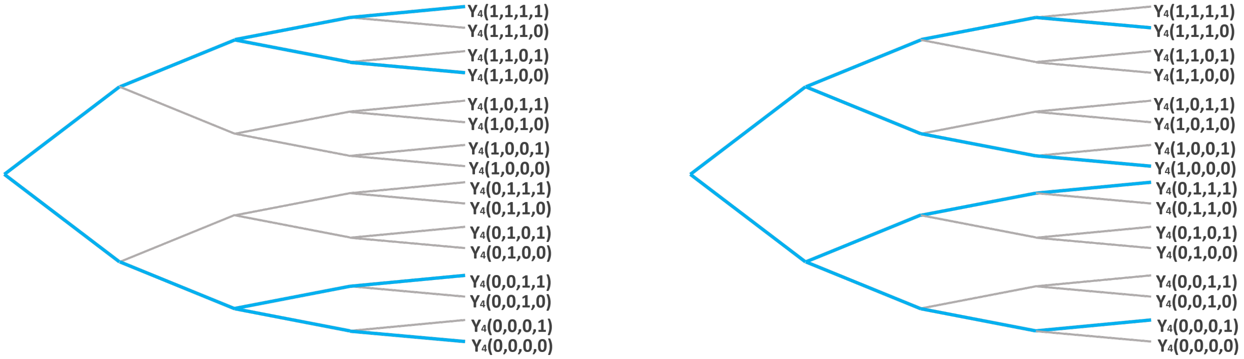


Figure 2 Two designs. The blue lines stand for the possible treatment assignments that a design could administer. Left: regular switchback experiment (Example 4); Right: irregular switchback experiment (Example 5).

In words, the manager decides on a collection of randomization points, which consists of flipping a fair coin at each period $t \in \{t_0, ..., t_K\}$. If the resulting flip at period t_k is heads, then the manager assigns the unit to treatment during periods $(t_k, t_k + 1, ..., t_{k+1} - 1)$; otherwise, if tails, then the manager assign the unit to control during periods $(t_k, t_k + 1, ..., t_{k+1} - 1)$. The reason behind fair coin flips are the inherent symmetry and ignorance of the manager. When $t \in \mathbb{T}$, the treatment probability that leads to the smallest variance is $1/2$.

Example 4. When $T = 4$, $\mathbb{T} = \{t_0 = 1, t_1 = 3\}$ corresponds to the following design: with probability $1/4$, $\boldsymbol{W}_{1:4} = (1,1,1,1)$; with probability $1/4$, $\boldsymbol{W}_{1:4} = (1,1,0,0)$; with probability $1/4$, $\boldsymbol{W}_{1:4} = (0,0,1,1)$; with probability $1/4$, $\boldsymbol{W}_{1:4} = (0,0,0,0)$. See Figure 2 (left figure). ☐

Example 5. Not all switchback experiments are regular. For example, when $T = 4$: with probability $1/4$, $\boldsymbol{W}_{1:4} = (1,1,1,0)$; with probability $1/4$, $\boldsymbol{W}_{1:4} = (1,0,0,0)$; with probability $1/4$, $\boldsymbol{W}_{1:4} = (0,1,1,1)$; with probability $1/4$, $\boldsymbol{W}_{1:4} = (0,0,0,1)$. See Figure 2 (right figure). ☐

Any design of switchback experiment induces a probabilistic distribution over assignment paths $\boldsymbol{w}_{1:T} \in \{0,1\}^T$. Define a design of switchback experiment to be any $\eta : \{0,1\}^N \to [0,1]$ such that

$$\sum_{\boldsymbol{w}_{1:T} \in \{0,1\}^T} \eta(\boldsymbol{w}_{1:T}) = 1, \qquad \eta(\boldsymbol{w}_{1:T}) \geq 0, \ \forall \boldsymbol{w}_{1:T} \in \{0,1\}^T.$$

Explicitly, $\eta(\cdot)$ is the underlying discrete distribution of the random assignment path $\boldsymbol{W}_{1:T}$. For any regular switchback experiment $\mathbb{T}$, we refer to the probability distribution using $\eta_{\mathbb{T}}(\cdot)$.

For any $\mathbb{T}$, there are a total of 2^{K+1} many assignment paths, which are uniquely determined by the values of $W_{t_0}, W_{t_1}, ..., W_{t_K}$. The assignment path is random, and follows the probability distribution $\eta_{\mathbb{T}}(\cdot)$:

$$\eta_{\mathbb{T}}(\boldsymbol{w}_{1:T}) = \begin{cases} 1/2^{K+1}, & \text{if } \ \forall k \in \{0,1,...,K\}, w_{t_k} = w_{t_k+1} = ... = w_{t_{k+1}-1}, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

In the remainder of this paper, unless explicitly noted, all probabilities and expectations are taken with respect to this probability distribution $\eta_{\mathbb{T}}(\cdot)$.

2.4. Estimation

Now that $\eta_{\mathbb{T}}(\cdot)$ is determined, following any realization of the assignment path $\boldsymbol{W}_{1:T} = \boldsymbol{w}_{1:T}$, we use the Horvitz-Thompson estimator to estimate the lag-p effect:

$$\hat{\tau}_p(\eta_{\mathbb{T}}, \boldsymbol{w}_{1:T}, \mathbb{Y}) = \frac{1}{T-p} \sum_{t=p+1}^{T} \left\{ Y_t^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{w}_{t-p:t} = \mathbf{1}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1})} - Y_t^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{w}_{t-p:t} = \mathbf{0}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t} = \mathbf{0}_{p+1})} \right\} \tag{4}$$

Since the assignment path $\boldsymbol{W}_{1:T}$ is random, this Horvitz-Thompson estimator is random, as well. We emphasize that estimator depends on (i) the probability distribution behind the random assignment path, (ii) the realization of the assignment path, and (iii) the potential outcomes.

Example 6. Suppose $T = 4, p = m = 1$. See Figure 1. Suppose the assignments are probabilistic and $\Pr(W_t = 1) = \Pr(W_t = 0) = 1/2, \forall t \in [4]$. With probability 1/16 the green assignment path is administered, $\boldsymbol{W}_{1:4} = (1, 1, 0, 0)$. The estimator is $\hat{\tau}_1 = \frac{1}{3} \{4Y_2(1,1) + 0 - 4Y_4(0,0)\}$. ☐

It is well-known that the Horvitz-Thompson estimator is unbiased under the probabilistic treatment assignment assumption, which is satisfied by regular switchback experiments.

Assumption 4 **(Probabilistic Treatment Assignment)**. *Assume for any* $t \in \{p+1 : T\}$, *both* $0 < \Pr(\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1}), \Pr(\boldsymbol{W}_{t-p:t} = \mathbf{0}_{p+1}) < 1$.

It is easy to verify that regular switchback experiments satisfy Assumption 4, since the assignment on the firs period is random, i.e. $\Pr(W_1 = 1) = 1/2$.

Theorem 1 **(Unbiasedness of the Horvitz-Thompson Estimator)**. *In a regular switchback experiment, under Assumptions 2 and 3, the Horvitz-Thompson estimator is unbiased for the average lag-p causal effect of consecutive treatments on outcome, i.e.,*

$$\mathbb{E}_{\boldsymbol{W}_{1:T} \sim \eta_{\mathbb{T}}}[\hat{\tau}_p(\eta_{\mathbb{T}}, \boldsymbol{W}_{1:T}, \mathbb{Y})] = \tau_p(\mathbb{Y}).$$

The proof to Theorem 1 is standard, by checking the expectations. We defer its proof to Section EC.3 in the Appendix. When m is misspecified, the above estimator is still meaningful with causal interpretations. See Section 5 for a discussion.

2.5. Evaluation of Experiments: the Decision-Theoretic Framework

To evaluate the quality of a design of experiment, we adopt the decision-theoretic framework (Berger 2013, Bickel and Doksum 2015). When the random design is $\eta_{\mathbb{T}}(\cdot)$, the assignment path $\boldsymbol{W}_{1:T}$ is random. For any realization of the assignment path $\boldsymbol{w}_{1:T}$ and any set of potential outcomes $\mathbb{Y}$, we define the loss function

$$L(\eta_{\mathbb{T}}, \boldsymbol{w}_{1:T}, \mathbb{Y}) = \left(\hat{\tau}_p(\eta_{\mathbb{T}}, \boldsymbol{w}_{1:T}, \mathbb{Y}) - \tau_p(\mathbb{Y})\right)^2$$

and the risk function

$$r(\eta_{\mathbb{T}}, \mathbb{Y}) = \mathbb{E}_{\boldsymbol{W}_{1:T} \sim \eta_{\mathbb{T}}(\cdot)} \left[L(\eta_{\mathbb{T}}, \boldsymbol{W}_{1:T}, \mathbb{Y}) \right]$$
$$= \sum_{\boldsymbol{w}_{1:T} \in \{0,1\}^T} \eta_{\mathbb{T}}(\boldsymbol{w}_{1:T}) \cdot \left(\hat{\tau}_p(\eta_{\mathbb{T}}, \boldsymbol{w}_{1:T}, \mathbb{Y}) - \tau_p(\mathbb{Y}) \right)^2 \tag{5}$$

Such a risk function quantifies the expected loss incurred by one design of experiment. Since the estimator is unbiased, the risk function also has a second interpretation: the variance of the estimator. A design with lower risk is also a design whose estimator has a lower variance.

Example 7 (Examples 4 and 6 Revisited). Suppose $T = 4$ and $p = m = 1$. As in Example 4, the experiment is $\mathbb{T} = \{1, 3\}$. With probability 1/4, $\boldsymbol{W}_{1:4} = (1, 1, 0, 0)$, and $\hat{\tau}_1(\mathbb{T}) = \frac{1}{3}\{2Y_2(1,1) - 2Y_4(0,0)\}$. So $L(\eta_{\tilde{\mathbb{T}}}, \boldsymbol{w}_{1:T}, \mathbb{Y}) = \frac{1}{9}\left(Y_2(1,1) + Y_2(0,0) - Y_3(1,1) + Y_3(0,0) - Y_4(1,1) - Y_4(0,0)\right)^2$. As in Example 6, $\tilde{\mathbb{T}} = \{1, 2, 3, 4\}$. With probability 1/16, $\boldsymbol{W}_{1:4} = (1, 1, 0, 0)$, and $\hat{\tau}_1(\tilde{\mathbb{T}}) = \frac{1}{3}\{4Y_2(1,1) - 4Y_4(0,0)\}$. So $L(\eta_{\mathbb{T}}, \boldsymbol{w}_{1:T}, \mathbb{Y}) = \frac{1}{9}\left(3Y_2(1,1) + Y_2(0,0) - Y_3(1,1) + Y_3(0,0) - Y_4(1,1) - 3Y_4(0,0)\right)^2$. □

Example 7 suggests that, even if the two realizations of the assignment path are the same and the potential outcomes are the same, since the probability distributions $\mathbb{T}, \tilde{\mathbb{T}}$ are different, the corresponding loss functions could be different. This motivates us to find a probability distribution with a smaller expected loss against some $\mathbb{Y}$, which we will detail in Section 3.

3. Design of Regular Switchback Experiments under Minimax Rule

The minimax decision rule (Berger 2013, Wu 1981) finds an optimal design of experiment, such that the worst-case risk against an adversarial selection of potential outcomes is minimized,

$$\min_{\mathbb{T} \in [T]} \max_{\mathbb{Y} \in \mathcal{Y}} r(\eta_{\mathbb{T}}, \mathbb{Y}) = \min_{\mathbb{T} \in [T]} \max_{\mathbb{Y} \in \mathcal{Y}} \sum_{\boldsymbol{w}_{1:T} \in \{0,1\}^T} \eta_{\mathbb{T}}(\boldsymbol{w}_{1:T}) \cdot \left(\hat{\tau}_p(\boldsymbol{w}_{1:T}, \mathbb{Y}) - \tau_p(\mathbb{Y}) \right)^2. \tag{6}$$

The goal of this section is to find the optimal $\mathbb{T}^* \subseteq [T]$. Throughout this section we assume perfect knowledge of m and assume $p = m$.

In practice, there is a trade-off between having too few randomization points (corresponding to small K) and too many (corresponding to large K). Intuitively, too many decreases the probability of observing an assignment path $\mathbf{1}_{m+1}$ or $\mathbf{0}_{m+1}$, which, in turn, decreases the amount of useful data. On the other hand, too few decreases the number of independent observations and reduces our ability to produce reliable results. Both of these scenarios reduce our ability to draw valid causal claims. To make switchback experiments useful in practice, we need to find the optimal number of randomization points that allows us to draw valid inference while minimizing the variance. We formalize this goal through the minimax framework, where we try to derive the best possible design for the worse possible set of potential outcomes.

To solve the minimax problem, we start by focusing on the inner maximization part of (6). We characterize the worst-case potential outcomes by identifying two dominating strategies for the adversarial selection of potential outcomes. Denote $\mathbb{Y}^+ = \{Y_t(\mathbf{1}_{m+1}) = Y_t(\mathbf{0}_{m+1}) = B\}_{t\in\{m+1:T\}}$ and $\mathbb{Y}^- = \{Y_t(\mathbf{1}_{m+1}) = Y_t(\mathbf{0}_{m+1}) = -B\}_{t\in\{m+1:T\}}$

Lemma 1. *Under Assumptions 1–3, $\mathbb{Y}^+$ and $\mathbb{Y}^-$ are the only two dominating strategies for the adversarial selection of potential outcomes. That is, for any $\mathbb{T} \subseteq [T]$ and for any $\mathbb{Y} \in \mathcal{Y}$,*

$$r(\eta_{\mathbb{T}}, \mathbb{Y}^+) \geq r(\eta_{\mathbb{T}}, \mathbb{Y}); \qquad\qquad r(\eta_{\mathbb{T}}, \mathbb{Y}^-) \geq r(\eta_{\mathbb{T}}, \mathbb{Y}).$$

Moreover, for any $\mathbb{Y} \in \mathcal{Y}$ such that $\mathbb{Y} \neq \mathbb{Y}^+$ or $\mathbb{Y}^-$, the above two inequalities are strict.

The proof of Lemma 1 and an implication of Lemma 1 can be found in Sections EC.4.2.2 and EC.4.2.3, respectively.

Example 8 (Example 4 Continued). Suppose $T = 4$, $p = m = 1$, and $\mathbb{T} = \{1, 3\}$. The risk function can be calculated by $r(\eta_{\mathbb{T}}, \mathbb{Y}) = \sum_{t=2}^{4}[(Y_t(1,1) + Y_t(0,0))^2] + 2Y_3(1,1)^2 + 2Y_3(0,0)^2 + 2\sum_{t=2}^{3}[(Y_t(1,1) + Y_t(0,0))(Y_{t+1}(1,1) + Y_{t+1}(0,0))]$. The risk function is maximized by $\mathbb{Y}$ (only) at $Y_t(1,1) = Y_t(0,0) = \pm B, \forall t \in \{2:4\}$. This is what Lemma 1 says. $\square$

Lemma 1 simplifies the minimax problem in (6). Instead of directly solving the minimax problem, we can now replace $\mathbb{Y}$ by either $\mathbb{Y}^+$ or $\mathbb{Y}^-$, and solve only a minimization problem.

Using Lemma 1, we now establish two structural results that limit the class of optimal designs of regular switchback experiments. Lemma 2 states the optimal starting and ending structure; Lemma 3 states the optimal middle-case structure. The proofs to Lemma 2 and Lemma 3 are deferred to Sections EC.4.3.1 and EC.4.3.2, respectively.

Lemma 2. *When $\mathbb{Y} = \mathbb{Y}^+$ or $\mathbb{Y} = \mathbb{Y}^-$, under Assumptions 1–3, any optimal design of experiment $\mathbb{T}$ must satisfy*

$$t_1 \geq m+2, \qquad \text{and} \qquad t_K \leq T - m.$$

Lemma 2 suggests that the first coin flip on period 1 should be followed by at least m periods that do not flip a coin, and that the last coin flip should be followed by at least m periods that do not flip a coin. It guarantees that the assignment path during $\{1 : m+1\}$ and during $\{T - m : T\}$ are both useful, i.e., $\mathbf{1}_{m+1}$ or $\mathbf{0}_{m+1}$.

Lemma 3. *When $\mathbb{Y} = \mathbb{Y}^+$ or $\mathbb{Y} = \mathbb{Y}^-$, under Assumptions 1–3, any optimal design of experiment $\mathbb{T}$ must satisfy*

$$t_{k+1} - t_{k-1} \geq m, \ \forall k \in [K].$$

Lemma 3 suggests that in every consecutive $m+1$ periods, there could be at most 3 randomization points. This is because too many randomization points in every consecutive $m+1$ periods decreases the chance of observing an useful assignment path of $\mathbf{1}_{m+1}$ or $\mathbf{0}_{m+1}$. Lemma 3 formalizes such intuition, and suggests that when m grows large, the optimal design randomizes less often.

Lemmas 2 and 3 restrict the space of possible optimal regular switchback experiment to a smaller class of switchback experiments, which we define below.

DEFINITION 2 (PERSISTENT SWITCHBACK EXPERIMENTS). We say a regular switchback experiment $\mathbb{T}$ is persistent, if it satisfies the following three conditions,

$$t_1 \geq m+2; \qquad\qquad t_K \leq T-m; \qquad\qquad t_{k+1}-t_{k-1} \geq m, \ \forall k \in [K].$$

For persistent switchback experiments, we can explicitly calculate the risk function $r(\eta_{\mathbb{T}}, \mathbb{Y})$.

THEOREM 2 **(Risk Function)**. *When* $\mathbb{Y}=\mathbb{Y}^+$ *or* $\mathbb{Y}=\mathbb{Y}^-$*, under Assumptions 1–3, the risk function for any persistent switchback experiment is given by*

$$r(\eta_{\mathbb{T}}, \mathbb{Y}) = \frac{1}{(T-m)^2}\left\{4\sum_{k=1}^{K+1}(t_k-t_{k-1})^2 + 8m(t_K-t_1) + 4m^2K - 4m^2 + 4\sum_{k=2}^{K}[(m-t_k+t_{k-1})^+]^2\right\}B^2 \tag{7}$$

Theorem 2 explicitly describes the risk function of any optimal design of regular switchback experiments, which lies in the class of persistent switchback experiments. The proof of Theorem 2 is deferred to Section EC.4.4 in the appendix.

To understand the risk function in (7), note that the first summation of the squares $\sum_{k=1}^{K+1}(t_k - t_{k-1})^2$ suggests that the gap between two consecutive randomization points should not be too big; while the last summation of the squares $\sum_{k=2}^{K}[(m-t_k+t_{k-1})^+]^2$ suggests that the gap should not be too small. Such a contrast formalized the trade-off that we have described earlier in this section.

Based on the risk function in (7), we are able to describe the optimal design, as we state in the next Theorem.

THEOREM 3 **(Optimal Design)**. *Under Assumptions 1–3, the optimal solution to the design of regular switchback experiment as we have introduced in* (6) *is equivalent to the optimal solution to the following subset selection problem.*

$$\min_{\mathbb{T}\subset[T]}\left\{4\sum_{k=0}^{K}(t_{k+1}-t_k)^2 + 8m(t_K-t_1) + 4m^2K - 4m^2 + 4\sum_{k=1}^{K-1}[(m-t_{k+1}+t_k)^+]^2\right\} \tag{8}$$

In particular, when $m=0$ *then* $\mathbb{T}^* = \{1,2,3,...,T\}$*; when* $m>0$*, and if there exists* $n \geq 4 \in \mathbb{N}$*, s.t.* $T = nm$*, then* $\mathbb{T}^* = \{1, 2m+1, 3m+1, ..., (n-2)m+1\}$.

The optimal design under two remarkable special cases are, when $m = 0$, $\mathbb{T}^* = \{1, 2, 3, ..., T\}$; and when $m = 1$, $\mathbb{T}^* = \{1, 3, 4, ..., T-1\}$. When managers believe there to be very little carryover effect, the optimal designs are almost the same. Moreover, Theorem 3 presents the optimal design in a class of perfect cases when the time horizon split into several epochs. In practice, selecting T is part of the design of the experiment; our recommendation is to pick a T that satisfies the conditions in Theorem 3. See Section 7.2.

We can also find the optimal design for other imperfect cases by solving (8); however, since there are integrality issues in the subset selection problem, the discussion of optimal design in such imperfect cases are rather technical. We defer to Section EC.4.5 in the appendix to discuss such details. The proof of Theorem 3 is deferred to Section EC.4.5.1 in the appendix.

EXAMPLE 9 (AN OPTIMAL DESIGN). When $T = 12$, $p = m = 2$, the optimal design of regular switchback experiment is $\mathbb{T}^* = \{1, 5, 7, 9\}$. See Table 1. □

Table 1 An example of the optimal design $\mathbb{T}^*$ versus an arbitrary design $\tilde{\mathbb{T}}$ when $T = 12$ and $p = m = 2$.

	1	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{T}^*$	✓	–	–	–	✓	–	✓	–	✓	–	–	–
$\tilde{\mathbb{T}}$	✓	–	–	✓	–	–	✓	–	–	✓	–	–

Each checkmark beneath a time period t indicates that t is a randomization point.

It is worth noting that both the causal estimand and the Horvitz-Thompson estimator involve consecutive treatments or controls for $m + 1$ periods. By contrast, Theorem 3 suggests that the optimal design have epochs of equal length m (ignoring the first and last epoch).

At a first sight this is counter-intuitive. Intuitively, each epoch should contain at least $m + 1$ periods so there exist periods that always have consecutive treatments $\mathbf{1}_{m+1}$ or $\mathbf{0}_{m+1}$ and always generate useful data; *e.g.*, periods $\tilde{t} = 4, 7, 10$ in the third row of Table 1. However, even if each epoch had $m + 1$ periods, there are still many periods that do not always generate useful data (*e.g.*, periods $\tilde{t} = 5, 6, 8, 9$); invalidating this intuition.

To understand the implications of Theorem 3, the correct intuition is the there is a trade-off between too many and too few randomization points. Due to the inherent underlying discrete convexity nature behind the minimax problem (6), the optimum balances randomization frequency between the two extremes. Theorem 3, quantifies this balance.

4. Inference and Statistical Testing

After designing and running the experiment, we obtain two time series. The first is the observed assignment path $\boldsymbol{w}_{1:T}^{\mathsf{obs}}$, and the second is the corresponding observed outcomes $\boldsymbol{Y}_{p+1:T}^{\mathsf{obs}}$. See Figure 3. To draw inference from this data we propose two methods, the exact inference and the asymptotic inference, as we detail below.

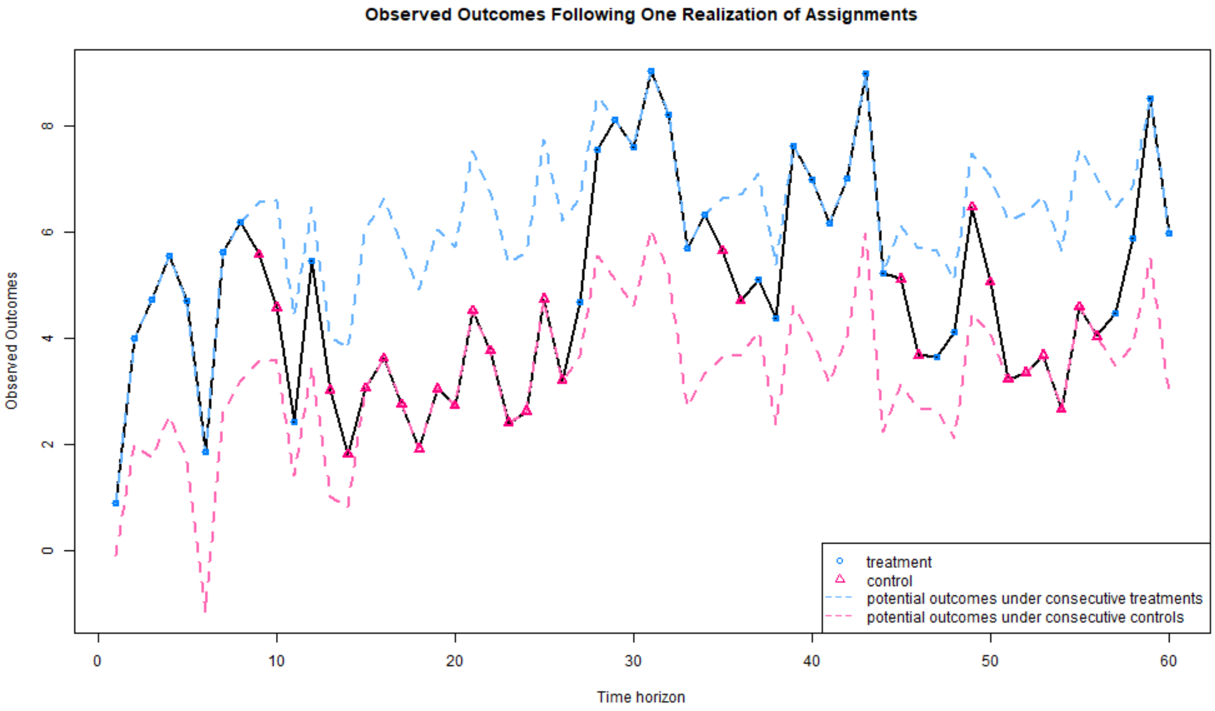


Figure 3 **Illustrator of the observed assignment path $w_{1:T}^{\text{obs}}$ (blue and red dots) and the observed outcomes $\boldsymbol{Y}_{p+1:T}^{\text{obs}}$ (black curve). The dashed lines are the potential outcomes under consecutive treatments / controls.**

Throughout this section we assume perfect knowledge of m, i.e., $p=m$, and we will write τ_m and $\hat{\tau}_m$ to stand for τ_p and $\hat{\tau}_p$, respectively. When $m \neq p$, our inference methods are still valid. See Section 5 for a discussion, and see Section 6.5 for a numerical example.

4.1. Exact Inference

We propose an exact non-parametric test for the sharp null of no causal effect at every time point

$$H_0 : Y_t(\mathbf{1}_{m+1}) - Y_t(\mathbf{0}_{m+1}) = 0 \quad \text{for all } t = 1, 2, ..., T. \tag{9}$$

This will be tested against a portmanteau alternative. The sharp null hypothesis implies that $Y_t(\boldsymbol{w}_{t-m:t}^{\text{obs}}) = Y_t(\boldsymbol{w}'_{t-m:t})$ for all $\boldsymbol{w}'_{t-m:t} \in \{0,1\}^t$; that is, regardless of the assignment path $\boldsymbol{w}'_{t-m:t}$ we would have observed the same outcomes.

We can conduct exact tests by using the known assignment mechanism to simulate new assignment paths. Algorithm 1 provides the details of how to implement it. In particular, under the sharp null hypothesis of no treatment effect (9), any assignment path $\boldsymbol{w}_{1:T}^{[i]}$ leads to the same observed outcomes. So in Step 3, we always plug in the same observed outcomes $Y_{p+1:T}^{\text{obs}}$. To obtain a confidence interval, we propose inverting a sequence of exact hypothesis tests to identify the region outside of which (9) is violated at the prespecified nominal level (Imbens and Rubin 2015, Chapter 5).

Algorithm 1 Algorithm for performing a sharp-null hypothesis test

```
Require: Fix $I$, total number of samples drawn.
for i in $1:I$ do
    Sample a new assignment path $\boldsymbol{w}_{1:T}^{[i]}$ according to the assignment mechanism.
    Hold $Y_{p+1:T}^{\mathsf{obs}}$ unchanged. Compute $\hat{\tau}^{[i]}$ according to (4),
        $\hat{\tau}^{[i]} = \frac{1}{T-m} \sum_{t=m+1}^{T} \left\{ Y_t^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{w}_{t-m:t}^{[i]} = \mathbf{1}_{m+1}\}}{\Pr(\boldsymbol{W}_{t-m:t} = \mathbf{1}_{m+1})} - Y_t^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{w}_{t-m:t}^{[i]} = \mathbf{0}_{m+1}\}}{\Pr(\boldsymbol{W}_{t-m:t} = \mathbf{0}_{m+1})} \right\}.$
end for
Compute $\hat{p}_{\mathsf{F}} = I^{-1} \sum_{i=1}^{I} \mathbb{1}\left\{ \left| \hat{\tau}^{[i]} \right| > |\hat{\tau}| \right\}$
return $\hat{p}_{\mathsf{F}}$, the estimated $p$-value. For large $I$, this is exact.
```

4.2. Asymptotic Inference

Now we introduce the asymptotic inference method, which tests the following null hypothesis

$$H_0 : \tau_m = \frac{1}{T-m} \sum_{t=m+1}^{T} \left[Y_t(\mathbf{1}_{m+1}) - Y_t(\mathbf{0}_{m+1}) \right] = 0. \tag{10}$$

The testing of such null hypothesis is based on the Horvitz-Thompson estimator in (4) being asymptotically Gaussian. Below we introduce how to conduct asymptotic inference.

Assume $n = T/, \geq 4$ is an integer, then under the optimal design as shown in Theorem 3, the assignment path is determined by the realizations at $W_1, W_{2m+1}, ..., W_{(n-2)m+1}$. To make the dependence on randomization clear, denote the following notations. Let $\mathbb{N}_0$ be the set of all non-negative integers. For any $k \in \mathbb{N}_0$, let $\bar{Y}_k(\mathbf{1}_{m+1}) = \sum_{t=(k+1)m+1}^{(k+2)m} Y_t(\mathbf{1}_{m+1})$ and $\bar{Y}_k(\mathbf{0}_{m+1}) = \sum_{t=(k+1)m+1}^{(k+2)m} Y_t(\mathbf{0}_{m+1})$. Moreover, let $\bar{Y}_k^{\mathsf{obs}} = \sum_{t=(k+1)m+1}^{(k+2)m} Y_t^{\mathsf{obs}}$ be the summation of observed outcomes.

Theorem 4 **(Variance of the Horvitz-Thompson Estimator)**. *Under Assumptions 1–3, if $n = T/m \geq 4$ is an integer, then under the optimal design as shown in Theorem 3, the variance of the Horvitz-Thompson estimator, $\mathsf{Var}(\hat{\tau}_m)$, is*

$$\begin{aligned} \mathsf{Var}(\hat{\tau}_m) = \frac{1}{(T-m)^2} \Bigg\{ & \bar{Y}_0(\mathbf{1}_{m+1})^2 + \bar{Y}_0(\mathbf{0}_{m+1})^2 + 2\bar{Y}_0(\mathbf{1}_{m+1})\bar{Y}_0(\mathbf{0}_{m+1}) \\ & + \sum_{k=1}^{n-3} \left[3\bar{Y}_k(\mathbf{1}_{m+1})^2 + 3\bar{Y}_k(\mathbf{0}_{m+1})^2 + 2\bar{Y}_k(\mathbf{1}_{m+1})\bar{Y}_k(\mathbf{0}_{m+1}) \right] \\ & + \bar{Y}_{n-2}(\mathbf{1}_{m+1})^2 + \bar{Y}_{n-2}(\mathbf{0}_{m+1})^2 + 2\bar{Y}_{n-2}(\mathbf{1}_{m+1})\bar{Y}_{n-2}(\mathbf{0}_{m+1}) \\ & + \sum_{k=0}^{n-3} 2 \left[\bar{Y}_k(\mathbf{1}_{m+1}) + \bar{Y}_k(\mathbf{0}_{m+1}) \right] \cdot \left[\bar{Y}_{k+1}(\mathbf{1}_{m+1}) + \bar{Y}_{k+1}(\mathbf{0}_{m+1}) \right] \Bigg\} \end{aligned} \tag{11}$$

Theorem 4 provides the variance of the estimator. Since we never observe all the potential outcomes, many of the cross-product terms from the variance can never be estimated. As an alternative, we provide the following two upper bounds, and detail two unbiased estimators of such two upper bounds, respectively.

Corollary 1. *Under the conditions in Theorem 4, there exist two upper bounds for the variance of the Horvitz-Thompson estimator,* $\mathsf{Var}(\hat{\tau}_m) \leq \mathsf{Var}^{\mathsf{U1}}(\hat{\tau}_m) \leq \mathsf{Var}^{\mathsf{U2}}(\hat{\tau}_m)$. *These two upper bounds* $\mathsf{Var}^{\mathsf{U1}}(\hat{\tau}_m)$ *and* $\mathsf{Var}^{\mathsf{U2}}(\hat{\tau}_m)$ *can be estimated by* $\hat{\sigma}^2_{\mathsf{U1}}$ *and* $\hat{\sigma}^2_{\mathsf{U2}}$*, respectively, where*

$$\hat{\sigma}^2_{\mathsf{U1}} = \frac{1}{(T-m)^2}\left\{6(\bar{Y}^{\mathsf{obs}}_0)^2 + \sum_{k=1}^{n-3} 24(\bar{Y}^{\mathsf{obs}}_k)^2 \mathbb{1}\{W_{km+1} = W_{(k+1)m+1}\} + 6(\bar{Y}^{\mathsf{obs}}_{n-2})^2 \right.$$
$$\left. + \sum_{k=0}^{n-3} 16\bar{Y}^{\mathsf{obs}}_k \bar{Y}^{\mathsf{obs}}_{k+1} \mathbb{1}\{W_{km+1} = W_{(k+1)m+1} = W_{(k+2)m+1}\}\right\},$$

and

$$\hat{\sigma}^2_{\mathsf{U2}} = \frac{1}{(T-m)^2}\left\{8(\bar{Y}^{\mathsf{obs}}_0)^2 + \sum_{k=1}^{n-3} 32(\bar{Y}^{\mathsf{obs}}_k)^2 \mathbb{1}\{W_{km+1} = W_{(k+1)m+1}\} + 8(\bar{Y}^{\mathsf{obs}}_{n-2})^2\right\}.$$

Moreover, $\hat{\sigma}^2_{\mathsf{U1}}$ *and* $\hat{\sigma}^2_{\mathsf{U2}}$ *are unbiased, i.e.,* $\mathbb{E}[\hat{\sigma}^2_{\mathsf{U1}}] = \mathsf{Var}^{\mathsf{U1}}(\hat{\tau}_m)$*, and* $\mathbb{E}[\hat{\sigma}^2_{\mathsf{U2}}] = \mathsf{Var}^{\mathsf{U2}}(\hat{\tau}_m)$.

Corollary 1 provides the foundation to make conservative inference. We make the following technical assumption for the asymptotic normal distribution to hold.

Assumption 5 **(Non-negligible Variance)**. *Assume that the randomization distribution has a non-negligible variance, i.e.,*

$$\mathsf{Var}(\hat{\tau}_m) \geq \Omega(n^{-1}) \tag{12}$$

In particular, one sufficient condition for (12) *is to assume that all the potential outcomes are positive, i.e., there exists some constant* $b > 0$*, such that* $\forall t \in [T], \forall \boldsymbol{w}_{1:T} \in \{0,1\}^T$, $Y_t(\boldsymbol{w}_{1:T}) \geq b$.

Intuitively, the key to Central Limit Theorem is that all the variables roughly have the same order of variance, i.e., there cannot be a small number of variables such that their variances make the majority of the sum. Assumption 5 suggests that the variance is large enough, such that it cannot come from only a few of the time periods.

Theorem 5 **(Asymptotic Normality)**. *Let* m *be fixed. For any* $n \geq 4 \in \mathbb{N}$*, define an* n*-replica experiment such that there are* $T = nm$ *time periods. We take the optimal design as in Theorem 3 whose randomization points are at* $\mathbb{T}^* = \{1, 2m+1, 3m+1, ..., (n-2)m+1\}$. *Under Assumptions 2–3, and under Assumption 5, the limiting distribution of the Horvitz-Thompson estimator in the* *n-replica experiment has an asymptotic normal distribution. That is, let* $\mathsf{Var}(\hat{\tau}_m)$ *be defined in Theorem 4. As* $n \to +\infty$,

$$\frac{\hat{\tau}_m - \tau_m}{\sqrt{\mathsf{Var}(\hat{\tau}_m)}} \xrightarrow{D} \mathcal{N}(0,1).$$

In particular, Theorem 5 does not require $\mathsf{Var}(\hat{\tau}_m)$ to converge as $n \to +\infty$.

To conduct inference, we replace $\mathsf{Var}(\hat{\tau}_m)$ by $\hat{\sigma}^2$, one of the two bounds provided in Corollary 1. Define the test statistic to be $z = |\hat{\tau}_m| / \sqrt{\hat{\sigma}^2}$. When the alternative hypothesis is two-sided, the estimated p-value is given by $\hat{p}_{\mathsf{N}} = 2 - 2\Phi(z)$, where Φ is the CDF of a standard normal distribution.

The proofs of Theorem 4, Corollary 1, and Theorem 5 are deferred to Sections EC.5.2, EC.5.3, and EC.5.4 in the Appendix, respectively.

5. A Discussion about Misspecified m

So far we have discussed cases when m is correctly specified. When m is misspecified, the estimation and inference are still valid and meaningful. We will detail below cases when m is either overestimated $m < p$, or underestimated $m > p$.

5.1. Causal Effects

When m is overestimated, $m < p$. Due to Assumption 3, $Y_t(\mathbf{1}_{p+1}) = Y_t(\mathbf{1}_{m+1}), \forall t \in \{p+1 : T\}$, so the lag-$p$ causal effect is essentially the lag-m causal effect.

When m is underestimated, $m > p$. The lag-p effect in (1) is not well defined. Instead, we define the m-misspecified lag-p causal effect that pads the $p+1$ assignments with the earlier observed treatments.

$$\tau_p^{(m)}(\mathbb{Y}) = \frac{1}{T-p} \left\{ \sum_{t=p+1}^{m} \left[Y_t(\boldsymbol{w}_{1:t-p-1}^{\mathsf{obs}}, \mathbf{1}_{p+1}) - Y_t(\boldsymbol{w}_{1:t-p-1}^{\mathsf{obs}}, \mathbf{0}_{p+1}) \right] + \sum_{t=m+1}^{T} \left[Y_t(\boldsymbol{w}_{t-m:t-p-1}^{\mathsf{obs}}, \mathbf{1}_{p+1}) - Y_t(\boldsymbol{w}_{t-m:t-p-1}^{\mathsf{obs}}, \mathbf{0}_{p+1}) \right] \right\}. \quad (13)$$

This is a special case of the weighted lag-p causal effect introduced in Bojinov and Shephard (2019). Similarly to the average lag-p causal effect, $\tau_p^{(m)}(\mathbb{Y})$ captures how administering $p+1$ consecutive treatments as opposed to $p+1$ consecutive controls impact the outcomes at time t, conditional on the observed assignment path up to time $t-p-1$. See Section 6.5 for numerical results.

5.2. Estimator

When m is overestimated $(m < p)$ Theorem 1 still holds, i.e., $\mathbb{E}[\hat{\tau}_p] = \tau_p(\mathbb{Y}) = \tau_m(\mathbb{Y})$. When m is underestimated $(m > p)$, sometimes we have to slightly augment the results and study the conditional expectation.

Define $f_{\mathbb{T}} : [T] \rightarrow \mathbb{T}$ to be the "determining randomization point of period t,"

$$f_{\mathbb{T}}(t) = \max\{j \,|\, j \in \mathbb{T}, j \leq t\}$$

such that, it is the realization at time $f_{\mathbb{T}}(t)$ that uniquely determines the assignment at time t, *i.e.* $W_t = W_{f_{\mathbb{T}}(t)}, \forall t \in [T]$. See Example 10 for an illustration of $f_{\mathbb{T}}(\cdot)$. When $\mathbb{T}$ is clear from the context we drop the subscript and use $f(\cdot) = f_{\mathbb{T}}(\cdot)$. Depending on if $f(t-p) \leq t-m$, we establish an analogy of Theorem 1 for the m-underestimated case.

Theorem 6 **(Unbiasedness of the Estimator when m is Misspecified)**. *Under Assumptions 2 and 3, for $m > p$, at each time $t \geq m+1$, the Horvitz-Thompson estimator is either unbiased for the lag-m causal effect when $f(t-p) \leq t-m$, or conditionally unbiased for the m-misspecified lag-p causal effect when $f(t-p) > t-m$. When $p+1 \leq t \leq m$, the Horvitz-Thompson estimator is either unbiased for the lag-t causal effect when $f(t-p) = 1$, or conditionally unbiased for the m-misspecified lag-t causal effect when $f(t-p) > 1$.*

To remove the conditional expectation, we can further take an outer loop of expectation averaged over the past assignment paths. Although this is somewhat different from the average lag-p effect introduced earlier in (1), it does capture the impact of a sequence of treatment relative to a sequence of controls.

All the mathematical expressions of Theorem 6, as well its proof, are all stated in Section EC.6.1 in the Appendix. See Example 10 below for a specific illustration of Theorem 6. For a numerical illustration of the estimand and estimator in more general setups, see Section 6.5.

Example 10 (Misspecified m). Suppose $T = 4, m = 2, p = 1, \mathbb{T} = \{1, 3\}$. Then the determining randomization points are $f_{\mathbb{T}}(1) = 1, f_{\mathbb{T}}(2) = 1, f_{\mathbb{T}}(3) = 3, f_{\mathbb{T}}(4) = 3$, and

$$\mathbb{E}\left[Y_2^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{W}_{1:2} = (1,1)\}}{\Pr(\boldsymbol{W}_{1:2} = (1,1))} - Y_2^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{W}_{1:2} = (0,0)\}}{\Pr(\boldsymbol{W}_{1:2} = (0,0))}\right] = Y_2(1,1) - Y_2(0,0)$$

$$\mathbb{E}\left[Y_3^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{W}_{2:3} = (1,1)\}}{\Pr(\boldsymbol{W}_{2:3} = (1,1))} - Y_3^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{W}_{2:3} = (0,0)\}}{\Pr(\boldsymbol{W}_{2:3} = (0,0))}\right] = Y_3(1,1,1) - Y_3(0,0,0)$$

$$\mathbb{E}\left[Y_4^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{W}_{3:4} = (1,1)\}}{\Pr(\boldsymbol{W}_{3:4} = (1,1))} - Y_4^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{W}_{3:4} = (0,0)\}}{\Pr(\boldsymbol{W}_{3:4} = (0,0))}\right] = \frac{1}{2}\left[Y_4(1,1,1) + Y_4(0,1,1) - Y_4(0,0,0) - Y_4(1,0,0)\right]$$

Note that this is the 2-misspecified lag-1 causal effect. □

5.3. Inference

The exact inference procedure as in Section 4.1 remains valid when m is misspecified. For the asymptotic inference procedure as in Section 4.2, Theorem 5 still holds when m is misspecified, as we state in Corollary 2. The proof is deferred to Section EC.6.2 in the Appendix.

Corollary 2 **(Asymptotic Normality when m is Misspecified)**. *For any $n \geq 4 \in \mathbb{N}$, define an n-replica experiment such that there are $T = np$ time periods. Take the optimal design as in Theorem 3 whose randomization points are at $\mathbb{T}^* = \{1, 2p+1, 3p+1, ..., (n-2)p+1\}$. We have the following two observations.*

i When $m < p$, under Assumptions 2–3, the variance of the Horvitz-Thompson estimator, $\mathsf{Var}(\hat{\tau}_p)$, is explicitly given by (11).

ii Furthermore, no matter if $m < p$ or $m > p$, under Assumptions 1–3 and assume $\mathsf{Var}(\hat{\tau}_p) \geq \Omega(n^{-1})$, the limiting distribution of the Horvitz-Thompson estimator in the n-replica experiment has an asymptotic normal distribution. That is, as $n \to +\infty$,

$$\frac{\hat{\tau}_p - \tau_p}{\sqrt{\mathsf{Var}(\hat{\tau}_p)}} \xrightarrow{D} \mathcal{N}(0,1).$$

Corollary 2, together with Theorem 5, is the key to identify m, the order of the carryover effect. In Section 7.1, we will provide methods to identify m.

5.4. Robustness to Misspecifications

So far we have discussed estimation and inference when the order of carryover effects are misspecified. We conclude with a short discussion on the robustness of our method.

First, the optimal design between $m = 0$ and $m = 1$ are almost the same. This suggests that when there is very little carryover effect, our proposed optimal design is robust. Second, as we will see in Section 6, when the order of carryover effect is slightly overestimated the variance is only a little larger. This adds an extra layer of robustness that a slight misspecification is often acceptable.

6. Simulation Study and Empirical Illustration

There are 5 goals for this simulation study. First, to illustrate how to conduct a switchback experiment for various outcome models. Second, to show that our proposed optimal design has the smallest risk, compared with two benchmarks. There are two dimensions for our comparison: the worst-case risk and the risk under a specific outcome model. Third, to verify the asymptotic normal distribution under a non-asymptotic setup, and to study the quality of the upper bound proposed in Corollary 1. Fourth, to understand the rejection rate and its dependence on the length of time horizon. Fifth, under randomly generated cases, to study the performance of the optimal design under a misspecified m, and to compare the difference of the two inference methods proposed in Section 4.

6.1. Outcome Models

The potential outcome framework is flexible. As we will see below, it is easy to use the potential outcome framework to describe many complex relationships between assignments and outcomes.

We start with a simple model which originates from Oman and Seiden (1988):

$$Y_t(\boldsymbol{w}_{1:t}) = \mu + \alpha_t + \delta w_t + \gamma w_{t-1} + \epsilon_t \tag{14}$$

where μ is a fixed effect; α_t is a fixed effect associated to period t; δw_t is the contemporaneous effect, and γw_{t-1} is the carryover effect from period $t-1$; ϵ_t is the random noise in period t. Such a model as well as a few very similar ones are widely used in the literature (Hedayat et al. 1978, Jones and Kenward 2014).

A more general variant from the above model is to consider carryover effects of any arbitrary order:

$$Y_t(\boldsymbol{w}_{1:t}) = \mu + \alpha_t + \delta^{(1)} w_t + \delta^{(2)} w_{t-1} + ... + \delta^{(t)} w_1 + \epsilon_t \tag{15}$$

where $\delta^{(1)}, \delta^{(2)}, ..., \delta^{(t)}$ are non-stochastic coefficients. The dotted terms are carryover effects of higher orders. And all the other parameters are as defined in (14). We will run simulations based on this more general model, which enables us to test the performance of our proposed optimal design under a misspecified m.

The autoregressive model (Arellano 2003) is even more general: $Y_1(w_1) = \delta_{1,1} w_1 + \epsilon_1$ and $\forall t > 1$

$$\begin{aligned} Y_t(\boldsymbol{w}_{1:t}) = \phi_{t,t-1} Y_{t-1}(\boldsymbol{w}_{1:t-1}) + \phi_{t,t-2} Y_{t-2}(\boldsymbol{w}_{1:t-2}) + ... + \phi_{t,1} Y_1(w_1) + \\ \delta_{t,t} w_t + \delta_{t,t-1} w_{t-1} + ... + \delta_{t,1} w_1 + \epsilon_t \end{aligned} \tag{16}$$

where $\phi_{t,\tilde{t}}$ and $\delta_{t,\tilde{t}}$ are non-stochastic coefficients; the dotted terms are carryover effects of higher orders; ϵ_t is the random noise in period t. We can iteratively replace $Y_t(w_t)$ using a linear combination of $w_t, w_{t-1}, ..., w_1$. So the autoregressive model in (16) can be written in a similar form of (15). The only difference is that the coefficients are different and dependent on t.

For all these models, we first decide what is the order of carryover effects, namely m. Then we use Theorem 3 to find the optimal design of experiment. Finally, we use the exact randomization test in Section 4 to conduct hypothesis test.

6.2. Comparison of the Risk Functions

6.2.1. Simulation setup. We consider two setups. The first setup is for the worst-case risk. We consider $T = 120, p = m = 2$ where m is correctly identified, and $Y_t(\mathbf{1}_3) = Y_t(\mathbf{0}_3) = 10$. We compare three different designs of switchback experiments. The first one is our proposed optimal design as in Theorem 3, such that $\mathbb{T}^* = \{1, 5, 7, ..., 117\}$. The second one is the most common and naive switchback experiment, which independently assign treatment/control in every period with half-half probability. It is parameterized by $\mathbb{T}^{\mathsf{H1}} = \{1, 2, 3, ..., 120\}$. The third one is the "intuitive" experiment discussed in Example 9, which divides the time horizon into several epochs each with length $m+1=3$. It is parameterized by $\mathbb{T}^{\mathsf{H2}}=\{1,4,7,...,118\}$.

Second, we run simulations based on the outcome model as in (15). We consider $T=120, p=m=2$ where m is correctly identified. For the outcome model, we consider $\mu=0$, $\alpha_t=\log(t)$, and $\epsilon_t \sim N(0,1)$ are i.i.d. standard normal distributions. For any $t>3$, let $\delta^{(t)}=0$. We will vary the values of $\delta^{(1)}, \delta^{(2)}, \delta^{(3)} \in \{1,2\}$ and conduct experiments under $2^3=8$ different scenarios. Again we compare the same three different designs of switchback experiments. $\mathbb{T}^*=\{1,5,7,...,117\}, \mathbb{T}^{\mathsf{H1}}=\{1,2,3,...,120\}, \mathbb{T}^{\mathsf{H2}}=\{1,4,7,...,118\}$.

We simulate one assignment path at a time, and conduct experiment following this assignment path. Since the outcome model is prescribed, we can calculate both the causal estimand and and the observed outcomes (along the simulated assignment path). Then we calculate the Horvitz-Thompson estimator based on the simulated assignment path and the simulated observed outcomes. With both the estimand and estimator, we can calculate the loss function. By repeating the above procedure enough (in this simulation, 100000) times we approximately have the risk function.

6.2.2. Simulation results. We calculate the worst-case risk functions via simulation. Notice that even though we could calculate the worst-case risk function explicitly via Theorem 2, we still run the simulation to confirm this result. See Table 2 for results.

The causal effect is $\tau_2=0$ because $Y_t(\mathbf{1}_3)=Y_t(\mathbf{0}_3)=10$. The simulated estimator is $\mathbb{E}[\hat{\tau}_2^*]=-0.0291$ for our proposed optimal design, and $\mathbb{E}[\hat{\tau}_2^{\mathsf{H1}}]=0.0104$ and $\mathbb{E}[\hat{\tau}_2^{\mathsf{H2}}]=-0.0478$ for the two benchmarks, respectively. It is worth noting that such estimation errors are quite small for all three designs, because the reported numbers are for total effects, not average effects. If we further divide them by $T-p$ then the errors are very clode to zero. The risk function is $r(\eta_{\mathbb{T}^*})=26.78$ for our proposed optimal design, and $r(\eta_{\mathbb{T}^{\mathsf{H1}}})=33.67$ and $r(\eta_{\mathbb{T}^{\mathsf{H1}}})=27.85$ for the two benchmarks, respectively. Such simulation results suggest that our proposed optimal design have the smallest risk, under the worst case outcome model.

Table 2 Simulation results for the worst-case risk function.

τ_2	$\mathbb{E}[\hat{\tau}_2^*]$	$\mathbb{E}[\hat{\tau}_2^{\mathsf{H1}}]$	$\mathbb{E}[\hat{\tau}_2^{\mathsf{H2}}]$	$r(\eta_{\mathbb{T}^*})$	$r(\eta_{\mathbb{T}^{\mathsf{H1}}})$	$r(\eta_{\mathbb{T}^{\mathsf{H2}}})$
0	0.0250	0.0200	0.0059	26.78	33.67	27.85

The optimal design $\mathbb{T}^*$ as suggested in Theorem 3 yields the smallest risk.

We also calculate the risk functions based on the outcome model in (15). See Table 3. As we vary the values of $\delta^{(1)}$, $\delta^{(2)}$ and $\delta^{(3)}$, the total lag-2 causal effect is being changed. All three estimators are able to reflect the change as the estimand changes. The risk function can be simulated and we see that the risk function associated with the first benchmark $\mathbb{T}^{\mathsf{H1}}$ is $28\% \sim 32\%$ larger than the optimal design; and the second benchmark $\mathbb{T}^{\mathsf{H2}}$ is $1\% \sim 2\%$ larger. Such simulation results suggest

Table 3 **Simulation results for the risk function based on the outcome model in** (15).

$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$	τ_2	$\mathbb{E}[\hat{\tau}_2^*]$	$\mathbb{E}[\hat{\tau}_2^{\mathsf{H1}}]$	$\mathbb{E}[\hat{\tau}_2^{\mathsf{H2}}]$	$r(\eta_{\mathbb{T}^*})$	$r(\eta_{\mathbb{T}^{\mathsf{H1}}})$	$r(\eta_{\mathbb{T}^{\mathsf{H2}}})$
1	1	1	3	3.016	3.012	3.002	7.96	10.22	8.11
1	1	2	4	4.018	4.013	4.002	9.57	12.39	9.74
1	2	1	4	4.018	4.013	4.002	9.57	12.39	9.74
2	1	1	4	4.018	4.013	4.002	9.57	12.39	9.74
1	2	2	5	5.020	5.015	5.003	11.34	14.81	11.52
2	1	2	5	5.020	5.015	5.003	11.34	14.81	11.52
2	2	1	5	5.020	5.015	5.003	11.34	14.81	11.52
2	2	2	6	6.022	6.016	6.003	13.28	17.48	13.47

For each row, the random seed that generates the simulation setup is fixed. The optimal design $\mathbb{T}^*$ as suggested in Theorem 3, though solved from a minimax program, still yields the smallest risk for the outcome model in (15).

again that our proposed optimal design have the smallest risk. Moreover, based on the fact that $r(\eta_{\mathbb{T}^{\mathsf{H2}}})$ is rather close to $r(\eta_{\mathbb{T}^*})$ and much smaller than $r(\eta_{\mathbb{T}^{\mathsf{H1}}})$, we suggest that a slight overestimate of m is more desirable than an underestimate.

As the magnitude of treatment effects increase, the associated risk functions also increase. The relative difference between risk functions of $r(\eta_{\mathbb{T}^{\mathsf{H1}}})$ and $r(\eta_{\mathbb{T}^*})$ increases, while the relative difference between $r(\eta_{\mathbb{T}^{\mathsf{H1}}})$ and $r(\eta_{\mathbb{T}^*})$ decreases. This coincides with the intuitions discussed in Section 3.

6.3. Asymptotic Normality

6.3.1. Simulation setup. We run simulations based on the outcome model as in (15). We consider $T = 120, m = 2$. We will consider three cases: (i) m is correctly specified so $p = 2$; (ii) m is overestimated to be 3 so $p = 3$, and we estimate lag-3 causal estimand as in (1); (iii) m is underestimated to be 1 so $p = 1$, and we pretend as if we estimated the lag-1 causal estimand. However, the lag-1 causal estimand is not well defined – and we instead estimate the 2-misspecified lag-1 causal estimand as in (13).

For the outcome model, we consider $\mu = 0$, $\alpha_t = \log(t)$, and $\epsilon_t \sim N(0, 1)$ are i.i.d. standard normal distributions. For any $t > 3$, let $\delta^{(t)} = 0$. For simplicity, let $\delta^{(1)} = \delta^{(2)} = \delta^{(3)} = \delta$. We vary $\delta \in \{1, 2, 3\}$ and conduct experiments under 3 different scenarios.

We simulate one assignment path at a time, and conduct experiment following this assignment path. Since the outcome model is prescribed, we calculate the observed outcomes based on the simulated assignment path. Then we calculate the Horvitz-Thompson estimator, and two conservative estimators of the randomization variance (Corollary 1), both based on the simulated assignment path and the simulated observed outcomes. On the other hand, the lag-p causal estimand is easy to calculate once the outcome model is prescribed. Yet the m-misspecified lag-p causal estimand has to be calculated in conjunction with the simulated assignment path. By repeating the above procedure enough (in this simulation, 100000) times we obtain a distribution of the estimator, and we calculate the mean value of the estimator (and the m-misspecified lag-p causal estimand).

6.3.2. Simulation results. Figure 4 shows approximate normality of the randomization distribution, under all 9 cases. There are three specifications of m: correctly specified when $p=2$; overestimated when $p=3$; underestimated when $p=1$. There are three specifications of $\delta=1,2,3$. In all 9 cases, the distributions are all centered around the red vertical lines, which are the mean values of the randomization distributions. Specifically, Figures (a) – (c) validates Theorem 5, and Figures (d) – (i) validates Corollary 2.

For all the above cases, see Table 4 for the expected values and the variances of the randomization distributions, as well two conservative estimators of the randomization variances. Note that the three cases all have the same underlying outcome model. It is the different knowledge of m that leads to three different designs.

Table 4 **Simulation results for the randomization distribution.**

		τ_p	$\tau_p^{[m]}$	$\mathbb{E}[\hat{\tau}_p]$	$\mathsf{Var}(\hat{\tau}_p)$	$\mathbb{E}[\hat{\sigma}^2_{\mathsf{U1}}]$	$\mathbb{E}[\hat{\sigma}^2_{\mathsf{U2}}]$
correct m	$\delta=1$	3	–	3.016	7.96	8.50	8.48
	$\delta=2$	6	–	6.022	13.28	15.24	15.16
	$\delta=3$	9	–	9.028	20.10	24.40	24.25
overestimated m	$\delta=1$	3	–	3.006	11.92	12.77	12.67
	$\delta=2$	6	–	6.009	19.89	22.91	22.70
	$\delta=3$	9	–	9.012	30.10	36.69	36.32
underestimated m	$\delta=1$	–	2	2.016	4.00	4.13	4.13
	$\delta=2$	–	4	4.026	6.69	7.09	7.06
	$\delta=3$	–	6	6.037	10.14	11.01	10.92

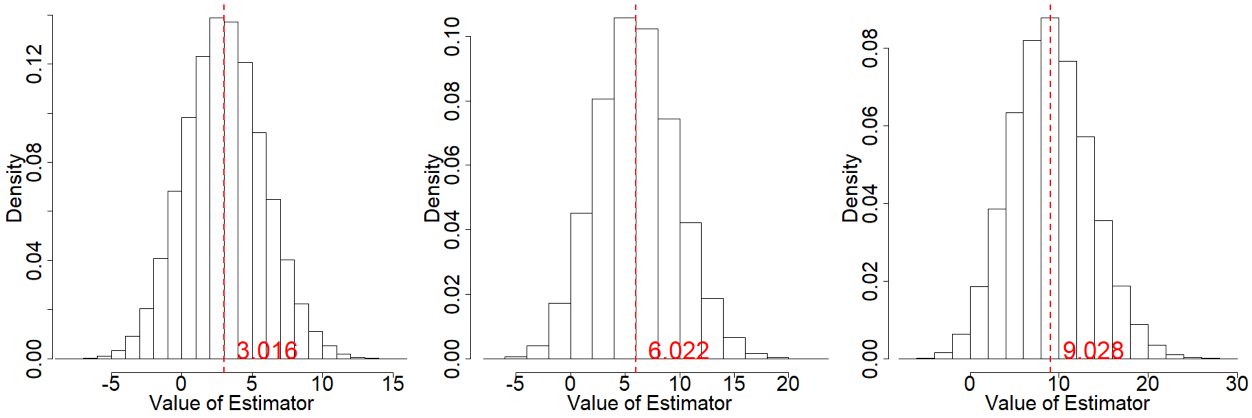
The randomization distributions in all 9 cases are unbiased. The conservative estimation of the variance upper bounds from Corollary 1 are close to the true variance.

From Table 4, we make the following two observations. **(i) Unbiasedness of the Horvitz-Thompson estimator**. When m is correctly specified, $\mathbb{R}[\hat{\tau}_p]$ is very close to τ_p, verifying the unbiasedness of the estimator. When m is overestimated, the estimand remains unchanged, and the estimator remains unbiased. But the variance of the estimator is larger. When m is underestimated, the estimand is the m-misspecified estimand, and the estimator is unbiased for this m-misspecified estimand.

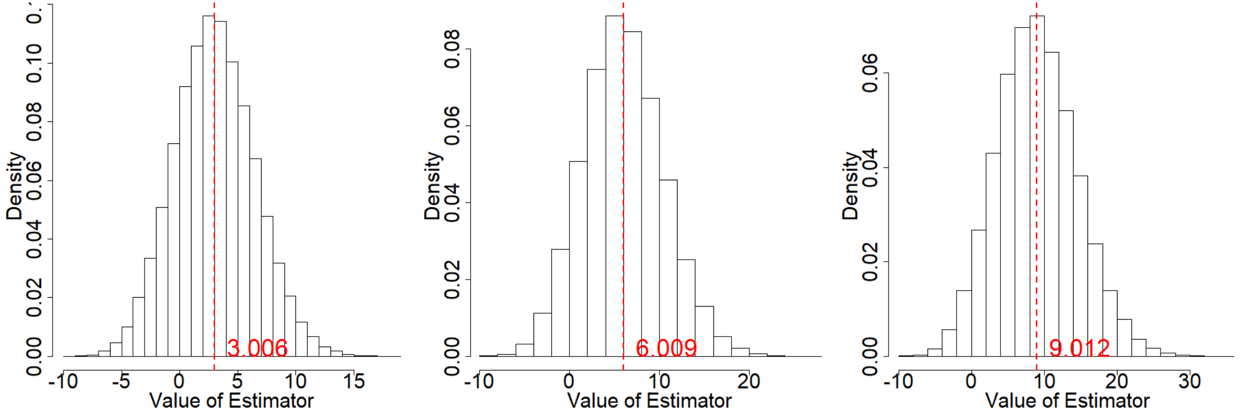
(ii) Quality of Corollary 1. As we increase δ, the variance of the randomization distribution also increases. The two conservative estimators of the randomization variance are very close to the true variance, which suggests that Corollary 1 approximates the true variance quite well. Even though the second upper bound $\mathsf{Var}^{\mathsf{U2}}(\hat{\tau}_p)$ is larger than the first one $\mathsf{Var}^{\mathsf{U1}}(\hat{\tau}_p)$, its estimator $\hat{\sigma}^2_{\mathsf{U2}}$ turns out to be smaller than $\hat{\sigma}^2_{\mathsf{U1}}$ in most cases.

6.4. Rejection Rates

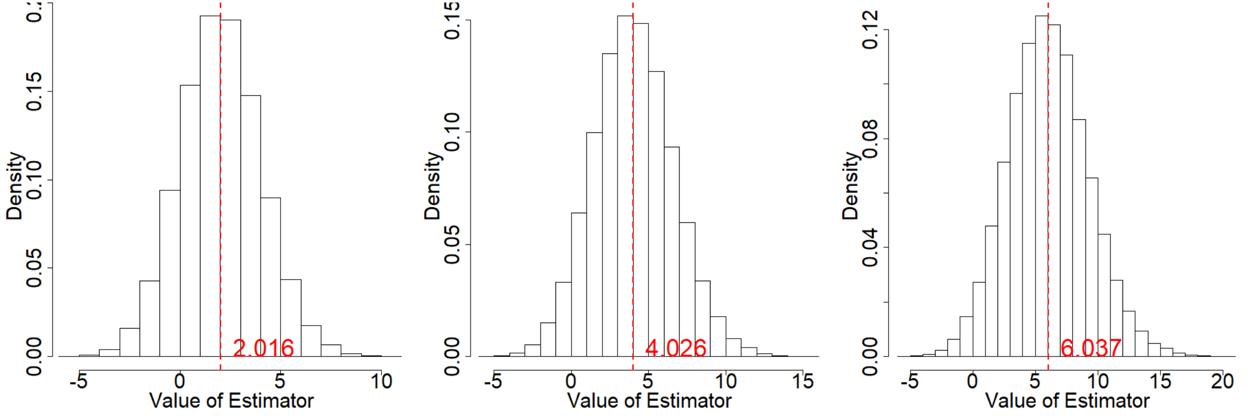
6.4.1. Simulation setup. We run simulations based on the outcome model as in (15). We very $T \in \{60, 120, 180, ..., 600\}$. We consider $p=m=2$ where m is correctly specified. Similar to



(a) m correctly specified, $\delta = 1$. (b) m correctly specified, $\delta = 2$. (c) m correctly specified, $\delta = 3$.



(d) m overestimated, $\delta = 1$. (e) m overestimated, $\delta = 2$. (f) m overestimated, $\delta = 3$.



(g) m underestimated, $\delta = 1$. (h) m underestimated, $\delta = 2$. (i) m underestimated, $\delta = 3$.

Figure 4 **Approximate normality of the randomization distributions in all 9 cases. The red vertical lines are the expected values of the randomization distributions.**

Section 6.3, we consider the same parameterization and conduct experiments under 3 different scenarios $\delta \in \{1, 2, 3\}$.

We simulate one assignment path at a time, and conduct experiment following this assignment path. We first calculate the observed outcomes and the Horvitz-Thompson estimator. Then we conduct the two inference methods as proposed in Section 4 (for the asymptotic inference method, we plug in the second upper bound $\hat{\sigma}^2_{\text{U}2}$) and obtain two estimated p-values. We reject the corresponding null hypothesis when the p-value is smaller than 0.1. By repeating the above procedure enough (in this simulation, 1000) times we obtain the frequency of a null hypothesis being rejected, which we refer to as the rejection rate.

6.4.2. Simulation Results. We calculate the rejection rates via simulation, and plot Figure 5. In all the simulations, $\delta \neq 0, \tau_p \neq 0$, so ideally we would wish to reject the null hypothesis (whether if it is (9) or (10)).

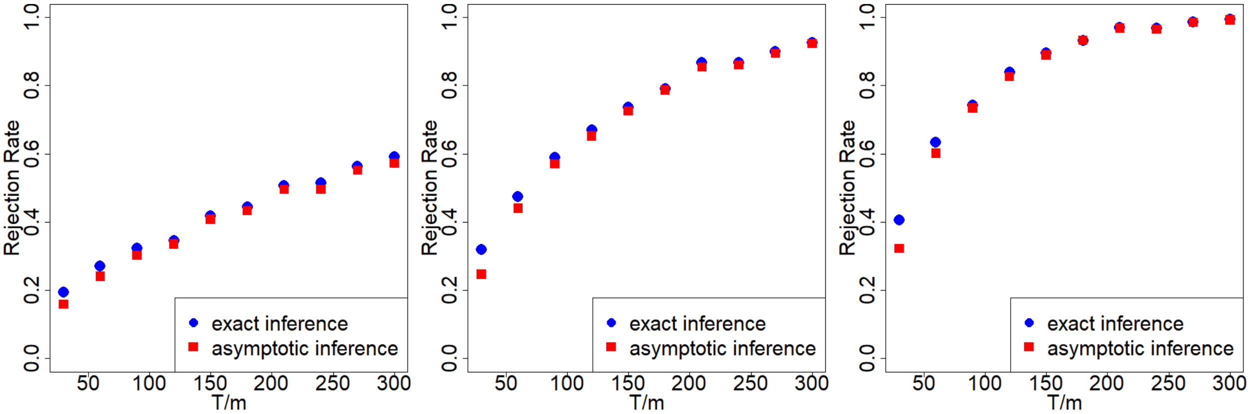


Figure 5 **Rejection rates and their dependence on T/m. The blue dots are rejection rates under exact inference; the red dots are under asymptotic inference. Left: $\delta = 1$; Middle: $\delta = 2$; Right: $\delta = 3$**

From Figure 5 we make the following three observations. **(i) Dependence on T/m.** The rejection rates increase as the length of the horizon increases – more specifically, as T/m the total number of epochs increases. In practice, when firms have choose the length of T and decide how much experimental budgets to allocate, they can refer to Figure 5 to choose T properly. Also see discussion in Section 7.2.

(ii) Between two inference methods. In all three cases, the rejection rate from testing a sharp null hypothesis (9) is slightly higher than that from testing the Neyman's null (10). This coincides with our intuition that a sharp null is more likely to be rejected. We discuss this in Section 6.5.2 together with the associated p-values.

(iii) Dependence on the signal-to-noise ratio. The rejection rates all increase as δ increases from 1 to 3 (while holding the noise from the model fixed). This suggests that when the treatment effect is relatively larger, we do not require a long experimental horizon to achieve a desired rejection rate.

6.5. Estimation under a Misspecified m

6.5.1. Simulation setup. We run simulations based on the outcome model as in (15). We consider $T = 120, m = 2$. We consider three cases: (i) m correctly specified $p = 2$; (ii) m is overestimated $p = 3$, and we estimate the lag-3 causal estimand as in (1); (iii) m is underestimated $p = 1$, and we pretend as if we estimated the lag-1 causal estimand. However, the lag-1 causal estimand is not well defined. Instead, we estimate the 2-misspecified lag-1 causal estimand as in (13).

For the outcome model, we consider the same parameterization as in Section 6.3, and conduct experiments under 3 different scenarios $\delta \in \{1, 2, 3\}$.

We only simulate one assignment path. Since the outcome model is prescribed, we calculate the observed outcomes. There is only one time series of such observed outcomes. We calculate the Horvitz-Thompson estimator based on the simulated assignment path and the simulated observed outcomes. We calculate the lag-p causal estimand directly, and also the m-misspecified lag-p causal estimand in conjunction with the simulated assignment path. Finally, we perform the two inference methods from Section 4, and report their associated estimated p-values. The conservative sampling variance we take is $\hat{\sigma}^2_{\mathsf{U2}}$. We choose $I = 100000$ to be the number of samples drawn in the exact inference method as shown in Algorithm 1.

6.5.2. Simulation results. Notice this is only one experiment under one simulated experimental setup from one simulated assignment path. So the estimators $\hat{\tau}_p$ we derive are different from τ_p. But they are still roughly following the true causal effects which they estimate. See Table 5.

Table 5 **Simulation results for correctly specified m, overestimated m, and underestimated m.**

		τ_p	$\tau_p^{[m]}$	$\hat{\tau}_p$	$\hat{\sigma}^2_{\mathsf{U2}}$	$\hat{p}_{\mathsf{F}}$	$\hat{p}_{\mathsf{N}}$
correct m	$\delta = 1$	3	–	1.35	8.81	0.626	0.648
	$\delta = 2$	6	–	4.30	15.16	0.231	0.269
	$\delta = 3$	9	–	7.25	23.88	0.101	0.138
overestimated m	$\delta = 1$	3	–	1.77	14.26	0.606	0.639
	$\delta = 2$	6	–	5.00	24.69	0.262	0.314
	$\delta = 3$	9	–	8.23	39.00	0.136	0.188
underestimated m	$\delta = 1$	–	2	-1.03	3.87	0.590	0.599
	$\delta = 2$	–	4	0.41	6.28	0.866	0.870
	$\delta = 3$	–	6	1.86	9.47	0.530	0.547

The simulation setup for the three $\delta = 1$ cases is the same; so are the $\delta = 2$ cases and $\delta = 3$ cases. The estimated p-values derived from the exact inference is slightly smaller than the p-values derived from the asymptotic inference.

From Table 5 we see that both our estimator and the estimated variance are well defined in all the cases when m is correctly specified, is overestimated, and is underestimated. In each case, as *delta* increases from 1 to 3, the associated p-values exhibit decreasing trends, suggesting a stronger rejection rate against the null hypothesis. Moreover, the p-values suggested by the exact inference is always slightly smaller than the p-values suggested by the asymptotic inference. This coincides with our intuition that: (i) the exact inference method possesses a stronger null hypothesis (9) which implies the null hypothesis of (10); (ii) in the asymptotic inference we replaced the true randomization variance by its conservative upper bound, which further leads to a larger p-value.

7. Practical Implications

We recap what would a manager do to practically run a switchback experiment. First, the granularity and the length of horizon are sometimes given to a manager; when the manager has more flexibility to choose the granularity and the length of horizon, see discussions in Section 7.2. Second, the manager either consults domain knowledge to decide what is the order of carryover effect, or when such knowledge is not perfect, run a first phase experiment to identify such an order. See discussions in Section 7.1. Third, the manager decides on a collection of randomization points, and draws one sample from the randomization distribution to be the assignment path. We recommend the optimal design as we discussed in Section 3. Finally, the manager collects data from the experiment, and draws causal conclusions using the methods in Section 4.

7.1. Identifying the Order of Carryover Effect

We borrow Theorem 5 and Corollary 2 to define a sub-routine, which, combined with searching method, identifies the order of carryover effect.

Suppose we have access to two i.i.d. experimental units. Such two experimental units could be two identical units. They can also be two time spans on one single experimental unit, such that the two spans are well separated and the carryover effect from one does not affect the outcomes of the other.

On the first experimental unit, we design an optimal experiment under $p = p_1$; while on the second unit, $p = p_2$. Without loss of generality let $p_1 < p_2$. We observe the outcomes, collect the data, and find the following statistics from the two experiments. In the first one, calculate $\hat{\tau}_{p_1}$, the sampling average, and $\hat{\sigma}^2_{p_1}$, the conservative sampling variance as suggested by Corollary 1. In the second one, calculate $\hat{\tau}_{p_2}$ and $\hat{\sigma}^2_{p_2}$.

Define a sub-routine that tests the following null hypothesis:

$$H_0: \ m \le p_1 \tag{17}$$

Under the null hypothesis (17), $\tau_{p_1} = \tau_{p_2}$. Furthermore, given that the two experimental units are independent, the difference between the two sample means should be a normal distribution centered around zero, i.e., $(\hat{\tau}_{p_1} - \hat{\tau}_{p_2})/\sqrt{\mathsf{Var}(\tau_{p_1}) + \mathsf{Var}(\tau_{p_2})} \xrightarrow{D} \mathcal{N}(0,1)$. To test the null hypothesis (17), define the test statistic to be $z = |\hat{\tau}_{p_1} - \hat{\tau}_{p_2}| / \sqrt{\hat{\sigma}^2_{p_1} + \hat{\sigma}^2_{p_2}}$. The estimated p-value is given by $\hat{p} = 2 - 2\Phi(z)$, where Φ is the CDF of a standard normal distribution.

Such a sub-routine enables us to test the null hypothesis (17). We can combine such a sub-routine with any searching method to identify m.

To take an example, suppose we are running an experiment whose setup is in Section 6.5, and that $\delta = 3$. Suppose we have narrowed down the range of the order of carryover effect to be $m \le 3$.

In the first round, we consult the sub-routine to test a null hypothesis $m \leq 2$. Then we would observe row 3 and 6 from Table 5, with $\hat{\tau}_2 = 7.25, \hat{\sigma}_2^2 = 23.88; \hat{\tau}_3 = 8.23, \hat{\sigma}_3^2 = 39.00$. So the estimated p-value for the null hypothesis $m \leq 2$ is estimated to be $\hat{p} = 0.902$, which is quite big to reject the null hypothesis. In the second round, we consult the sub-routine to test a null hypothesis $m \leq 1$. Then we would observe row 3 and 9 from Table 5, with $\hat{\tau}_1 = 1.86, \hat{\sigma}_3^2 = 9.47; \hat{\tau}_2 = 7.25, \hat{\sigma}_2^2 = 23.88$. The estimated p-value for the null hypothesis $m \leq 1$ is estimated to be $\hat{p} = 0.350$. This is still rather big, yet a significant difference from the previous 0.902. In practice, we suggest to increase the length of the horizon to a degree such that $T/p \approx 100$.

7.2. Choosing the Granularity and the Length of Horizon

The selection of T depends on the following two components: first, the granularity of one single time period; and second, the number of total epochs T/p included in the time horizon.

The granularity of each time period refers to how long in the physical world a single time period corresponds to. As long as each time period is granular enough such that its length is smaller than the length of the carryover effect, and that the length of the carryover effect is some multiple times of the length of one time unit, then the selection of granularity makes no difference to the optimal design and analysis of switchback experiments. See Example 11.

Example 11 (Two Granularity Levels). In the ride-sharing application, suppose the firms have two options to either treat one single time period as 15 minutes, or as 30 minutes. See Figure 6. Each smallest time period stands for 15 minutes, and the carryover effect lasts for 1 hour. In the time granularity as shown in blue, each time period lasts for 15 minutes, and the carryover effect lasts for $m = 4$ time periods. In the time granularity as shown in red, each time period lasts for 30 minutes, and the carryover effect lasts for $m = 2$ time periods.

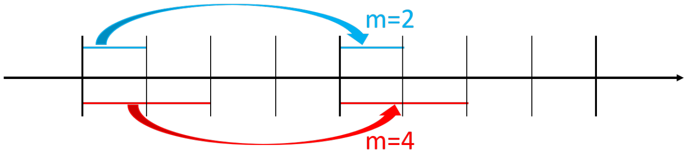


Figure 6 Illustrator of two granularity levels. Blue: each period is 15 minutes; Red: each period is 30 minutes.

From Theorem 3, the optimal design exhibits an optimal structure that randomizes once every m time periods (except for the first and last epoch, which lasts for $2m$ time periods each). In both cases, the optimal design would randomize once every 1 hour. Furthermore, from Theorem 1 we know that under both cases the mean value of the Horvitz-Thompson estimator remains unchanged. From Theorem 4, the variance consists of terms like $\bar{Y}_k(\mathbf{1}_{m+1}) = \sum_{t=(k+1)m+1}^{(k+2)m} Y_t(\mathbf{1}_{m+1})$, which are the sum across all the outcomes within 1 hour. So under both cases the variance of the estimator remains unchanged. $\square$

On the other hand, when the length of each time period is longer than the length of the carryover effect, this will cause unnecessary overestimation of the length of the carryover effect. In an extreme case, the length of carryover effect is 1 minute while each period is selected to be 1 hour. Then the minimum order of carryover effect would be 1, which overestimates the length of the carryover effect to be 1 hour. So we suggest that the length of each time period should be smaller than the length of the carryover effect.

Second, after we have decided about the granularity of each time period, we can consult the procedures from Section 7.1 to identify m. The next to choose is how long the experiment should last, $n = T/m$. We choose n by referring to the rejection rate curve as shown in Section 6.4. We first prescribe which inference method to use (exact inference or asymptotic inference). Then we borrow our domain knowledge to get a sense of the signal-to-noise ratio. Finally, we choose any desired rejection rate and find out the length of horizon required.

8. Concluding Remarks, Limitations and Future Work

We studied the design and analysis of switchback experiments. We formulated and solved a minimax problem for the design of optimal switchback experiments. We then analyzed our proposed optimal design and proposed two inferential methods. In particular, we showed asymptotic normality of the randomization distribution. We discussed cases when the order of carryover effect, m, is misspecified, and detailed a method to identify the order of carryover effect. We gave empirical suggestions that a slight overestimate of m is acceptable and better than an underestimate.

We point out two limitations of our paper. First, when m the order of carryover effect is as large as comparable to T the length of horizon, our method, though still unbiased in theory, incurs a huge variance that typically prohibits the experimental designer to make any inference. This is due to the fact that our method is general and requires the minimum amount of modeling assumptions. When the outcome model has some structures, for example the equilibrium effects as in Wager and Xu (2019), utilizing such structures will lead to difference designs. Second, our method does not take advantage of covariate information to further reduce the randomization variance. This could lead to future work.

Finally, we do encourage empirical researchers who apply our method to use domain knowledge to narrow down m first, before using the procedure in Section 7.1 to identify m. This is because, empirically, each sub-routine to identify (17) needs to consume experimental resources at the scale of $T/p \approx 100$ to well distinguish two candidate values, which could be luxurious when the resource is scarce.

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Online Appendix

EC.1. Recap of Notations

Within this paper, let $\mathbb{N}, \mathbb{N}_0$ be the set of positive integers and non-negative integers, respectively. For any $T \in \mathbb{N}$, let $[T] = \{1, ..., T\}$ be the set of positive integers no larger than T. For any $t < t' \in \mathbb{N}$, let $\{t : t'\} = \{t, t+1, ..., t'\}$ be the set of integers between (including) t and t'. For any $m \in \mathbb{N}$, let $\mathbf{1}_m = (1, 1, ..., 1)$ be a vector of m ones; let $\mathbf{0}_m = (0, 0, ..., 0)$ be a vector of m zeros. We use parentheses for probabilities, i.e., $\Pr(\cdot)$; brackets for expectations, i.e., $\mathbb{E}[\cdot]$; and curly brackets for indicators, i.e., $\mathbb{1}\{\cdot\}$. For any $a \in \mathbb{R}$, let $(a)^+ = \max\{a, 0\}$.

EC.2. Theorems Used

We summarize here the preliminaries that we have directly used in our proof.

Definition EC.1 (ϕ-Dependent Random Variables, Hoeffding and Robbins (1948)). For any sequence $\{X_1, X_2, ...\}$, if there exists ϕ such that for any $s - r > \phi$, the two sets

$$(X_1, X_2, ..., X_r), \quad (X_s, X_{s+1}, ..., X_n)$$

are independent, then the sequence is said to be ϕ-dependent.

Lemma EC.1 **(Romano and Wolf (2000), Theorem 2.1)**. *Let $\{X_{n,i}\}$ be a triangular array of zero-mean random variables. Let $\phi \in \mathbb{N}$ be a fixed constant. For each $n = 1, 2, ...$, let $d = d_n$, and suppose that $X_{n,1}, X_{n,2}, ..., X_{n,d}$ is an ϕ-dependent sequence of random variables. Define*

$$B^2_{n,k,a} = \mathsf{Var}\left(\sum_{i=a}^{a+k-1} X_{n,i}\right), \qquad B^2_n = B^2_{n,d,1} = \mathsf{Var}\left(\sum_{i=1}^{d} X_{n,i}\right)$$

For some $\delta > 0$ and $-1 \le \gamma \le 1$, if the following conditions hold:

1. $\mathbb{E}|X_{n,i}|^{2+\delta} \le \Delta_n$, *for all i;*
2. $B^2_{n,k,a}/k^{1+\gamma} \le K_n$, *for all a and $k \ge \phi$;*
3. $B^2_n/(d\phi^\gamma) \ge L_n$*;*
4. $K_n/L_n = O(1)$*;*
5. $\Delta/L_n^{(2+\delta)/2} = O(1)$,

then

$$\frac{\sum_{i=1}^d X_{n,i}}{B_n} \xrightarrow{D} \mathcal{N}(0,1).$$

We explain Lemma EC.1. The $\xrightarrow{D}$ notation stands for convergence in distribution. The definition of a sequence of ϕ-dependent random variables is given in Definition EC.1. To check if the conditions in Lemma EC.1 hold, we will first calculate $B^2_{n,k,a}$ for any k and a, and then construct some proper Δ_n, K_n, and L_n.

EC.3. Proof of Unbiasedness of the Horvitz-Thompson Estimator

Proof of Theorem 1. First observe that for regular switchback experiments, both $0 < \Pr(\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1}), \Pr(\boldsymbol{W}_{t-p:t} = \mathbf{0}_{p+1}) < 1$. So Assumption 4 is naturally satisfied.

So for any $t \in \{p+1:T\}$, with probability $\Pr(\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1}) \neq 0$, $\mathbb{1}\{\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1}\} = 1$, and $Y_t^{\mathsf{obs}} = Y_t(\mathbf{1}_{p+1})$. So $\mathbb{E}\left[Y_t^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t}=\mathbf{1}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t}=\mathbf{1}_{p+1})}\right] = Y_t(\mathbf{1}_{m+1})$. Similarly $\mathbb{E}\left[Y_t^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t}=\mathbf{0}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t}=\mathbf{0}_{p+1})}\right] = Y_t(\mathbf{0}_{p+1})$. Sum them up for any $t \in \{p+1:T\}$ we finish the proof. □

EC.4. Proofs and Discussions from Section 3

In Section 3 we focus on the case when $p = m$. Throughout this section in the appendix, we use only m instead of p.

EC.4.1. Extra Notations Used in the Proofs from Section 3

For any $t \in \{m+1:T\}$, denote

$$\begin{aligned}\mathbb{1}_t(\mathbb{T},\mathbb{Y}) = Y_t(\mathbf{1}_{m+1}) &\left[\mathbb{1}\{\boldsymbol{W}_{t-m:t} = (\mathbf{1}_{m+1})\} \cdot 2 \cdot \prod_{l=1}^{m} 2^{\mathbb{1}\{t-l+1\in\mathbb{T}\}} - 1\right] \\ &- Y_t(\mathbf{0}_{m+1}) \left[\mathbb{1}\{\boldsymbol{W}_{t-m:t} = (\mathbf{0}_{m+1})\} \cdot 2 \cdot \prod_{l=1}^{m} 2^{\mathbb{1}\{t-l+1\in\mathbb{T}\}} - 1\right] \quad \text{(EC.1)}\end{aligned}$$

where we use $2\prod_{l=1}^{m} 2^{\mathbb{1}\{t-l+1\in\mathbb{T}\}}$ to calculate the inverse propensity score. When $\mathbb{T}$ and $\mathbb{Y}$ are clear from the context we omit them and use $\mathbb{1}_t$ for $\mathbb{1}_t(\mathbb{T},\mathbb{Y})$.

Using the above notation, we could re-write

$$\hat{\tau}_m - \tau_m = \frac{1}{T-m} \sum_{t=m+1}^{T} \mathbb{1}_t(\mathbb{T},\mathbb{Y})$$

Note that $\forall t \in \{m+1, m+2, ..., T\}$,

$$\mathbb{E}[\mathbb{1}_t(\mathbb{T},\mathbb{Y})] = 0. \quad \text{(EC.2)}$$

Recall that any regular switchback experiment can be represented by $\mathbb{T} = \{t_0, t_1, ..., t_K\} \subseteq [T]$. Define $f_{\mathbb{T}} : [T] \to \mathbb{T}$ to be the "determining randomization point of period t", i.e.

$$f_{\mathbb{T}}(t) = \max\{j \,|\, j \in \mathbb{T}, j \leq t\}$$

such that $W_{f_{\mathbb{T}}(t)}$ uniquely determines the distribution of W_t, i.e. $W_t = W_{f_{\mathbb{T}}(t)}$. When $\mathbb{T}$ is clear from the context we also omit the subscript and use $f(t)$ for $f_{\mathbb{T}}(t)$.

Similarly, we define $f_{\mathbb{T}}^m(t) : [T] \to \{0,1\}^{\mathbb{T}}$, which maps a time period to a subset of the set $\mathbb{T}$, to be the "determining randomization points of periods $\{t-m, t-m+1, ..., t\}$", i.e.

$$f_{\mathbb{T}}^m(t) = \{j \,|\, \exists i \in \{t-m, ..., t\}, s.t. j = f_{\mathbb{T}}(i)\}$$

such that $f^m_{\mathbb{T}}(t) \subseteq \mathbb{T} \subseteq [T]$. And $f^m_{\mathbb{T}}(t)$ contains all the time periods whose coin flips determine the distributions of $W_{t-m}, W_{t-m+1}, ..., W_t$. Denote $|f^m_{\mathbb{T}}(t)| = J$. We keep in mind that J depends on m, t and $\mathbb{T}$, yet they are all omitted for brevity. Since $W_{t-m}, W_{t-m+1}, ..., W_t$ are determined by at least one randomization point $f(t-m)$, we know that $f^m_{\mathbb{T}}(t) \neq \emptyset$ is non-empty, i.e.,

$$|f^m_{\mathbb{T}}(t)| = J \geq 1. \tag{EC.3}$$

Finally, define "overlapping randomization points of periods $\{t-m, t-m+1, ..., t\}$ and $\{t'-m, t'-m+1, ..., t'\}$" to be

$$O_{\mathbb{T}}(t,t') = f^m_{\mathbb{T}}(t) \cap f^m_{\mathbb{T}}(t')$$

Denote $|O_{\mathbb{T}}(t,t')| = J^{\circ}$. We keep in mind that J° depends on m, t, t' and $\mathbb{T}$, yet they are all omitted for brevity.

EC.4.2. Lemma 1: Adversarial Selection of Potential Outcomes

EC.4.2.1. Preliminaries. We first introduce two Lemmas for the proof of Lemma 1.

LEMMA EC.2. *Under Assumptions 2–3, for any $t \in [T]$, let $|f^m_{\mathbb{T}}(t)| = J$.*

$$\mathbb{E}[\mathbb{1}_t^2] = (2^J - 1) Y_t(\mathbf{1}_{m+1})^2 + 2 Y_t(\mathbf{1}_{m+1}) Y_t(\mathbf{0}_{m+1}) + (2^J - 1) Y_t(\mathbf{0}_{m+1})^2. \tag{EC.4}$$

Proof of Lemma EC.2. Denote $|f^m_{\mathbb{T}}(t)| = J$. Let the elements be $f^m_{\mathbb{T}}(t) = \{u_1, u_2, ..., u_J\}$. Let $u_1 < u_2 < ... < u_J$.

Using the notations defined earlier in Section EC.4,

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_t^2] =& \Pr\left(W_{t-m:t} = \mathbf{1}_{m+1}\right) \cdot \left\{ Y_t(\mathbf{1}_{m+1})(2^J \cdot 1 - 1) - Y_t(\mathbf{0}_{m+1})(2^J \cdot 0 - 1) \right\}^2 \\
&+ \Pr\left(W_{t-m:t} = \mathbf{0}_{m+1}\right) \cdot \left\{ Y_t(\mathbf{1}_{m+1})(2^J \cdot 0 - 1) - Y_t(\mathbf{0}_{m+1})(2^J \cdot 1 - 1) \right\}^2 \\
&+ \Pr\left(W_{t-m:t} \neq \mathbf{1}_J \text{ or } \mathbf{0}_J\right) \cdot \left\{ Y_t(\mathbf{1}_{m+1})(2^J \cdot 0 - 1) - Y_t(\mathbf{0}_{m+1})(2^J \cdot 0 - 1) \right\}^2 \\
=& \Pr\left((W_{u_1}, ..., W_{u_J}) = \mathbf{1}_J\right) \cdot \left\{ (2^J - 1) Y_t(\mathbf{1}_{m+1}) + Y_t(\mathbf{0}_{m+1}) \right\}^2 \\
&+ \Pr\left((W_{u_1}, ..., W_{u_J}) = \mathbf{0}_J\right) \cdot \left\{ -Y_t(\mathbf{1}_{m+1}) - (2^J - 1) Y_t(\mathbf{0}_{m+1}) \right\}^2 \\
&+ \Pr\left((W_{u_1}, ..., W_{u_J}) \neq \mathbf{1}_J \text{ or } \mathbf{0}_J\right) \cdot \left\{ -Y_t(\mathbf{1}_{m+1}) + Y_t(\mathbf{0}_{m+1}) \right\}^2 \\
=& (2^J - 1) Y_t(\mathbf{1}_{m+1})^2 + 2 Y_t(\mathbf{1}_{m+1}) Y_t(\mathbf{0}_{m+1}) + (2^J - 1) Y_t(\mathbf{0}_{m+1})^2
\end{aligned}$$

which finishes the proof. □

LEMMA EC.3. *Under Assumptions 2–3, for any $t < t' \in [T]$, when $|O_{\mathbb{T}}(t,t')| = J^{\circ} = 0$,*

$$\mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] = 0. \tag{EC.5}$$

When $|O_{\mathbb{T}}(t,t')| = J^{\circ} \geq 1$,

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] =& (2^{J^{\circ}} - 1) Y_t(\mathbf{1}_{m+1}) Y_{t'}(\mathbf{1}_{m+1}) + Y_t(\mathbf{1}_{m+1}) Y_{t'}(\mathbf{0}_{m+1}) \\
&+ Y_t(\mathbf{0}_{m+1}) Y_{t'}(\mathbf{1}_{m+1}) + (2^{J^{\circ}} - 1) Y_t(\mathbf{0}_{m+1}) Y_{t'}(\mathbf{0}_{m+1}).
\end{aligned} \tag{EC.6}$$

Proof of Lemma EC.3. Denote $|f_{\mathbb{T}}^m(t)| = J$, $|f_{\mathbb{T}}^m(t')| = J'$, and $|O_{\mathbb{T}}(t,t')| = J^\circ$. Let the elements be $f_{\mathbb{T}}^m(t) = \{u_1, u_2, ..., u_J\}$, $f_{\mathbb{T}}^m(t') = \{u'_1, u'_2, ..., u'_{J'}\}$, and $O_{\mathbb{T}}(t,t') = \{u^\circ_1, u^\circ_2, ..., u^\circ_{J^\circ}\}$. Let $u_1 < u_2 < ... < u_J$, $u'_1 < u'_2 < ... < u'_{J'}$, and $u^\circ_1 < u^\circ_2 < ... < u^\circ_{J^\circ}$.

One time period could have different numberings in $f_{\mathbb{T}}^m(t)$, $f_{\mathbb{T}}^m(t')$, and $O_{\mathbb{T}}(t,t')$. For example, $u_{J-J^\circ+1} = u'_1 = u^\circ_1$, and $u_J = u'_{J^\circ} = u^\circ_{J^\circ}$. See Table EC.1 for an illustrator of the determining randomization points and the overlapping randomization points.

Table EC.1 Illustrator of the determining randomization points and the overlapping randomization points.

u_1	u_2	...	$u_{J-J^\circ+1}$	...	u_J			
			u°_1	...	$u^\circ_{J^\circ}$			
			u'_1	...	u'_{J°	$u'_{J^\circ+1}$	...	$u'_{J'}$

Each columns stands for one time period. The first row stands for the determining randomization points of $f_{\mathbb{T}}^m(t)$; the second row for the overlapping randomization points of $O_{\mathbb{T}}(t,t')$; and the third row for the determining randomization points of $f_{\mathbb{T}}^m(t')$.

First, when $J^\circ = 0$, this implies that $\mathbb{1}_t$ and $\mathbb{1}_{t'}$ are independent. Then $\mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] = \mathbb{E}[\mathbb{1}_t]\mathbb{E}[\mathbb{1}_{t'}] = 0$, where the second equality is due to (EC.2).

When $J^\circ \geq 1$, this implies that $\mathbb{1}_t$ and $\mathbb{1}_{t'}$ are correlated. Using the notations defined above,

$$\begin{aligned}\mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] =& \mathbb{E}_{W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}}} \left[\mathbb{E}\left[\mathbb{1}_t \mathbb{1}_{t'} \middle| W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}} \right] \right] \\ =& \Pr\left((W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}}) = \mathbf{1}_{J^\circ} \right) \mathbb{E}\left[\mathbb{1}_t \mathbb{1}_{t'} \middle| (W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}}) = \mathbf{1}_{J^\circ} \right] \\ &+ \Pr\left((W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}}) = \mathbf{0}_{J^\circ} \right) \mathbb{E}\left[\mathbb{1}_t \mathbb{1}_{t'} \middle| (W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}}) = \mathbf{0}_{J^\circ} \right] \\ &+ \Pr\left((W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}}) \neq \mathbf{1}_{J^\circ} \text{ or } \ \mathbf{0}_{J^\circ} \right) \mathbb{E}\left[\mathbb{1}_t \mathbb{1}_{t'} \middle| (W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}}) \neq \mathbf{1}_{J^\circ} \text{ or } \ \mathbf{0}_{J^\circ} \right]\end{aligned} \tag{EC.7}$$

Next we go over the three cases of $(W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}})$ as decomposed above. Note that conditional on $(W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}})$, $\mathbb{1}_t$ and $\mathbb{1}_{t'}$ are independent, i.e.,

$$\mathbb{E}\left[\mathbb{1}_t \mathbb{1}_{t'} \middle| W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}} \right] = \mathbb{E}\left[\mathbb{1}_t \middle| W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}} \right] \mathbb{E}\left[\mathbb{1}_{t'} \middle| W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}} \right]$$

(1) With probability $1/2^{J^\circ}$, $(W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}}) = \mathbf{1}_{J^\circ}$, in which case

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_t \middle| W_{u^\circ_1}, ..., W_{u^\circ_{J^\circ}} \right] =& \Pr\left(W_{t-m:t} = \mathbf{1}_{m+1} \right) \cdot \left\{ Y_t(\mathbf{1}_{m+1})(2^J \cdot 1 - 1) + Y_t(\mathbf{0}_{m+1}) \right\} \\ &+ \Pr\left(W_{t-m:t} \neq \mathbf{1}_{m+1} \right) \cdot \left\{ Y_t(\mathbf{1}_{m+1})(2^J \cdot 0 - 1) + Y_t(\mathbf{0}_{m+1}) \right\} \\ =& \Pr\left((W_{u_1}, W_{u_2}, ..., W_{u_{J-J^\circ}}) = \mathbf{1}_{J-J^\circ} \right) \cdot \left\{ Y_t(\mathbf{1}_{m+1})(2^J - 1) + Y_t(\mathbf{0}_{m+1}) \right\} \\ &+ \Pr\left((W_{u_1}, W_{u_2}, ..., W_{u_{J-J^\circ}}) \neq \mathbf{1}_{J-J^\circ} \right) \cdot \left\{ -Y_t(\mathbf{1}_{m+1}) + Y_t(\mathbf{0}_{m+1}) \right\} \\ =& \frac{1}{2^{J-J^\circ}} \cdot \left\{ Y_t(\mathbf{1}_{m+1})(2^J - 1) + Y_t(\mathbf{0}_{m+1}) \right\} + \left(1 - \frac{1}{2^{J-J^\circ}} \right) \cdot \left\{ -Y_t(\mathbf{1}_{m+1}) + Y_t(\mathbf{0}_{m+1}) \right\} \\ =& (2^{J^\circ} - 1) Y_t(\mathbf{1}_{m+1}) + Y_t(\mathbf{0}_{m+1})\end{aligned}$$

where the third equality is due to (2). Similarly,

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{t'}\,\middle|\,W_{u_1^\circ},...,W_{u_{J^\circ}^\circ}\right] =& \Pr\left(W_{t'-m:t'}=\mathbf{1}_{m+1}\right)\cdot\left\{Y_{t'}(\mathbf{1}_{m+1})(2^{J'}\cdot 1-1)+Y_{t'}(\mathbf{0}_{m+1})\right\}\\ &+\Pr\left(W_{t'-m:t'}\neq\mathbf{1}_{m+1}\right)\cdot\left\{Y_{t'}(\mathbf{1}_{m+1})(2^{J'}\cdot 0-1)+Y_{t'}(\mathbf{0}_{m+1})\right\}\\ =&\Pr\left((W_{u'_{J^\circ+1}},W_{u'_{J^\circ+2}},...,W_{u'_{J'}})=\mathbf{1}_{J'-J^\circ}\right)\cdot\left\{Y_{t'}(\mathbf{1}_{m+1})(2^{J'}-1)+Y_{t'}(\mathbf{0}_{m+1})\right\}\\ &+\Pr\left((W_{u'_{J^\circ+1}},W_{u'_{J^\circ+2}},...,W_{u'_{J'}})\neq\mathbf{1}_{J'-J^\circ}\right)\cdot\left\{-Y_{t'}(\mathbf{1}_{m+1})+Y_{t'}(\mathbf{0}_{m+1})\right\}\\ =&\frac{1}{2^{J'-J^\circ}}\cdot\left\{Y_{t'}(\mathbf{1}_{m+1})(2^{J'}-1)+Y_{t'}(\mathbf{0}_{m+1})\right\}+\left(1-\frac{1}{2^{J'-J^\circ}}\right)\cdot\left\{-Y_{t'}(\mathbf{1}_{m+1})+Y_{t'}(\mathbf{0}_{m+1})\right\}\\ =&(2^{J^\circ}-1)Y_{t'}(\mathbf{1}_{m+1})+Y_{t'}(\mathbf{0}_{m+1})\end{aligned}$$

(2) With probability $1/2^{J^\circ}$, $(W_{u_1^\circ},...,W_{u_{J^\circ}^\circ})=\mathbf{0}_{J^\circ}$, in which case

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{t}\,\middle|\,W_{u_1^\circ},...,W_{u_{J^\circ}^\circ}\right] =& \Pr\left(W_{t-m:t}=\mathbf{0}_{m+1}\right)\cdot\left\{-Y_{t}(\mathbf{1}_{m+1})-Y_{t}(\mathbf{0}_{m+1})(2^{J}\cdot 1-1)\right\}\\ &+\Pr\left(W_{t-m:t}\neq\mathbf{1}_{m+1}\right)\cdot\left\{-Y_{t}(\mathbf{1}_{m+1})-Y_{t}(\mathbf{0}_{m+1})(2^{J}\cdot 0-1)\right\}\\ =&\frac{1}{2^{J-J^\circ}}\cdot\left\{-Y_{t}(\mathbf{1}_{m+1})-Y_{t}(\mathbf{0}_{m+1})(2^{J}-1)\right\}+\left(1-\frac{1}{2^{J-J^\circ}}\right)\cdot\left\{-Y_{t}(\mathbf{1}_{m+1})+Y_{t}(\mathbf{0}_{m+1})\right\}\\ =&-Y_{t}(\mathbf{1}_{m+1})-(2^{J^\circ}-1)Y_{t}(\mathbf{0}_{m+1})\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{t'}\,\middle|\,W_{u_1^\circ},...,W_{u_{J^\circ}^\circ}\right] =& \Pr\left(W_{t'-m:t'}=\mathbf{0}_{m+1}\right)\cdot\left\{-Y_{t'}(\mathbf{1}_{m+1})-Y_{t'}(\mathbf{0}_{m+1})(2^{J'}\cdot 1-1)\right\}\\ &+\Pr\left(W_{t'-m:t'}\neq\mathbf{1}_{m+1}\right)\cdot\left\{-Y_{t'}(\mathbf{1}_{m+1})-Y_{t'}(\mathbf{0}_{m+1})(2^{J'}\cdot 0-1)\right\}\\ =&\frac{1}{2^{J'-J^\circ}}\cdot\left\{-Y_{t'}(\mathbf{1}_{m+1})-Y_{t'}(\mathbf{0}_{m+1})(2^{J'}-1)\right\}+\left(1-\frac{1}{2^{J'-J^\circ}}\right)\cdot\left\{-Y_{t'}(\mathbf{1}_{m+1})+Y_{t'}(\mathbf{0}_{m+1})\right\}\\ =&-Y_{t'}(\mathbf{1}_{m+1})-(2^{J^\circ}-1)Y_{t'}(\mathbf{0}_{m+1})\end{aligned}$$

(3) With probability $1-2\cdot(1/2^{J^\circ})$, $(W_{u_1^\circ},...,W_{u_{J^\circ}^\circ})\neq\mathbf{1}_{J^\circ}$ or $\mathbf{0}_{J^\circ}$, in which case

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{t}\,\middle|\,W_{u_1^\circ},...,W_{u_{J^\circ}^\circ}\right]&=-Y_{t}(\mathbf{1}_{m+1})+Y_{t}(\mathbf{0}_{m+1})\\ \mathbb{E}\left[\mathbb{1}_{t'}\,\middle|\,W_{u_1^\circ},...,W_{u_{J^\circ}^\circ}\right]&=-Y_{t'}(\mathbf{1}_{m+1})+Y_{t'}(\mathbf{0}_{m+1})\end{aligned}$$

Finally, putting all above together into (EC.7), we have

$$\begin{aligned}\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]=&\frac{1}{2^{J^\circ}}\cdot\left\{(2^{J^\circ}-1)Y_t(\mathbf{1}_{m+1})+Y_t(\mathbf{0}_{m+1})\right\}\cdot\left\{(2^{J^\circ}-1)Y_{t'}(\mathbf{1}_{m+1})+Y_{t'}(\mathbf{0}_{m+1})\right\}\\ &+\frac{1}{2^{J^\circ}}\cdot\left\{-Y_t(\mathbf{1}_{m+1})-(2^{J^\circ}-1)Y_t(\mathbf{0}_{m+1})\right\}\cdot\left\{-Y_{t'}(\mathbf{1}_{m+1})-(2^{J^\circ}-1)Y_{t'}(\mathbf{0}_{m+1})\right\}\\ &+\left\{1-\frac{2}{2^{J^\circ}}\right\}\cdot\left\{-Y_t(\mathbf{1}_{m+1})+Y_t(\mathbf{0}_{m+1})\right\}\cdot\left\{-Y_{t'}(\mathbf{1}_{m+1})+Y_{t'}(\mathbf{0}_{m+1})\right\}\\ =&(2^{J^\circ}-1)Y_t(\mathbf{1}_{m+1})Y_{t'}(\mathbf{1}_{m+1})+Y_t(\mathbf{1}_{m+1})Y_{t'}(\mathbf{0}_{m+1})+Y_t(\mathbf{0}_{m+1})Y_{t'}(\mathbf{1}_{m+1})+(2^{J^\circ}-1)Y_t(\mathbf{0}_{m+1})Y_{t'}(\mathbf{0}_{m+1})\end{aligned}$$

which finishes the proof. □

EC.4.2.2. Proof of Lemma 1. The proof of Lemma 1 is through careful expansion of the risk function, the expected square loss.

Proof of Lemma 1. From Lemma EC.2 and Lemma EC.3, all the terms are quadratic, and all the coefficients are non-negative. That is, after multiplying the constant $(T-m)^2$, for any $\mathbb{T} \in [T], \mathbb{Y} \in \mathcal{Y}$ we can express in canonical form the following:

$$\begin{aligned}
&(T-m)^2 \cdot \mathbb{E}_{\boldsymbol{W}_{1:T} \sim \eta(\cdot)} \left[\left(\hat{\tau}_m(\boldsymbol{W}_{1:T}, \mathbb{Y}) - \tau_m(\mathbb{Y}) \right)^2 \right] \\
=& \sum_{t=m+1}^{T} \left\{ a_t(11) Y_t(\mathbf{1}_{m+1})^2 + a_t(10) Y_t(\mathbf{1}_{m+1}) Y_t(\mathbf{0}_{m+1}) + a_t(00) Y_t(\mathbf{0}_{m+1})^2 \right\} \\
&+ \sum_{m+1 \leq t < t' \leq T} \{ b_{tt'}(11) Y_t(\mathbf{1}_{m+1}) Y_{t'}(\mathbf{1}_{m+1}) + b_{tt'}(10) Y_t(\mathbf{1}_{m+1}) Y_{t'}(\mathbf{0}_{m+1}) \\
&\qquad + b_{tt'}(01) Y_t(\mathbf{0}_{m+1}) Y_{t'}(\mathbf{1}_{m+1}) + b_{tt'}(00) Y_t(\mathbf{0}_{m+1}) Y_{t'}(\mathbf{0}_{m+1}) \}
\end{aligned}$$

Combining Lemma EC.2 and inequality (EC.3), $a_t(11), a_t(10), a_t(00)$ are all strictly positive. From Lemma EC.3 $b_{tt'}(11), b_{tt'}(10), b_{tt'}(01), b_{tt'}(00)$ are all non-negative.

For the squared terms, $y^2 \leq B^2$, where the inequality is due to convexity. For the cross-product terms, $y_1 \cdot y_2 \leq (y_1^2 + y_2^2)/2 \leq B^2$ where the first inequality is due to Cauchy-Schwarz, and the second inequality is due to convexity. Combining that fact that all coefficients are non-negative, $r(\eta_{\mathbb{T}}, \mathbb{Y}) \leq r(\eta_{\mathbb{T}}, \mathbb{Y}^+) = r(\eta_{\mathbb{T}}, \mathbb{Y}^-)$.

Moreover, for any $\mathbb{Y} \in \mathcal{Y}$ such that $\mathbb{Y} \neq \mathbb{Y}^+$ or $\mathbb{Y}^-$, if $\exists t \in \{m+1, ..., T\}$ such that $-B < Y_t(\mathbf{1}_{m+1}) < B$. Then from inequality (EC.3), $a_t(11) > 0$, so the inequality is strict. Similarly, if $\exists t \in \{m+1, ..., T\}$ such that $-B < Y_t(\mathbf{0}_{m+1}) < B$, then combine $a_t(00) > 0$, so the inequality is strict. ☐

EC.4.2.3. Implications of Lemma 1. Lemma 1 simplifies the minimax problem in (6). Instead of thinking it as a minimax problem, we can now replace $\mathbb{Y}$ by either $\mathbb{Y}^+$ or $\mathbb{Y}^-$, and solve only a minimization problem.

Here we state Lemma EC.4 that is a direct implication of Lemma 1. It will be frequently used later on.

LEMMA EC.4. *When $\mathbb{Y} = \mathbb{Y}^+$ or $\mathbb{Y} = \mathbb{Y}^-$, under Assumptions 1–3, for any $t \in [T]$,*

$$\mathbb{E}[\mathbb{1}_t^2] = 2^{J+1} B^2.$$

For any $t < t' \in [T]$, when $|O_{\mathbb{T}}(t, t')| = J^\circ = 0$,

$$\mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] = 0$$

When $|O_{\mathbb{T}}(t, t')| = J^\circ \geq 1$,

$$\mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] = 2^{J^\circ + 1} B^2$$

Proof of Lemma EC.4. Replace $Y_t(\mathbf{1}_{m+1}) = Y_t(\mathbf{0}_{m+1})$ by B or $-B$, into the expressions in Lemmas EC.2 and EC.3. ☐

EC.4.3. Structural Results of the Optimal Design

First note that when we focus on the optimal design, we treat T and m both as constants. So the constant of $1/(T-m)$ in the expression of the risk function does not affect the optimal design.

EC.4.3.1. Proof of Lemma 2.

Proof of Lemma 2. We prove the two parts separately, both by contradiction.

(1) Suppose there exists an optimal design $\mathbb{T} = \{t_0 = 1, t_1, t_2, ..., t_K\}$ such that $t_1 \leq m+1$. Then we try to construct another design $\tilde{\mathbb{T}}$, such that $\left|\tilde{\mathbb{T}}\right| = K = |\mathbb{T}| - 1$. And the K elements are $\tilde{\mathbb{T}} = \{\tilde{t}_0 = 1, \tilde{t}_1 = t_2, \tilde{t}_2 = t_3, ..., \tilde{t}_{K-1} = t_K\}$.

Table EC.2 An example of two regular switchback experiments $\mathbb{T}$ and $\tilde{\mathbb{T}}$ when $m = 4$ and $t_1 = 3$.

	1	2	3	4	5	6	...
$\mathbb{T}$	✓	–	✓	–	–	✓	...
$\tilde{\mathbb{T}}$	✓	–	–	–	–	✓	...

Each checkmark beneath a number indicates that this number is within that set; and each dash beneath a number indicates that this number is not within that set. For example, the checkmark ✓ beneath number 3 indicates that $3 \in \mathbb{T}$; and the dash – beneath number 3 indicates that $3 \neq \tilde{\mathbb{T}}$.

Next we argue that when $\mathbb{Y} = \mathbb{Y}^+$ or $\mathbb{Y} = \mathbb{Y}^-$,

$$r(\mathbb{T}, \mathbb{Y}) > r(\tilde{\mathbb{T}}, \mathbb{Y}),$$

which suggests that $\mathbb{T}$ is not the optimal design.

First, focus on the squared terms. For any $m+1 \leq t \leq t_1+m-1$, $t_1 \in f^m_{\mathbb{T}}(t), t_1 \neq f^m_{\tilde{\mathbb{T}}}(t)$. Moreover, $t-m \leq t_1 - 1$, so that $t_0 \in f^m_{\tilde{\mathbb{T}}}(t)$. So $f^m_{\mathbb{T}}(t) - \{t_1\} = f^m_{\tilde{\mathbb{T}}}(t)$, and $\left|f^m_{\tilde{\mathbb{T}}}(t)\right| \geq 1$. As a result,

$$\mathbb{E}[\mathbb{1}_t(\mathbb{T})^2] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})^2] \geq (2^{2+1} - 2^{1+1})B^2 = 4B^2.$$

For any $t \geq t_1 + m$, either (i) $f_{\mathbb{T}}(t-m) = t_1$, in which case $f_{\tilde{\mathbb{T}}}(t-m) = t_0$. This is the only difference between $f^m_{\mathbb{T}}(t)$ and $f^m_{\tilde{\mathbb{T}}}(t)$, i.e., $f^m_{\mathbb{T}}(t) - \{t_1\} = f^m_{\tilde{\mathbb{T}}}(t) - \{t_0\}$. So $|f^m_{\mathbb{T}}(t)| = \left|f^m_{\tilde{\mathbb{T}}}(t)\right|$. The second case is (ii) $f_{\mathbb{T}}(t-m) \geq t_2$, in which case $f^m_{\mathbb{T}}(t) = f^m_{\tilde{\mathbb{T}}}(t)$. Both cases suggest that

$$\mathbb{E}[\mathbb{1}_t(\mathbb{T})^2] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})^2] = 0.$$

So we have

$$\begin{aligned}
\sum_{t=m+1}^{T} \mathbb{E}\left[\mathbb{1}_t(\mathbb{T})^2\right] - \sum_{t=m+1}^{T} \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})^2\right] =& \sum_{t=m+1}^{t_1+m-1} \left(\mathbb{E}\left[\mathbb{1}_t(\mathbb{T})^2\right] - \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})^2\right]\right) + \sum_{t=t_1+m}^{T} \left(\mathbb{E}\left[\mathbb{1}_t(\mathbb{T})^2\right] - \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})^2\right]\right) \\
\geq& \sum_{t=m+1}^{t_1+m-1} (4B^2) + 0 \\
=& 4(t_1 - 1)B^2 \\
>& 0
\end{aligned}$$

Second, focus on the cross product terms. For any t and t' such that $m+1 \leq t < t' \leq t_1+m-1$, $t_1 \in O_{\mathbb{T}}(t,t'), t_1 \neq O_{\tilde{\mathbb{T}}}(t,t')$. Moreover, $t-m \leq t_1-1$, so that $t_0 \in O_{\mathbb{T}}(t,t')$. So $O_{\mathbb{T}}(t,t') - \{t_1\} = O_{\tilde{\mathbb{T}}}(t,t')$, and $|O_{\tilde{\mathbb{T}}}(t,t')| \geq 1$. As a result,

$$\mathbb{E}[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})] \geq (2^{2+1} - 2^{1+1})B^2 = 4B^2 > 0.$$

For any $m+1 \leq t < t' \leq T$ such that $t' \geq t_1+m$, either (i) $f_{\mathbb{T}}(t'-m) = t_1$, in which case $f_{\tilde{\mathbb{T}}}(t'-m) = t_0$. So $O_{\mathbb{T}}(t,t') - \{t_1\} = O_{\tilde{\mathbb{T}}}(t,t') - \{t_0\}$. So $|O_{\mathbb{T}}(t,t')| = |O_{\tilde{\mathbb{T}}}(t,t')|$. The second case is (ii) $f_{\mathbb{T}}(t'-m) \geq t_2$, in which case $O_{\mathbb{T}}(t,t') = O_{\tilde{\mathbb{T}}}(t,t')$. Both cases suggest that

$$\mathbb{E}[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})] = 0.$$

So we have

$$\begin{aligned}
&\sum_{m+1\leq t<t'\leq T} \mathbb{E}\left[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})\right] - \sum_{m+1\leq t<t'\leq T} \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})\right] \\
=&\sum_{m+1\leq t<t'\leq t_1+m-1} \left(\mathbb{E}\left[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})\right] - \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})\right]\right) + \sum_{\substack{m+1\leq t<t'\leq T \\ t'\geq t_1+m}} \left(\mathbb{E}\left[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})\right] - \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})\right]\right) \\
\geq& 0
\end{aligned}$$

Combine both square terms and cross-product terms we know that

$$r(\mathbb{T},\mathbb{Y}) > r(\tilde{\mathbb{T}},\mathbb{Y}).$$

(2) Suppose there exists an optimal design $\mathbb{T} = \{t_0 = 1, t_1, t_2, ..., t_K\}$ such that $t_K \geq T-m+1$. Then we try to construct another design $\tilde{\mathbb{T}}$, such that $\left|\tilde{\mathbb{T}}\right| = K = |\mathbb{T}|-1$. And the K elements are $\tilde{\mathbb{T}} = \{\tilde{t}_0 = 1, \tilde{t}_1 = t_1, \tilde{t}_2 = t_2, ..., \tilde{t}_{K-1} = t_{K-1}\}$.

Table EC.3 An example of two regular switchback experiments $\mathbb{T}$ and $\tilde{\mathbb{T}}$ when $m=4$ and $t_K = T-2$.

	...	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
$\mathbb{T}$	...	✓	−	✓	✓	−	−
$\tilde{\mathbb{T}}$	...	✓	−	✓	−	−	−

Each checkmark beneath a number indicates that this number is within that set; and each dash beneath a number indicates that this number is not within that set. For example, the checkmark ✓ beneath number $T-2$ indicates that $T-2 \in \mathbb{T}$; and the dash − beneath number $T-2$ indicates that $T-2 \neq \tilde{\mathbb{T}}$.

Next we argue that when $\mathbb{Y} = \mathbb{Y}^+$ or $\mathbb{Y} = \mathbb{Y}^-$,

$$r(\mathbb{T},\mathbb{Y}) > r(\tilde{\mathbb{T}},\mathbb{Y}),$$

which suggests that $\mathbb{T}$ is not the optimal design.

First focus on the squared terms. For any $m+1 \le t \le t_K - 1$, $f_{\mathbb{T}}^m(t) = f_{\tilde{\mathbb{T}}}^m(t)$ is totally unchanged.

$$\mathbb{E}[\mathbb{1}_t(\mathbb{T})^2] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})^2] = 0.$$

For any $t_K \le t \le T$, $t_K \notin f_{\tilde{\mathbb{T}}}^m(t), t_K \in f_{\mathbb{T}}^m(t)$. And all the other determining randomization points are unchanged. So $f_{\tilde{\mathbb{T}}}^m(t) \subset f_{\mathbb{T}}^m(t)$ and $f_{\mathbb{T}}^m(t) - \{t_K\} = f_{\tilde{\mathbb{T}}}^m(t)$ and $\left|f_{\tilde{\mathbb{T}}}^m(t)\right| \ge 1$.

$$\mathbb{E}[\mathbb{1}_t(\mathbb{T})^2] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})^2] \ge (2^{2+1} - 2^{1+1})B^2 = 4B^2.$$

So we have

$$\begin{aligned}
\sum_{t=m+1}^{T} \mathbb{E}\left[\mathbb{1}_t(\mathbb{T})^2\right] - \sum_{t=m+1}^{T} \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})^2\right] &= \sum_{t=m+1}^{t_K-1} \left(\mathbb{E}\left[\mathbb{1}_t(\mathbb{T})^2\right] - \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})^2\right]\right) + \sum_{t=t_K}^{T} \left(\mathbb{E}\left[\mathbb{1}_t(\mathbb{T})^2\right] - \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})^2\right]\right) \\
&\ge \sum_{t=t_K}^{T} (4B^2) + 0 \\
&= 4(T - t_K + 1)B^2 \\
&> 0
\end{aligned}$$

Next we focus on the cross-product terms. For any $m+1 \le t < t' \le T$ such that $t \le t_K - 1$, $O_{\mathbb{T}}(t,t') = O_{\tilde{\mathbb{T}}}(t,t')$ is totally unchanged.

$$\mathbb{E}[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})] = 0.$$

For any $t_K \le t < t' \le T$, since $t' - m \le T - m \le t_K - 1$, so $f_{\tilde{\mathbb{T}}}(t'-m) < t_K$ and $|O_{\tilde{\mathbb{T}}}(t,t')| \ge 1$ must contain an element. Moreover, $O_{\tilde{\mathbb{T}}}(t,t') \subset O_{\mathbb{T}}(t,t')$. So

$$\mathbb{E}[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})] \ge (2^{2+1} - 2^{1+1})B^2 \ge 4B^2 > 0.$$

So we have

$$\begin{aligned}
&\sum_{m+1 \le t < t' \le T} \mathbb{E}\left[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})\right] - \sum_{m+1 \le t < t' \le T} \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})\right] \\
=& \sum_{\substack{m+1 \le t < t' \le T \\ t \le t_K - 1}} \left(\mathbb{E}\left[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})\right] - \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})\right]\right) + \sum_{t_K \le t < t' \le T} \left(\mathbb{E}\left[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})\right] - \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})\right]\right) \\
\ge& 0
\end{aligned}$$

Combine both square terms and cross-product terms we know that

$$r(\mathbb{T}, \mathbb{Y}) > r(\tilde{\mathbb{T}}, \mathbb{Y}).$$

□

EC.4.3.2. Proof of Lemma 3.

Proof of Lemma 3. Recall that we denote $t_0 = 1$ and $t_{K+1} = T + 1$. First, from Lemma 2, $t_1 \geq m + 2, t_K \leq T - m$. So $k = 1$ and $k = K$ cases both hold. Next, when $2 \leq k \leq K - 1$, we prove by contradiction.

Suppose there exists some optimal design $\mathbb{T}$, such that $\exists 2 \leq k \leq K - 1, s.t.\ t_{k+1} - t_{k-1} \leq m - 1$. Denote

$$\mathbb{K} = \{k \in \{2 : K - 1\} \,|\, t_{k+1} - t_{k-1} \leq m - 1\}.$$

Since $\mathbb{K} \neq \emptyset$, pick $j = \max \mathbb{K}$ to be the largest element in $\mathbb{K}$. Apparently $j \leq K - 1$ since $j \in \{2 : K - 1\}$. We also know that $t_{j+2} \geq t_j + m$, because otherwise $j + 1 \in \mathbb{K}$, which contradicts the maximality of j.

We now construct another design $\tilde{\mathbb{T}}$ such that $\left|\tilde{\mathbb{T}}\right| = K = |\mathbb{T}| - 1$, and the K elements are $\tilde{\mathbb{T}} = \{\tilde{t_0} = 1, \tilde{t_1} = t_1, ..., \tilde{t}_{j-1} = t_{j-1}, \tilde{t_j} = t_{j+1}, ..., \tilde{t}_{K-1} = t_K\}$.

Table EC.4 An example of two regular switchback experiments $\mathbb{T}$ and $\tilde{\mathbb{T}}$ when $m = 4$ and $t_j = t_{j+1} - 1 = t_{j-1} + 2$.

	...	t_{j-1}	$t_{j-1} + 1$	t_j	t_{j+1}	$t_{j+1} + 1$	$t_{j+1} + 2$	t_{j+2}	...
$\mathbb{T}$	...	✓	–	✓	✓	–	–	✓	...
$\tilde{\mathbb{T}}$	...	✓	–	–	✓	–	–	✓	...

Each checkmark beneath a number indicates that this number is within that set; and each dash beneath a number indicates that this number is not within that set. For example, the checkmark ✓ beneath number t_j indicates that $t_j \in \mathbb{T}$; and the dash – beneath number t_j indicates that $t_j \neq \tilde{\mathbb{T}}$.

Next we argue that when $\mathbb{Y} = \mathbb{Y}^+$ or $\mathbb{Y} = \mathbb{Y}^-$,

$$r(\mathbb{T}, \mathbb{Y}) > r(\tilde{\mathbb{T}}, \mathbb{Y}),$$

which suggests that $\mathbb{T}$ is not the optimal design.

First focus on the squared terms. When $t \leq t_j - 1$, $f^m_{\mathbb{T}}(t) = f^m_{\tilde{\mathbb{T}}}(t)$ is totally unchanged.

$$\mathbb{E}[\mathbb{1}_t(\mathbb{T})^2] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})^2] = 0.$$

When $t_j \leq t \leq t_j + m - 1$, this suggests that $t - m \leq t_J - 1$ so that $f_{\tilde{\mathbb{T}}} \leq t_j - 1$. So $t_j \notin f^m_{\tilde{\mathbb{T}}}(t), t_j \in f^m_{\mathbb{T}}(t)$. And all the other determining randomization points are unchanged. So $f^m_{\tilde{\mathbb{T}}}(t) \subset f^m_{\mathbb{T}}(t)$ and $f^m_{\mathbb{T}}(t) - \{t_j\} = f^m_{\tilde{\mathbb{T}}}(t)$ and $\left|f^m_{\tilde{\mathbb{T}}}(t)\right| \geq 1$.

$$\mathbb{E}[\mathbb{1}_t(\mathbb{T})^2] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})^2] \geq (2^{2+1} - 2^{1+1})B^2 = 4B^2.$$

When $t_j + m \leq t \leq T$, either (i) $f_{\mathbb{T}}(t - m) = t_j$, in which case $f_{\tilde{\mathbb{T}}}(t - m) = t_{j-1}$. This is the only difference between $f^m_{\mathbb{T}}(t)$ and $f^m_{\tilde{\mathbb{T}}}(t)$, i.e., $f^m_{\mathbb{T}}(t) - \{t_j\} = f^m_{\tilde{\mathbb{T}}}(t) - \{t_{j-1}\}$. So $|f^m_{\mathbb{T}}(t)| = \left|f^m_{\tilde{\mathbb{T}}}(t)\right|$. The second case is (ii) $f_{\mathbb{T}}(t - m) \geq t_{j+1}$, in which case $f^m_{\mathbb{T}}(t) = f^m_{\tilde{\mathbb{T}}}(t)$. Both cases suggest that

$$\mathbb{E}[\mathbb{1}_t(\mathbb{T})^2] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})^2] = 0.$$

So we have

$$
\begin{aligned}
&\sum_{t=m+1}^{T} \mathbb{E}\left[\mathbb{1}_t(\mathbb{T})^2\right] - \sum_{t=m+1}^{T} \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})^2\right] \\
=&\sum_{t=m+1}^{t_j-1} \left(\mathbb{E}\left[\mathbb{1}_t(\mathbb{T})^2\right] - \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})^2\right]\right) + \sum_{t=t_j}^{t_j+m-1} \left(\mathbb{E}\left[\mathbb{1}_t(\mathbb{T})^2\right] - \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})^2\right]\right) + \sum_{t=t_j+m}^{T} \left(\mathbb{E}\left[\mathbb{1}_t(\mathbb{T})^2\right] - \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})^2\right]\right) \\
\geq& 0 + \sum_{t=t_j}^{t_j+m-1} (4B^2) + 0 \\
=&4(m-1)B^2 \\
>&0
\end{aligned}
$$

Next we focus on the cross-product terms. Let $m+1 \leq t < t' \leq T$. There are many cases which we summarize in Table EC.5

Table EC.5 Summary of the differences between cross-product terms under two regular switchback experiments $\mathbb{T}$ and $\tilde{\mathbb{T}}$.

	$\mathbb{T}$	$\tilde{\mathbb{T}}$
$m+1 \leq t \leq t_{j-1}, t < t' \leq T$	unchanged	
$t_{j-1} \leq t \leq t_j - 1, t < t' \leq t_j + m - 1$	unchanged	
$t_{j-1} \leq t \leq t_j - 1, t_j + m \leq t' \leq t_{j+1} + m - 1$	0	$4B^2$
$t_{j-1} \leq t \leq t_j - 1, t_{j+1} + m \leq t' \leq T$	unchanged	
$t_j \leq t < t' \leq t_j + m - 1$	$2^{\left\lvert O_{\mathbb{T}}(t,t')\right\rvert+1}B^2$	$2^{\left\lvert O_{\tilde{\mathbb{T}}}(t,t')\right\rvert+1}B^2$
$t_j \leq t \leq t_j + m - 1, t_j + m \leq t' \leq T$	unchanged	
$t_j + m \leq t < t' \leq T$	unchanged	

We explain Table EC.5.

When $m+1 \leq t \leq t_{j-1}, t < t' \leq T$, all the overlapping randomization points are earlier than $t_{j-1} - 1$, i.e., $\forall a \in O_{\mathbb{T}}(t,t'), a \leq t_{j-1} - 1; \forall a \in O_{\tilde{\mathbb{T}}}(t,t'), a \leq t_{j-1} - 1$. So $t_j \notin O_{\mathbb{T}}(t,t')$, and the overlapping randomization points are unchanged, i.e., $O_{\mathbb{T}}(t,t') = O_{\tilde{\mathbb{T}}}(t,t')$.

When $t_{j-1} \leq t \leq t_j - 1, t < t' \leq t_j + m - 1$, all the overlapping randomization points are earlier than t_{j-1}, i.e., $\forall a \in O_{\mathbb{T}}(t,t'), a \leq t_{j-1}; \forall a \in O_{\tilde{\mathbb{T}}}(t,t'), a \leq t_{j-1}$. So $t_j \notin O_{\mathbb{T}}(t,t')$, and the overlapping randomization points are unchanged, i.e., $O_{\mathbb{T}}(t,t') = O_{\tilde{\mathbb{T}}}(t,t')$.

When $t_{j-1} \leq t \leq t_j - 1, t_j + m \leq t' \leq t_{j+1} + m - 1$, changing from $\mathbb{T}$ to $\tilde{\mathbb{T}}$ increases the expected values. This is because $t' - m \geq t_j > t$. So first, $O_{\mathbb{T}}(t,t') = \emptyset$. But $f_{\tilde{\mathbb{T}}}(t'-m) = t_{j-1}$ and $t_{j-1} \in f_{\mathbb{T}}^m(t)$, which suggests that $t_{j-1} \in O_{\tilde{\mathbb{T}}}(t,t')$. Also, $\forall a \in f_{\mathbb{T}}^m(t'), a \geq t_{j-1}; \forall a \in f_{\mathbb{T}}^m(t), a \leq t_{j-1}$, which suggests that t_{j-1} is the only overlapping element. So, $O_{\tilde{\mathbb{T}}}(t,t') = \{t_{j-1}\}$. In this case,

$$\mathbb{E}[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})] = (0 - 2^{1+1})B^2 = -4B^2.$$

When $t_{j-1} \leq t \leq t_j - 1, t_{j+1} + m \leq t' \leq T$, since $t' - m \geq t_{j+1} > t_j > t$, $O_{\mathbb{T}}(t,t') = O_{\tilde{\mathbb{T}}}(t,t') = \emptyset$.

When $t_j \leq t < t' \leq t_j + m - 1$, $t_j \in O_{\mathbb{T}}(t,t')$ and $t_j \notin O_{\tilde{\mathbb{T}}}(t,t')$. And all the other overlapping randomization points are unchanged, so $O_{\mathbb{T}}(t,t') - \{t_j\} = O_{\tilde{\mathbb{T}}}(t,t')$ and $|O_{\tilde{\mathbb{T}}}(t,t')| \geq 1$. In this case,

$$\mathbb{E}[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})] \geq (2^{2+1} - 2^{1+1})B^2 = 4B^2.$$

When $t_j \leq t \leq t_j + m - 1, t_j + m \leq t' \leq T$, either (i) $f^m_{\mathbb{T}}(t'-m) = t_j$, in which case $f_{\tilde{\mathbb{T}}}(t'-m) = t_{j-1}$. This is the only difference between $O_{\mathbb{T}}(t,t')$ and $O_{\tilde{\mathbb{T}}}(t,t')$, i.e., $O_{\mathbb{T}}(t,t') - \{t_j\} = O_{\tilde{\mathbb{T}}}(t,t') - \{t_{j-1}\}$. $|O_{\mathbb{T}}(t,t')| = |O_{\tilde{\mathbb{T}}}(t,t')|$. The second case is (ii) $f_{\mathbb{T}}(t'-m) \geq t_{j+1}$, in which case $O_{\mathbb{T}}(t,t') = O_{\tilde{\mathbb{T}}}(t,t')$ is unchanged. Both cases suggest that $\mathbb{E}[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})] = 0$.

When $t_j + m \leq t < t' \leq T$, either (i) $f^m_{\mathbb{T}}(t'-m) = t_j$, in which case $f_{\tilde{\mathbb{T}}}(t'-m) = t_{j-1}$. This is the only difference between $O_{\mathbb{T}}(t,t')$ and $O_{\tilde{\mathbb{T}}}(t,t')$, i.e., $O_{\mathbb{T}}(t,t') - \{t_j\} = O_{\tilde{\mathbb{T}}}(t,t') - \{t_{j-1}\}$. $|O_{\mathbb{T}}(t,t')| = |O_{\tilde{\mathbb{T}}}(t,t')|$. The second case is (ii) $f_{\mathbb{T}}(t'-m) \geq t_{j+1}$, in which case $O_{\mathbb{T}}(t,t') = O_{\tilde{\mathbb{T}}}(t,t')$ is unchanged. Both cases suggest that $\mathbb{E}[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})] - \mathbb{E}[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})] = 0$.

So we have

$$\begin{aligned}
&\sum_{m+1 \leq t < t' \leq T} \mathbb{E}\left[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})\right] - \sum_{m+1 \leq t < t' \leq T} \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})\right] \\
=& \sum_{\substack{t_{j-1} \leq t \leq t_j - 1 \\ t_j + m \leq t' \leq t_{j+1} + m - 1}} \left(\mathbb{E}\left[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})\right] - \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})\right]\right) + \sum_{t_j \leq t < t' \leq t_j + m - 1} \left(\mathbb{E}\left[\mathbb{1}_t(\mathbb{T})\mathbb{1}_{t'}(\mathbb{T})\right] - \mathbb{E}\left[\mathbb{1}_t(\tilde{\mathbb{T}})\mathbb{1}_{t'}(\tilde{\mathbb{T}})\right]\right) \\
\geq& \sum_{\substack{t_{j-1} \leq t \leq t_j - 1 \\ t_j + m \leq t' \leq t_{j+1} + m - 1}} \left(-4B^2\right) + \sum_{t_j \leq t < t' \leq t_j + m - 1} \left(4B^2\right) \\
=& -(t_j - t_{j-1})(t_{j+1} - t_j)4B^2 + \frac{m(m-1)}{2}4B^2 \\
\geq& 0
\end{aligned}$$

where the last inequality is because $j \in \mathbb{K}$, $t_{j+1} - t_{j-1} \leq m - 1$, so $(t_j - t_{j-1})(t_{j+1} - t_j) \leq \frac{(m-1)^2}{4} \leq \frac{m(m-1)}{2}$.

Combine both square terms and cross-product terms we know that

$$r(\mathbb{T}, \mathbb{Y}) > r(\tilde{\mathbb{T}}, \mathbb{Y}).$$

□

EC.4.4. Proof of Theorem 2

Proof of Theorem 2. Think of $\mathbb{E}[\mathbb{1}_t^2]$ as $\mathbb{E}[\mathbb{1}_t\mathbb{1}_t]$, so that $r(\eta_{\mathbb{T}}, \mathbb{Y}) = \sum_{t=m+1}^{T}\sum_{t'=m+1}^{T} \mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]$. Then we can decompose the risk function to be

$$(T-m)^2 \cdot r(\eta_{\mathbb{T}}, \mathbb{Y}) = \sum_{\substack{m+1 \leq t, t' \leq T \\ \min\{t,t'\} \leq t_1 - 1}} \mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}] + \sum_{k=1}^{K-1}\left(\sum_{\substack{t_k \leq t, t' \leq T \\ \min\{t,t'\} \leq t_{k+1} - 1}} \mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]\right) + \sum_{t_K \leq t, t' \leq T} \mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}] \tag{EC.8}$$

The core of this proof is to carefully count how many values can each $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}], \forall t,t' \in \{m+1:T\}$ take. See Table EC.6 for an illustration.

Table EC.6 Illustrator of the different values of $\mathbb{E}[\mathbb{1}_t\mathbb{1}_t]$, when $T=17, m=4, \mathbb{T}=\{1,6,8,13\}$.

(1	2	3	4)	5	6	7	8	9	10	11	12	13	14	15	16	17
(✓	–	–	–)	–	✓	–	✓	–	–	–	–	✓	–	–	–	–
			–	4	4	4	4	4								
			✓	4	**8**	**8**	**8**	**8**	4	4						
			–	4	**8**	**8**	**8**	**8**	4	4						
			✓	4	**8**	**8**	**16**	**16**	**8**	**8**	4	4	4	4	4	
			–	4	**8**	**8**	**16**	**16**	**8**	**8**	4	4	4	4	4	
			–		4	4	**8**	**8**	**8**	**8**	4	4	4	4	4	
			–		4	4	**8**	**8**	**8**	**8**	4	4	4	4	4	
			–				4	4	4	4	4	4	4	4	4	
			✓				4	4	4	4	4	**8**	**8**	**8**	**8**	4
			–				4	4	4	4	4	**8**	**8**	**8**	**8**	4
			–				4	4	4	4	4	**8**	**8**	**8**	**8**	4
			–				4	4	4	4	4	**8**	**8**	**8**	**8**	4
			–									4	4	4	4	4

In the second line, each checkmark beneath number t indicates that period $t \in \mathbb{T}$, i.e. there is a randomization point at period t. This table illustrates different values of $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]$ when $t,t' \in \{m+1,T\}$, where the zero values are omitted. The B^2 magnitudes are also omitted.

First we calculate the first block from equation (EC.8). Because $t_1 \geq m+2$, for any t,t' such that $m+1 \leq \min\{t,t'\} \leq t_1 - 1$, $m+1 \leq \max\{t,t'\} \leq t_1+m-1$, we know that the only overlapping randomization point is t_0. So $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}] = 4B^2$. For any t,t' such that $m+1 \leq \min\{t,t'\} \leq t_1 - 1$, $t_1+m \leq \max\{t,t'\} \leq T$, there is no overlapping randomization point so $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}] = 0$.

$$\sum_{\substack{m+1\leq t,t'\leq T \\ \min\{t,t'\}\leq t_1-1}} \mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}] = B^2\left(4\cdot((t_1-1)^2-m^2)\right)$$

Then we calculate the second block from equation (EC.8). For any $k \in [K-1]$, consider $t_k - t_{k-1}$ and $t_{k+1} - t_k$, which jointly determine the values of $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]$ for any t,t', such that $t_k \leq \min\{t,t'\} \leq t_{k+1} - 1$ and $t_k \leq \max\{t,t'\} \leq T$. We will go over each of the four cases below.

(1) When $t_k - t_{k-1} \geq m, t_{k+1} - t_k \geq m$. Due to Lemma EC.4, for all $t,t' \in \{t_k : t_k+m-1\}$, $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}] = 8B^2$, because both $t_{k-1} \leq t-m \leq t_k - 1$ and $t_{k-1} \leq t'-m \leq t_k-1$, and both t_{k-1} and t_k are overlapping randomization points. For all t,t' such that $t_k \leq \min\{t,t'\} \leq t_{k+1}-1$ and $t_k+m \leq \max\{t,t'\} \leq t_{k+1}+m-1$, $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}] = 4B^2$, because $t_k \leq \min\{t,t'\} \leq t_{k+1}-1$ and $t_k \leq \max\{t,t'\} - m \leq t_{k+1}-1$ so only t_k is the overlapping randomization point. For all t,t' such that $t_k \leq \min\{t,t'\} \leq t_{k+1}-1$ and $t_{k+1}+m \leq \max\{t,t'\} \leq T$, $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}] = 0$.

In this case,

$$\sum_{\substack{t_k\le t,t'\le T\\ \min\{t,t'\}\le t_{k+1}-1}} \mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}] = B^2\left(8\cdot m^2+4\cdot((m+t_{k+1}-t_k)^2-2m^2)\right)$$

(2) When $t_k-t_{k-1}\ge m, t_{k+1}-t_k<m$. Due to Lemma EC.4, for all t, t' such that $t_k\le\min\{t,t'\}\le t_{k+1}-1$, $t_k\le\max\{t,t'\}\le t_k+m-1$, $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]=8B^2$, because both $t,t'\le t_k+m-1$, so $t_{k-1}\le t-m\le t_k-1$ and $t_{k-1}\le t'-m\le t_k-1$, and both t_{k-1} and t_k are overlapping randomization points. For all t,t' such that $t_k\le\min\{t,t'\}\le t_{k+1}-1$ and $t_k+m\le\max\{t,t'\}\le t_{k+1}+m-1$, $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]=4B^2$, because $t_k\le\min\{t,t'\}\le t_{k+1}-1$ and $t_k\le\max\{t,t'\}-m\le t_{k+1}-1$ so only t_k is the overlapping randomization point. For all t,t' such that $t_k\le\min\{t,t'\}\le t_{k+1}-1$ and $t_{k+1}+m\le\max\{t,t'\}\le T$, $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]=0$.

In this case,

$$\sum_{\substack{t_k\le t,t'\le T\\ \min\{t,t'\}\le t_{k+1}-1}} \mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}] = B^2\left(8\cdot(m^2-(m-t_{k+1}+t_k)^2)+4\cdot((m+t_{k+1}-t_k)^2-2m^2+(m-t_{k+1}-t_k)^2)\right)$$

(3) When $t_k-t_{k-1}<m, t_{k+1}-t_k\ge m$. Due to Lemma EC.4, for all $t,t'\in\{t_k:t_{k-1}+m-1\}$, $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]=16B^2$, because $t-m\le t_{k-1}-1\le t_k\le t$ and $t'-m\le t_{k-1}-1\le t_k\le t'$ so t_{k-2},t_{k-1},t_k are three determining randomization points. Also $t_k-t_{k-2}\ge m$ so $t_{k-2}\le\min\{t,t'\}-m$ and t_{k-3} is not a determining randomization point. For all t,t' such that $t_k\le\min\{t,t'\}\le t_k+m-1, t_{k-1}+m\le\max\{t,t'\}\le t_k+m-1$, $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]=8B^2$, because $\min\{t,t'\}-m\le t_k-1$ and $t_{k-1}\le\max\{t,t'\}-m\le t_k-1$ so t_{k-1} and t_k are two determining randomization point. For all t,t' such that $t_k\le\min\{t,t'\}\le t_{k+1}-1, t_k+m\le\max\{t,t'\}\le t_{k+1}+m-1$, $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]=4B^2$, because $t_k\le\max\{t,t'\}-m$ so t_k is the only determining randomization point. For all t,t' such that $t_k\le\min\{t,t'\}\le t_{k+1}-1, t_{k+1}+m\le\max\{t,t'\}\le T$, $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]=0$.

In this case,

$$\begin{aligned}\sum_{\substack{t_k\le t,t'\le T\\ \min\{t,t'\}\le t_{k+1}-1}} \mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}] =&B^2\left(16\cdot(m-t_k+t_{k-1})^2+8\cdot(m^2-(m-t_k+t_{k-1})^2)\right.\\ &\left.+4\cdot((m+t_{k+1}-t_k)^2-2m^2)\right)\end{aligned}$$

(4) When $t_k-t_{k-1}<m, t_{k+1}-t_k<m$. Due to Lemma EC.4, for all $t,t'\in\{t_k:t_{k-1}+m-1\}$, $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]=16B^2$, because $t-m\le t_{k-1}-1\le t_k\le t$ and $t'-m\le t_{k-1}-1\le t_k\le t'$ so t_{k-2},t_{k-1},t_k are three determining randomization points. Also $t_k-t_{k-2}\ge m$ so $t_{k-2}\le\min\{t,t'\}-m$ and t_{k-3} is not a determining randomization point. For all t,t' such that $t_k\le\min\{t,t'\}\le t_{k+1}-1, t_{k-1}+m\le\max\{t,t'\}\le t_k+m-1$, $\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]=8B^2$, because $\min\{t,t'\}-m<t_k-1$ and $t_{k-1}\le\max\{t,t'\}-m\le$

$t_k - 1$ so t_{k-1} and t_k are two determining randomization points. For all t, t' such that $t_k \le \min\{t,t'\} \le t_{k+1} - 1, t_k + m \le \max\{t,t'\} \le t_{k+1} + m - 1$, $\mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] = 4B^2$, because $t_k \le \max\{t,t'\} - m$ so t_k is the only determining randomization point. For all t, t' such that $t_k \le \min\{t,t'\} \le t_{k+1} - 1, t_{k+1} + m \le \max\{t,t'\} \le T$, $\mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] = 0$.

In this case,

$$\sum_{\substack{t_k \le t,t' \le T \\ \min\{t,t'\} \le t_{k+1}-1}} \mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] = B^2 \big(16 \cdot (m - t_k + t_{k-1})^2 + 8 \cdot (m^2 - (m - t_k + t_{k-1})^2 - (m - t_{k+1} + t_k)^2) \\ + 4 \cdot ((m + t_{k+1} - t_k)^2 - 2m^2 + (m - t_{k+1} + t_k)^2)\big)$$

Finally we calculate the third block from equation (EC.8). Observe that $T - t_K \ge m$. **(1)** When $t_K - t_{K-1} \ge m$. Due to Lemma EC.4, for all $t, t' \in \{t_K : t_K + m - 1\}$, $\mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] = 8B^2$, because both $t_{K-1} \le t - m \le t_K - 1$ and $t_{K-1} \le t' - m \le t_K - 1$, and both t_{K-1} and t_K are overlapping randomization points. For all t, t' such that $t_K \le \min\{t,t'\} \le T, t_K + m \le \max\{t,t'\} \le T$, $\mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] = 4B^2$, because $t_K \le \max\{t,t'\} - m$ so t_K is the only determining randomization point.

In this case,

$$\sum_{t_K \le t,t' \le T} \mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] = B^2 \left(8 \cdot m^2 + 4 \cdot ((T + 1 - t_K)^2 - m^2)\right)$$

(2) When $t_K - t_{K-1} < m$. Due to Lemma EC.4, for all $t, t' \in \{t_K : t_{K-1} + m - 1\}$, $\mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] = 16B^2$, because $t - m \le t_{K-1} - 1 \le t_K \le t$ and $t' - m \le t_{K-1} - 1 \le t_K \le t'$ so t_{K-2}, t_{K-1}, t_K are three determining randomization points. Also $t_K - t_{K-2} \ge m$ so $t_{K-2} \le \min\{t,t'\} - m$ and t_{K-3} is not a determining randomization point. For all t, t' such that $t_K \le \min\{t,t'\} \le t_K + m - 1, t_{K-1} + m \le \max\{t,t'\} \le t_K + m - 1$, $\mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] = 8B^2$, because $\min\{t,t'\} - m \le t_K - 1$ and $t_{K-1} \le \max\{t,t'\} - m \le t_K - 1$ so t_{K-1} and t_K are two determining randomization points. For all t, t' such that $t_K \le \min\{t,t'\} \le T, t_K + m \le \max\{t,t'\} \le T$, $\mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] = 4B^2$, because $t_K \le \max\{t,t'\} - m$ so t_K is the only determining randomization point.

In this case,

$$\sum_{t_K \le t,t' \le T} \mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}] = B^2 \left(16 \cdot (m - t_K + t_{K-1})^2 + 8 \cdot (m^2 - (m - t_K + t_{K-1})^2) + 4 \cdot ((T + 1 - t_K)^2 - m^2)\right)$$

Now we combine all above together.

Note that whenever there exists $k \in \{2 : K\}$ such that $(t_k - t_{k-1}) < m$, this suggests that in $\sum_{\substack{t_k \le t,t' \le T \\ \min\{t,t'\} \le t_{k+1}-1}} \mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}]$ there is a $16(m - t_k + t_{k-1})^2$; but in $\sum_{\substack{t_{k-1} \le t,t' \le T \\ \min\{t,t'\} \le t_k - 1}} \mathbb{E}[\mathbb{1}_t \mathbb{1}_{t'}]$ there is a

$8(-(m-t_k+t_{k-1})^2)$. So when we sum them up, we break $16(m-t_k+t_{k-1})^2$ into two $8(m-t_k+t_{k-1})^2$, which cancels in two sumations. By telescoping,

$$\begin{aligned}
(T-m)^2\cdot r(\eta_{\mathbb{T}},\mathbb{Y}) &= \sum_{\substack{m+1\le t,t'\le T\\ \min\{t,t'\}\le t_1-1}} \mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}] + \sum_{k=1}^{K-1}\left(\sum_{\substack{t_k\le t,t'\le T\\ \min\{t,t'\}\le t_{k+1}-1}} \mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}]\right) + \sum_{t_K\le t,t'\le T}\mathbb{E}[\mathbb{1}_t\mathbb{1}_{t'}] \\
&= 4B^2\cdot\left((t_1-1)^2-m^2\right) + \sum_{k=1}^{K-1} B^2\cdot\left(8m^2+4\left((m+t_{k+1}-t_k)^2-2m^2+\left((m-t_{k+1}+t_k)^+\right)^2\right)\right) \\
&\quad + B^2\cdot\left(8m^2+4\left((T+1-t_K)^2-m^2\right)\right) \\
&= B^2\cdot\left\{4\sum_{k=0}^{K}(t_{k+1}-t_k)^2+8m(t_K-t_1)+4m^2K-4m^2+4\sum_{k=1}^{K-1}[(m-t_{k+1}+t_k)^+]^2\right\}
\end{aligned}$$

which finishes the proof. $\square$

EC.4.5. Optimal Solutions to the Subset Selection Problem in Theorem 3

EC.4.5.1. Proof of Theorem 3.

Proof of Theorem 3. Consider the problem as we have introduced in (6). Due to Lemma 1, $\mathbb{Y}^+=\{Y_t(\mathbf{1}_{m+1})=Y_t(\mathbf{0}_{m+1})=B\}_{t\in\{m+1:T\}}$ and $\mathbb{Y}^-=\{Y_t(\mathbf{1}_{m+1})=Y_t(\mathbf{0}_{m+1})=-B\}_{t\in\{m+1:T\}}$ are the only two dominating strategies for the adversarial selection of potential outcomes.

Then due to Lemma 2 and Lemma 3, the optimal design of switchback experiment must satisfy the following three conditions.

$$t_1\ge m+2,\qquad t_K\le T-m\qquad t_{k+1}-t_{k-1}\ge m,\ \forall k\in[K].$$

Due to Theorem 2, the risk function of the optimal design of experiment is given by

$$r(\eta_{\mathbb{T}},\mathbb{Y})=\frac{1}{(T-m)^2}\left\{4\sum_{k=1}^{K+1}(t_k-t_{k-1})^2+8m(t_K-t_1)+4m^2K-4m^2+4\sum_{k=2}^{K}[(m-t_k+t_{k-1})^+]^2\right\}B^2$$

So if we further take minimum over $\mathbb{T}\subset[T]$ in the above risk function, we find the optimal solution to the original problem introduced in (6). Note that B^2 is a constant and irrelevant to our decisions, and that T and m are inputs. So we solve

$$\min_{\mathbb{T}\subset[T]}\left\{4\sum_{k=0}^{K}(t_{k+1}-t_k)^2+8m(t_K-t_1)+4m^2K-4m^2+4\sum_{k=1}^{K-1}[(m-t_{k+1}+t_k)^+]^2\right\}$$

as stated in (8).

In particular, if there exists some constant $n\in\mathbb{N}, n\ge 4$, such that $T=nm$, we can explicitly find the optimal design of experiment. Take the continuous relaxation of this problem, such that for any K, $\{1<t_1<t_2<...<t_K<T+1\}\in[1,T+1]^K$.

$$\min_{\substack{K\in\mathbb{N},\\ \{1<t_1<t_2<...<t_K<T+1\}\in[1,T+1]^K}}\left\{4\sum_{k=0}^{K}(t_{k+1}-t_k)^2+8m(t_K-t_1)+4m^2K-4m^2+4\sum_{k=1}^{K-1}[(m-t_{k+1}+t_k)^+]^2\right\}$$

The relaxed problem provides a lower bound to the original subset selection problem as stated in (8). We will argue later that it is a lucky coincidence that the optimal solution to this relaxed problem is also an integer solution.

First we argue that $t_1 - t_0 = t_{K+1} - t_K$. This is because otherwise if $t_1 - t_0 \neq t_{K+1} - t_K$ then denote $a = \frac{t_1 - t_0 + t_{K+1} - t_K}{2}$. We could always pick for any $k \in \{1:K\}$, $\tilde{t}_k = t_k + a - t_1 + 1$, such that $t_{k+1} - t_k$ is unchanged for any $k \in \{1:K-1\}$. The only change in the objective value comes from

$$\left(2a^2\right) - \left((t_1 - t_0)^2 + (t_{K+1} - t_K)^2\right) < 0,$$

which suggests that $t_1 - t_0 \neq t_{K+1} - t_K$ is not optimal.

Second, similarly, we argue that for any $k' < k'' \in [K-1]$, $t_{k'+1} - t_{k'} = t_{k''+1} - t_{k''}$ This is because otherwise if $t_{k'+1} - t_{k'} \neq t_{k''+1} - t_{k''}$ then denote $b = \frac{t_{k'+1} - t_{k'} + t_{k''+1} - t_{k''}}{2}$. We could always pick for any $k \in \{k'+1 : k''\}$, $\tilde{t}_k = t_k + b - (t_{k'+1} - t_{k'})$, such that $t_{k+1} - t_k$ is unchanged for any $k \in \{k'+1 : k''-1\}$. The only change in the objective value comes from

$$\left(2b^2 + 2((m-b)^+)^2\right) - \left((t_{k'+1} - t_{k'})^2 + (t_{k''+1} - t_{k''})^2 + ((m - t_{k'+1} + t_{k'})^+)^2 + ((m - t_{k''+1} + t_{k''})^+)^2\right) < 0,$$

where $x^2 + ((m-x)^+)^2$ is convex and the inequality holds due to Jensen's Inequality. This inequality suggests that $t_{k'+1} - t_{k'} \neq t_{k''+1} - t_{k''}$ is not optimal.

With the above two structural results, we can assume that there exists $a, b > 0$, such that $t_1 - t_0 = t_{K+1} - t_K = a$, and $t_{k+1} - t_k = b, \forall k \in [K-1]$ Also, it must be satisfied that $2a + (K-1)b = T$. Next we replace $K - 1 = \frac{T-2a}{b}$ into the relaxed problem, to have

$$\begin{aligned}
&\min_{a,b>0} \left\{4(2a^2 + (K-1)b^2) + 8m(K-1)b + 4m^2(K-1) + 4(K-1)((m-b)^+)^2\right\} \\
=&\min_{a,b>0} \left\{8a^2 + 4(T-2a)b + 8m(T-2a) + 4m^2\frac{T-2a}{b} + 4\frac{T-2a}{b}((m-b)^+)^2\right\}
\end{aligned}$$

Either when $b \geq m$, the above is to minimize

$$\min_{a,b>0} \left\{8a^2 + 4(T-2a)b + 8m(T-2a) + 4m^2\frac{T-2a}{b}\right\}$$

Note that

$$\begin{aligned}
8a^2 + 4(T-2a)b + 8m(T-2a) + 4m^2\frac{T-2a}{b} =& 8a^2 + 8m(T-2a) + 4(T-2a)\left(b + \frac{m^2}{b}\right) \\
\geq& 8a^2 + 16m(T-2a) \\
=& 8(a-2m)^2 + 16mT - 32m^2 \\
\geq& 16mT - 32m^2
\end{aligned}$$

where the first inequality takes equality if and only if $b = \frac{m^2}{b}$, which suggests $b = m$; the second inequality takes equality if and only if $a = 2m$.

Or when $b \leq m$, the above is to minimize

$$\min_{a,b>0}\left\{8a^2+4(T-2a)b+8m(T-2a)+4m^2\frac{T-2a}{b}+4\frac{T-2a}{b}(m-b)^2\right\}$$

Note that

$$\begin{aligned}8a^2+4(T-2a)b+8m(T-2a)+4m^2\frac{T-2a}{b}+4\frac{T-2a}{b}(m-b)^2 =&8a^2+8(T-2a)\left(b+\frac{m^2}{b}\right)\\ \geq&8a^2+16m(T-2a)\\ =&8(a-2m)^2+16mT-32m^2\\ \geq&16mT-32m^2\end{aligned}$$

where the first inequality takes equality if and only if $b=\frac{m^2}{b}$, which suggests $b=m$; the second inequality takes equality if and only if $a=2m$.

Combining both cases, the optimal solution is when $a=2m$ and $b=m$, which happens to be an integer solution, thus optimal for the subset selection problem. Translating into $t_1,...,t_K$ this suggests that $t_1=2m+1, t_2=3m+1,...,t_K=(n-2)m+1$. □

EC.4.5.2. Solutions in the Imperfect Cases. It is always worth noting that we are taking a design of experiments perspective. So when practically we have control of T, we can pick T to be some multiples of m, which fits our Theorem 3 perfectly. If we do not have control of T, we can always pick a smaller T' such that $T'=\lfloor T/m\rfloor\cdot m$ is some multiples of m.

Nonetheless, from an optimization perspective, we establish the following optimal structures for the subset selection problem as in (8). Recall that $t_{K+1}=T+1$.

LEMMA EC.5. *Under Assumptions 1–3, the optimal design of regular switchback experiment must satisfy the following two conditions,*

$$|(t_1-t_0)-(t_{K+1}-t_K)|\leq 1, \qquad |(t_{j+1}-t_j)-(t_{j'+1}-t_{j'})|\leq 1, \forall 1\leq j,j'\leq K-1.$$

Proof of Lemma EC.5. Prove by contradiction.

Case 1. Suppose there exists some optimal design $\mathbb{T}$, such that $(t_1-t_0)-(t_{K+1}-t_K)\geq 2$. We now construct another design $\tilde{\mathbb{T}}$, such that $\left|\tilde{\mathbb{T}}\right|=K=|\mathbb{T}|$, and the K elements are $\tilde{\mathbb{T}}=\{\tilde{t}_0=1,\tilde{t}_1=t_1-1,\tilde{t}_2=t_2-1,...,\tilde{t}_K=t_K-1\}$. Now check the expression as in (8). Note that $\tilde{t}_{k+1}-\tilde{t}_k=t_{k+1}-t_k$ is unchanged for any $k\in[K-1]$; $\tilde{t}_K-\tilde{t}_1=t_K-t_1$ is unchanged; and $m-\tilde{t}_{k+1}-\tilde{t}_k=m-t_{k+1}-t_k$ in unchanged for any $k\in[K-1]$. But $(\tilde{t}_1-\tilde{t}_0)^2+(\tilde{t}_{K+1}-\tilde{t}_K)^2=(t_1-t_0-1)^2+(t_{K+1}-t_K+1)^2\leq(t_1-t_0)^2+(t_{K+1}-t_K)^2$, because $(t_1-t_0)-(t_{K+1}-t_K)\geq 2$ and due to convexity.

Similarly, if there exists some optimal design $\mathbb{T}$, such that $(t_{K+1}-t_K)-(t_1-t_0)\geq 2$, then construct another design $\tilde{\mathbb{T}}=\{\tilde{t}_0=1,\tilde{t}_1=t_1+1,\tilde{t}_2=t_2+1,...,\tilde{t}_K=t_K+1\}$.

Case 2. Suppose there exists some optimal design $\mathbb{T}$, and there exists $1 \le j < j' \le K-1$ such that $(t_{j+1}-t_j)-(t_{j'+1}-t_{j'}) \ge 2$. We now construct another design $\tilde{\mathbb{T}}$, such that $\left|\tilde{\mathbb{T}}\right| = K = |\mathbb{T}|$, and the K elements are $\tilde{\mathbb{T}} = \{\tilde{t}_0 = 1, \tilde{t}_1 = t_1, ..., \tilde{t}_j = t_j, \tilde{t}_{j+1} = t_{j+1}-1, ..., \tilde{t}_{j'} = t_{j'}-1, \tilde{t}_{j'+1} = t_{j'+1}, ..., \tilde{t}_K = t_K\}$. Now check the expression as in (8). Note that $\tilde{t}_{k+1}-\tilde{t}_k = t_{k+1}-t_k$ is unchanged for any $k \in \{0:K\}$ except j and j'; $\tilde{t}_K - \tilde{t}_1 = t_K - t_1$ is unchanged; and $m-\tilde{t}_{k+1}-\tilde{t}_k = m-t_{k+1}-t_k$ in unchanged for any $k \in [K-1]$ except j and j'. Now focus on j and j'.

$$\begin{aligned}
&(\tilde{t}_{j+1}-\tilde{t}_j)^2 + (\tilde{t}_{j'+1}-\tilde{t}_{j'})^2 + [(m-\tilde{t}_{j+1}+\tilde{t}_j)^+]^2 + [(m-\tilde{t}_{j'+1}+\tilde{t}_{j'})^+]^2 \\
=&(t_{j+1}-t_j-1)^2 + (t_{j'+1}-t_{j'}+1)^2 + [(m-t_{j+1}+t_j+1)^+]^2 + [(m-t_{j'+1}+t_{j'}-1)^+]^2 \\
\le&(t_{j+1}-t_j)^2 + (t_{j'+1}-t_{j'})^2 + [(m-t_{j+1}+t_j)^+]^2 + [(m-t_{j'+1}+t_{j'})^+]^2
\end{aligned}$$

To see why this inequality holds, define $g(x) = x^2 + [(m-x)^+]^2$ and note that $g(x)$ is a univariate convex function. The inequality holds due to $(t_{j+1}-t_j)-(t_{j'+1}-t_{j'}) \ge 2$ and convexity.

Similarly, if there exists some optimal design $\mathbb{T}$, and there exists $1 \le j < j' \le K-1$ such that $(t_{j'+1}-t_{j'})-(t_{j+1}-t_j) \ge 2$. Then construct another design $\tilde{\mathbb{T}} = \{\tilde{t}_0 = 1, \tilde{t}_1 = t_1, ..., \tilde{t}_j = t_j, \tilde{t}_{j+1} = t_{j+1}+1, ..., \tilde{t}_{j'} = t_{j'}+1, \tilde{t}_{j'+1} = t_{j'+1}, ..., \tilde{t}_K = t_K\}$.

Combine both cases we finish the proof. ☐

EC.5. Proofs and Discussions from Section 4

In Section 4 we focus on the case when $p = m$. Throughout this section in the appendix, we use only m instead of p.

EC.5.1. Extra Notations Used in the Proofs from Section 4

For any $t \in \{m+1:T\}$, we use the notations of $\mathbb{1}_t$ as defined in (EC.1). Denote

$$\begin{aligned}
\bar{\mathbb{1}}_0 &= \sum_{t=m+1}^{2m} \mathbb{1}_t \\
\bar{\mathbb{1}}_k &= \sum_{t=(k+1)m+1}^{(k+2)m} \mathbb{1}_t, && \forall k \in [K] \\
\bar{\mathbb{1}}_{K+1} &= \sum_{t=(K+2)m+1}^{(K+3)m} \mathbb{1}_t
\end{aligned}$$

It is worth noting that under the optimal design as suggested by Theorem 3, when $T/m = n \in \mathbb{N}$ is an integer, we have $K = n-3$. So $(K+3)m = T$. See Example EC.1 below.

Example EC.1 (An Optimal Design and Its $\bar{\mathbb{1}}_k$ Notations). When $T = 12$, $p = m = 2$, the optimal design of regular switchback experiment is $\mathbb{T}^* = \{1, 5, 7, 9\}$, and $K = 3$. The $\bar{\mathbb{1}}_k$ notations are defined below. Each $\bar{\mathbb{1}}_k$ spans $m = 2$ periods. See Table EC.7. ☐

Table EC.7 **An example of the optimal design $\mathbb{T}^*$ and its $\bar{\mathbb{1}}_k$ notations when $T = 12$ and $p = m = 2$.**

	1	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{T}^*$	✓	–	–	–	✓	–	✓	–	✓	–	–	–
$\{\bar{\mathbb{1}}_k\}_{k=0}^{K+1}$	–		$\bar{\mathbb{1}}_0$		$\bar{\mathbb{1}}_1$		$\bar{\mathbb{1}}_2$		$\bar{\mathbb{1}}_3$		$\bar{\mathbb{1}}_4$	

Using the above notation, we could write

$$\hat{\tau}_m - \tau_m = \frac{1}{T-m}\sum_{k=0}^{K+1} \bar{\mathbb{1}}_k,$$

and so

$$\mathsf{Var}(\hat{\tau}_m) = \frac{1}{(T-m)^2}\mathsf{Var}\left(\sum_{k=0}^{K+1} \bar{\mathbb{1}}_k\right).$$

EC.5.2. Proof of Theorem 4

The proof of Theorem 4 resembles the proof of Lemmas EC.2 and EC.3. The trick here is to observe that for any $k \in [K]$, the values of all the variables $\mathbb{1}_t$, where $(k+1)m+1 \le t \le (k+2)m$, are all determined by the randomization at time $km+1$ and $(k+1)m+1$. Since they are all correlated, we can use $\bar{\mathbb{1}}_k$ to stand for $\sum_{t=(k+1)m+1}^{(k+2)m} \mathbb{1}_t$ for short.

Proof of Theorem 4. First observe that $\bar{\mathbb{1}}_k$ has zero mean for each $k \in \{0 : K+1\}$. So we can decompose the variance into squared terms and cross-product terms,

$$(T-m)^2\mathsf{Var}(\hat{\tau}_m) = \mathsf{Var}\left(\sum_{k=0}^{K+1} \bar{\mathbb{1}}_k\right) = \sum_{k=0}^{K+1} \mathbb{E}\left[\bar{\mathbb{1}}_k^2\right] + \sum_{0 \le k < k' \le K+1} 2\mathbb{E}\left[\bar{\mathbb{1}}_k \bar{\mathbb{1}}_{k'}\right].$$

We focus on the variance of the squared terms first,

$$\mathbb{E}\left[\bar{\mathbb{1}}_k^2\right] = \begin{cases} \bar{Y}_0(\mathbf{1}_{m+1})^2 + \bar{Y}_0(\mathbf{0}_{m+1})^2 + 2\bar{Y}_0(\mathbf{1}_{m+1})\bar{Y}_0(\mathbf{0}_{m+1}), & \text{if } \ k = 0 \\ 3\bar{Y}_k(\mathbf{1}_{m+1})^2 + 3\bar{Y}_k(\mathbf{0}_{m+1})^2 + 2\bar{Y}_k(\mathbf{1}_{m+1})\bar{Y}_k(\mathbf{0}_{m+1}), & \text{if } \ 1 \le k \le K \\ \bar{Y}_{K+1}(\mathbf{1}_{m+1})^2 + \bar{Y}_{K+1}(\mathbf{0}_{m+1})^2 + 2\bar{Y}_{K+1}(\mathbf{1}_{m+1})\bar{Y}_{K+1}(\mathbf{0}_{m+1}), & \text{if } \ k = K+1 \end{cases}$$

This is because when $k = 0$ or $k = K+1$, then with probability 1/2, $\bar{\mathbb{1}}_k = \bar{Y}_0(\mathbf{1}_{m+1}) + \bar{Y}_0(\mathbf{0}_{m+1})$; with probability 1/2, $\bar{\mathbb{1}}_k = -\bar{Y}_0(\mathbf{1}_{m+1}) - \bar{Y}_0(\mathbf{0}_{m+1})$. When $k \in [K]$, with probability 1/4, $\bar{\mathbb{1}}_k = 3\bar{Y}_0(\mathbf{1}_{m+1}) + \bar{Y}_0(\mathbf{0}_{m+1})$; with probability 1/2, $\bar{\mathbb{1}}_k = -\bar{Y}_0(\mathbf{1}_{m+1}) + \bar{Y}_0(\mathbf{0}_{m+1})$; with probability 1/4, $\bar{\mathbb{1}}_k = -\bar{Y}_0(\mathbf{1}_{m+1}) - 3\bar{Y}_0(\mathbf{0}_{m+1})$.

Then for the cross-product terms, if $k' - k \ge 2$, then $\bar{\mathbb{1}}_k$ and $\bar{\mathbb{1}}_{k'}$ are independent, i.e., $\mathbb{E}\left[\bar{\mathbb{1}}_k \bar{\mathbb{1}}_{k'}\right] = 0$. If $k' - k = 1$, then

$$\mathbb{E}\left[\bar{\mathbb{1}}_k \bar{\mathbb{1}}_{k+1}\right] = (\bar{Y}_k(\mathbf{1}_{m+1}) + \bar{Y}_k(\mathbf{0}_{m+1})) \cdot (\bar{Y}_{k+1}(\mathbf{1}_{m+1}) + \bar{Y}_{k+1}(\mathbf{0}_{m+1}))$$

This is because the values of $\bar{\mathbb{1}}_k$ and $\bar{\mathbb{1}}_{k+1}$ are determined by the realization at 3 randomization points, $W_{km+1}, W_{(k+1)m+1}, W_{(k+2)m+1}$. With probability 1/8, $\bar{\mathbb{1}}_k \bar{\mathbb{1}}_{k+1} = (3\bar{Y}_k(\mathbf{1}_{m+1}) +$

$\bar{Y}_k(\mathbf{0}_{m+1}))\cdot(3\bar{Y}_{k+1}(\mathbf{1}_{m+1})+\bar{Y}_{k+1}(\mathbf{0}_{m+1}))$; with probability 1/8, $\bar{\mathbb{1}}_k\bar{\mathbb{1}}_{k+1}=(3\bar{Y}_k(\mathbf{1}_{m+1})+\bar{Y}_k(\mathbf{0}_{m+1}))\cdot(-\bar{Y}_{k+1}(\mathbf{1}_{m+1})+\bar{Y}_{k+1}(\mathbf{0}_{m+1}))$; with probability 1/8, $\bar{\mathbb{1}}_k\bar{\mathbb{1}}_{k+1}=(-\bar{Y}_k(\mathbf{1}_{m+1})+\bar{Y}_k(\mathbf{0}_{m+1}))\cdot(3\bar{Y}_{k+1}(\mathbf{1}_{m+1})+\bar{Y}_{k+1}(\mathbf{0}_{m+1}))$; with probability 1/8, $\bar{\mathbb{1}}_k\bar{\mathbb{1}}_{k+1}=(-\bar{Y}_k(\mathbf{1}_{m+1})+\bar{Y}_k(\mathbf{0}_{m+1}))\cdot(-\bar{Y}_{k+1}(\mathbf{1}_{m+1})+\bar{Y}_{k+1}(\mathbf{0}_{m+1}))$; with probability 1/8, $\bar{\mathbb{1}}_k\bar{\mathbb{1}}_{k+1}=(-\bar{Y}_k(\mathbf{1}_{m+1})+\bar{Y}_k(\mathbf{0}_{m+1}))\cdot(-\bar{Y}_{k+1}(\mathbf{1}_{m+1})-3\bar{Y}_{k+1}(\mathbf{0}_{m+1}))$; with probability 1/8, $\bar{\mathbb{1}}_k\bar{\mathbb{1}}_{k+1}=(-\bar{Y}_k(\mathbf{1}_{m+1})-3\bar{Y}_k(\mathbf{0}_{m+1}))\cdot(-\bar{Y}_{k+1}(\mathbf{1}_{m+1})+\bar{Y}_{k+1}(\mathbf{0}_{m+1}))$; with probability 1/8, $\bar{\mathbb{1}}_k\bar{\mathbb{1}}_{k+1}=(-\bar{Y}_k(\mathbf{1}_{m+1})-3\bar{Y}_k(\mathbf{0}_{m+1}))\cdot(-\bar{Y}_{k+1}(\mathbf{1}_{m+1})-3\bar{Y}_{k+1}(\mathbf{0}_{m+1}))$.

Combining the squared terms and the cross-product terms we finish the proof. □

EC.5.3. Discssions and proof of Corollary 1

We first provide the details of the two variance upper bounds here.

$$\begin{aligned}\mathsf{Var}^{\mathsf{U1}}(\hat{\tau}_m)=\frac{1}{(T-m)^2}&\Bigg\{3\left[\bar{Y}_0(\mathbf{1}_{m+1})^2+\bar{Y}_0(\mathbf{0}_{m+1})^2\right]+\sum_{k=1}^{n-3}6\left[\bar{Y}_k(\mathbf{1}_{m+1})^2+\bar{Y}_k(\mathbf{0}_{m+1})^2\right]\\&+4\left[\bar{Y}_{n-2}(\mathbf{1}_{m+1})^2+\bar{Y}_{n-2}(\mathbf{0}_{m+1})^2\right]+\sum_{k=0}^{n-3}2\left[\bar{Y}_k(\mathbf{1}_{m+1})\cdot\bar{Y}_{k+1}(\mathbf{1}_{m+1})+\bar{Y}_k(\mathbf{0}_{m+1})\cdot\bar{Y}_{k+1}(\mathbf{0}_{m+1})\right]\Bigg\},\end{aligned}$$

and

$$\begin{aligned}\mathsf{Var}^{\mathsf{U2}}(\hat{\tau}_m)=\frac{1}{(T-m)^2}&\Bigg\{4\left[\bar{Y}_0(\mathbf{1}_{m+1})^2+\bar{Y}_0(\mathbf{0}_{m+1})^2\right]+\sum_{k=1}^{n-3}8\left[\bar{Y}_k(\mathbf{1}_{m+1})^2+\bar{Y}_k(\mathbf{0}_{m+1})^2\right]\\&+4\left[\bar{Y}_{n-2}(\mathbf{1}_{m+1})^2+\bar{Y}_{n-2}(\mathbf{0}_{m+1})^2\right]\Bigg\}.\end{aligned}$$

We prove Corollary 1 using the basic inequality that $2xy\leq x^2+y^2$. Such an inequality is commonly used to find a conservative upper bound of the variance.

Proof of Corollary 1. From Theorem 4, the variance of the estimator is given by

$$\begin{aligned}&(T-m)^2\mathsf{Var}(\hat{\tau}_m)\\\leq&2\left\{\bar{Y}_0(\mathbf{1}_{m+1})^2+\bar{Y}_0(\mathbf{0}_{m+1})^2\right\}+\sum_{k=1}^{n-3}4\left\{\bar{Y}_k(\mathbf{1}_{m+1})^2+\bar{Y}_k(\mathbf{0}_{m+1})^2\right\}+2\left\{\bar{Y}_{n-2}(\mathbf{1}_{m+1})^2+\bar{Y}_{n-2}(\mathbf{0}_{m+1})^2\right\}\\&+\sum_{k=0}^{n-3}2\left[\bar{Y}_k(\mathbf{1}_{m+1})+\bar{Y}_k(\mathbf{0}_{m+1})\right]\cdot\left[\bar{Y}_{k+1}(\mathbf{1}_{m+1})+\bar{Y}_{k+1}(\mathbf{0}_{m+1})\right]\\\leq&2\left\{\bar{Y}_0(\mathbf{1}_{m+1})^2+\bar{Y}_0(\mathbf{0}_{m+1})^2\right\}+\sum_{k=1}^{n-3}4\left\{\bar{Y}_k(\mathbf{1}_{m+1})^2+\bar{Y}_k(\mathbf{0}_{m+1})^2\right\}+2\left\{\bar{Y}_{n-2}(\mathbf{1}_{m+1})^2+\bar{Y}_{n-2}(\mathbf{0}_{m+1})^2\right\}\\&+\sum_{k=0}^{n-3}\left\{2\bar{Y}_k(\mathbf{1}_{m+1})\bar{Y}_{k+1}(\mathbf{1}_{m+1})+2\bar{Y}_k(\mathbf{0}_{m+1})\bar{Y}_{k+1}(\mathbf{0}_{m+1})+\bar{Y}_k(\mathbf{1}_{m+1})^2+\bar{Y}_k(\mathbf{0}_{m+1})^2+\bar{Y}_{k+1}(\mathbf{1}_{m+1})^2+\bar{Y}_{k+1}(\mathbf{0}_{m+1})^2\right\}\\\leq&3\left\{\bar{Y}_0(\mathbf{1}_{m+1})^2+\bar{Y}_0(\mathbf{0}_{m+1})^2\right\}+\sum_{k=1}^{n-3}6\left\{\bar{Y}_k(\mathbf{1}_{m+1})^2+\bar{Y}_k(\mathbf{0}_{m+1})^2\right\}+3\left\{\bar{Y}_{n-2}(\mathbf{1}_{m+1})^2+\bar{Y}_{n-2}(\mathbf{0}_{m+1})^2\right\}\end{aligned}$$

$$+\sum_{k=0}^{n-3}\left\{\bar{Y}_k(\mathbf{1}_{m+1})^2+\bar{Y}_k(\mathbf{0}_{m+1})^2+\bar{Y}_{k+1}(\mathbf{1}_{m+1})^2+\bar{Y}_{k+1}(\mathbf{0}_{m+1})^2\right\}$$

$$=4\left\{\bar{Y}_0(\mathbf{1}_{m+1})^2+\bar{Y}_0(\mathbf{0}_{m+1})^2\right\}+\sum_{k=1}^{n-3}8\left\{\bar{Y}_k(\mathbf{1}_{m+1})^2+\bar{Y}_k(\mathbf{0}_{m+1})^2\right\}+4\left\{\bar{Y}_{n-2}(\mathbf{1}_{m+1})^2+\bar{Y}_{n-2}(\mathbf{0}_{m+1})^2\right\}$$

where the first inequality suggests $\mathsf{Var}(\hat{\tau}_m)\leq\mathsf{Var}^{\mathsf{U1}}(\hat{\tau}_m)$, and the last inequality suggests $\mathsf{Var}^{\mathsf{U1}}(\hat{\tau}_m)\leq\mathsf{Var}^{\mathsf{U2}}(\hat{\tau}_m)$.

The unbiasedness part is due to the estimator of the variances being Horvitz-Thompson type estimators, and that regular switchback experiments naturally satisfy Assumption 4. ☐

EC.5.4. Proof of Theorem 5

We prove Theorem 5 by using Lemma EC.1. In particular, we derive $B^2_{n,k,a}$, and then construct some proper Δ_n, K_n, and L_n.

Proof of Theorem 5. In the n-replica experiment, $\hat{\tau}_m-\tau_m=\frac{1}{(n-1)m}\sum_{k=0}^{n-2}\bar{\mathbb{1}}_k$, and $\mathsf{Var}(\hat{\tau}_m)=\frac{1}{(n-1)^2m^2}\mathsf{Var}\left(\sum_{k=0}^{n-2}\bar{\mathbb{1}}_k\right)$. To use the language from Lemma EC.1, denote $d=n-1$. Denote for any $i\in[n-1]$, $X_{n,i}=\frac{1}{(n-1)m}\bar{\mathbb{1}}_{i-1}$ so we know that $\phi=1$, i.e., $\{X_{n,1},X_{n,2},...\}$ is a sequence of 1-dependent random variables.

First note that $B^2_n=\mathsf{Var}(\hat{\tau}_m)$, and we calculate $B^2_{n,k,a}$ as follows.

$$\begin{aligned}
B^2_{n,k,a}&=\frac{1}{(n-1)^2m^2}\mathsf{Var}\left(\sum_{i=a}^{a+k-1}\bar{\mathbb{1}}_{i-1}\right)\\
&\leq\frac{1}{(n-1)^2m^2}\left\{\sum_{i=a}^{a+k-1}\left[3\bar{Y}_{i-1}(\mathbf{1}_{m+1})^2+3\bar{Y}_{i-1}(\mathbf{0}_{m+1})^2+2\bar{Y}_{i-1}(\mathbf{1}_{m+1})\bar{Y}_{i-1}(\mathbf{0}_{m+1})\right]\right.\\
&\qquad\left.+\sum_{i=a}^{a+k-2}2[\bar{Y}_{i-1}(\mathbf{1}_{m+1})+\bar{Y}_{i-1}(\mathbf{0}_{m+1})]\cdot[\bar{Y}_i(\mathbf{1}_{m+1})+\bar{Y}_i(\mathbf{0}_{m+1})]\right\}\\
&\leq\frac{8km^2B^2+8(k-1)m^2B^2}{(n-1)^2m^2}\\
&\leq\frac{16kB^2}{(n-1)^2}
\end{aligned}$$

Pick $\gamma=0,\delta=1$, then $\Delta_n=B^3/(n-1)^3$, $K_n=16B^2/(n-1)^2$, and $L_n=\mathsf{Var}(\hat{\tau}_m)/(n-1)$.

We check that all the five conditions from Lemma EC.1 are satisfied.

1. $\mathbb{E}|X_{n,i}|^3\leq\Delta_n=B^3/(n-1)^3$, because all the potential outcomes are bounded by B, so that $X_{n,i}\leq B/(n-1)$.
2. $B^2_{n,k,a}/k\leq K_n=16B^2/(n-1)^2$.
3. $B^2_n/(n-1)\geq L_n=\mathsf{Var}(\hat{\tau}_m)/(n-1)$.
4. $K_n/L_n=16B^2/(n-1)\mathsf{Var}(\hat{\tau}_m)=O(1)$, where the last equality is due to Assumption 5.
5. $\Delta_n/L_n^{3/2}=B^3/(n-1)^{3/2}\mathsf{Var}(\hat{\tau}_m)^{3/2}=O(1)$, where the last equality is due to Assumption 5.

Due to Lemma EC.1,

$$\frac{\hat{\tau}_m - \tau_m}{\sqrt{\mathsf{Var}(\hat{\tau}_m)}} \xrightarrow{D} \mathcal{N}(0,1).$$

□

EC.6. Proofs and Discussions from Section 5

In Section 5 we discuss the cases when m is misspecified. Throughout this section in the appendix, we use both p and m. Recall that m is the order of carryover effect, and p is the experimenter's knowledge of m.

EC.6.1. Unbiasedness of the Horvitz-Thompson Estimator when m is Misspecified

We state here the omitted mathematics in Theorem 6.

Under Assumptions 2 and 3, for $m > p$, at each time $t \geq m+1$, the Horvitz-Thompson estimator is either unbiased for the lag-m causal effect when $f(t-p) \leq t-m$, i.e.,

$$\mathbb{E}_{\boldsymbol{W}_{1:T}\sim\eta_{\mathbb{T}}}\left[Y_t^{\mathsf{obs}}\frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t}=\mathbf{1}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t}=\mathbf{1}_{p+1})} - Y_t^{\mathsf{obs}}\frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t}=\mathbf{0}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t}=\mathbf{0}_{p+1})}\right] = Y_t(\mathbf{1}_{m+1}) - Y_t(\mathbf{0}_{m+1}),$$

or conditionally unbiased for the m-misspecified lag-p causal effect when $f(t-p) > t-m$, i.e.,

$$\mathbb{E}_{\boldsymbol{W}_{1:T}\sim\eta_{\mathbb{T}}}\left[\left\{Y_t^{\mathsf{obs}}\frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t}=\mathbf{1}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t}=\mathbf{1}_{p+1})} - Y_t^{\mathsf{obs}}\frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t}=\mathbf{0}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t}=\mathbf{0}_{p+1})}\right\} - \right.$$
$$\left.\left\{Y_t(\boldsymbol{w}^{\mathsf{obs}}_{t-m:f(t-p)-1}, \mathbf{1}_{t-f(t-p)+1}) - Y_t(\boldsymbol{w}^{\mathsf{obs}}_{t-m:f(t-p)-1}, \mathbf{0}_{t-f(t-p)+1})\right\} \middle| \boldsymbol{W}_{t-m:f(t-p)-1} = \boldsymbol{w}^{\mathsf{obs}}_{t-m:f(t-p)-1}\right] = 0.$$

When $p+1 \leq t \leq m$, the Horvitz-Thompson estimator is either unbiased for the lag-t causal effect when $f(t-p) = 1$, i.e.,

$$\mathbb{E}_{\boldsymbol{W}_{1:T}\sim\eta_{\mathbb{T}}}\left[Y_t^{\mathsf{obs}}\frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t}=\mathbf{1}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t}=\mathbf{1}_{p+1})} - Y_t^{\mathsf{obs}}\frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t}=\mathbf{0}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t}=\mathbf{0}_{p+1})}\right] = Y_t(\mathbf{1}_t) - Y_t(\mathbf{0}_t),$$

or conditionally unbiased for the m-misspecified lag-t causal effect when $f(t-p) > 1$, i.e.,

$$\mathbb{E}_{\boldsymbol{W}_{1:T}\sim\eta_{\mathbb{T}}}\left[\left\{Y_t^{\mathsf{obs}}\frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t}=\mathbf{1}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t}=\mathbf{1}_{p+1})} - Y_t^{\mathsf{obs}}\frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t}=\mathbf{0}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t}=\mathbf{0}_{p+1})}\right\} - \right.$$
$$\left.\left\{Y_t(\boldsymbol{w}^{\mathsf{obs}}_{1:f(t-p)-1}, \mathbf{1}_{t-f(t-p)+1}) - Y_t(\boldsymbol{w}^{\mathsf{obs}}_{1:f(t-p)-1}, \mathbf{0}_{t-f(t-p)+1})\right\} \middle| \boldsymbol{W}_{1:f(t-p)-1} = \boldsymbol{w}^{\mathsf{obs}}_{1:f(t-p)-1}\right] = 0.$$

To remove the conditional expectation, we can further take an outer loop of expectation averaged over the past assignment paths. So the estimator is estimating a weighted average of lag-p effects. When $t \geq m+1$,

$$\sum_{\boldsymbol{w}_{t-m:f(t-p)-1}} \Pr(\boldsymbol{W}_{t-m:f(t-p)-1} = \boldsymbol{w}_{t-m:f(t-p)-1})(Y_t(\boldsymbol{w}_{t-m:f(t-p)-1}, \mathbf{1}_{t-f(t-p)+1}) - Y_t(\boldsymbol{w}_{t-m:f(t-p)-1}, \mathbf{0}_{t-f(t-p)+1})),$$

and when $p+1 \leq t \leq m$,

$$\sum_{\boldsymbol{w}_{1:f(t-p)-1}} \Pr(\boldsymbol{W}_{1:f(t-p)-1} = \boldsymbol{w}_{1:f(t-p)-1})(Y_t(\boldsymbol{w}_{1:f(t-p)-1}, \mathbf{1}_{t-f(t-p)+1}) - Y_t(\boldsymbol{w}_{1:f(t-p)-1}, \mathbf{0}_{t-f(t-p)+1})).$$

We prove Theorem 6 as follows.

Proof of Theorem 6. Focus on any specific $t \in \{m+1:T\}$.

When $f(t-p) \leq t-m$, both $0 < \Pr(\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1}), \Pr(\boldsymbol{W}_{t-p:t} = \mathbf{0}_{p+1}) < 1$. With probability $\Pr(\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1}) \neq 0$, $\mathbb{1}\{\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1}\} = 1$, and $Y_t^{\mathsf{obs}} = Y_t(\mathbf{1}_{m+1})$. So $\mathbb{E}\left[Y_t^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t}=\mathbf{1}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t}=\mathbf{1}_{p+1})}\right] = Y_t(\mathbf{1}_{m+1})$. Similarly $\mathbb{E}\left[Y_t^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t}=\mathbf{0}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t}=\mathbf{0}_{p+1})}\right] = Y_t(\mathbf{0}_{m+1})$. So

$$\mathbb{E}_{\boldsymbol{W}_{1:T} \sim \eta_{\mathbb{T}}} \left[\left\{ Y_t^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1})} - Y_t^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t} = \mathbf{0}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t} = \mathbf{0}_{p+1})} \right\}\right] = Y_t(\mathbf{1}_{m+1}) - Y_t(\mathbf{0}_{m+1}).$$

When $f(t-p) > t-m$, both $0 < \Pr\left(\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1} \middle| \boldsymbol{W}_{t-m:f(t-p)-1} = \boldsymbol{w}^{\mathsf{obs}}_{t-m:f(t-p)-1}\right) < 1$ and $0 < \Pr\left(\boldsymbol{W}_{t-p:t} = \mathbf{0}_{p+1} \middle| \boldsymbol{W}_{t-m:f(t-p)-1} = \boldsymbol{w}^{\mathsf{obs}}_{t-m:f(t-p)-1}\right) < 1$. Conditional on $\boldsymbol{W}_{t-m:f(t-p)-1} = \boldsymbol{w}^{\mathsf{obs}}_{t-m:f(t-p)-1}$, we know that with probability $\Pr\left(\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1} \middle| \boldsymbol{W}_{t-m:f(t-p)-1} = \boldsymbol{w}^{\mathsf{obs}}_{t-m:f(t-p)-1}\right) \neq 0$, $\mathbb{1}\{\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1}\} = 1$, and $Y_t^{\mathsf{obs}} = Y_t(\boldsymbol{w}^{\mathsf{obs}}_{t-m:f(t-p)-1}, \mathbf{1}_{t-f(t-p)+1})$. So

$$\mathbb{E}_{\boldsymbol{W}_{1:T} \sim \eta_{\mathbb{T}}} \left[Y_t^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t} = \mathbf{1}_{p+1})} - Y_t(\boldsymbol{w}^{\mathsf{obs}}_{t-m:f(t-p)-1}, \mathbf{1}_{t-f(t-p)+1}) \middle| \boldsymbol{W}_{t-m:f(t-p)-1} = \boldsymbol{w}^{\mathsf{obs}}_{t-m:f(t-p)-1} \right] = 0.$$

Similarly, we have

$$\mathbb{E}_{\boldsymbol{W}_{1:T} \sim \eta_{\mathbb{T}}} \left[Y_t^{\mathsf{obs}} \frac{\mathbb{1}\{\boldsymbol{W}_{t-p:t} = \mathbf{0}_{p+1}\}}{\Pr(\boldsymbol{W}_{t-p:t} = \mathbf{0}_{p+1})} - Y_t(\boldsymbol{w}^{\mathsf{obs}}_{t-m:f(t-p)-1}, \mathbf{0}_{t-f(t-p)+1}) \middle| \boldsymbol{W}_{t-m:f(t-p)-1} = \boldsymbol{w}^{\mathsf{obs}}_{t-m:f(t-p)-1} \right] = 0,$$

which finishes the proof. ☐

EC.6.2. Asymptotic Normality when m is Misspecified

The proof of Corollary 2 consists of two parts: $m < p$ and $m > p$. When $m < p$ we consult Theorems 4 and 5. When $m > p$ we prove Corollary 2 by using Lemma EC.1. In particular, we derive $B^2_{n,k,a}$, and then construct some proper Δ_n, K_n, and L_n.

Proof of Corollary 2. The proof consists of two parts: $m < p$ and $m > p$. First, when $m < p$, we know that $\hat{\tau}_p = \hat{\tau}_m, \tau_p = \tau_m, \mathsf{Var}(\hat{\tau}_p) = \mathsf{Var}(\hat{\tau}_m)$. Due to Theorems 4 we prove part (i) the expression in (11). Due to Theorem 5 we know that

$$\frac{\hat{\tau}_p - \tau_p}{\sqrt{\mathsf{Var}(\hat{\tau}_p)}} = \frac{\hat{\tau}_m - \tau_m}{\sqrt{\mathsf{Var}(\hat{\tau}_m)}} \xrightarrow{D} \mathcal{N}(0,1).$$

Second, when $m > p$, then we follow the same trick as in Theorem 5. In the n-replica experiment, $\hat{\tau}_p - \mathbb{E}[\tau_p^{[m]}] = \frac{1}{(n-1)p} \sum_{k=0}^{n-2} \bar{\mathbb{1}}_k$, and $\mathsf{Var}(\hat{\tau}_p) = \frac{1}{(n-1)^2 p^2} \mathsf{Var}\left(\sum_{k=0}^{n-2} \bar{\mathbb{1}}_k\right)$. To use the language from Lemma EC.1, denote $d = n-1$. Denote for any $i \in [n-1]$, $X_{n,i} = \frac{1}{(n-1)p} \bar{\mathbb{1}}_{i-1}$. We know that

Table EC.8 An illustration of ϕ when $m = 5, p = 3$.

$\xrightarrow{\hspace{12em}}$

carryover effect

	...	13	14	15	16	17	18	19	20	21	22	23	24	...
$\mathbb{T}^*$		✓	–	–	✓	–	–	✓	–	–	✓	–	–	
$\{\bar{\mathbb{1}}_k\}_{k=0}^{K+1}$		$\bar{\mathbb{1}}_3$			$\bar{\mathbb{1}}_4$			$\bar{\mathbb{1}}_5$			$\bar{\mathbb{1}}_6$			

In this example $\phi = \lceil \frac{m}{p} \rceil = 2$. The arrow above numbers 17 through 22 means that the assignment on period 17 affects the outcome on period 22. So that $\bar{\mathbb{1}}_4$ and $\bar{\mathbb{1}}_6$ are correlated, but $\bar{\mathbb{1}}_3$ and $\bar{\mathbb{1}}_6$ are independent.

$\phi = \lceil \frac{m}{p} \rceil$, so that $\{X_{n,1}, X_{n,2}, ...\}$ is a sequence of ϕ-dependent random variables. See Table EC.8 for an illustration of ϕ.

First note that $B_n^2 = \mathsf{Var}(\hat{\tau}_p)$, and we calculate $B_{n,k,a}^2$ as follows. Note that $k \geq \phi + 1$.

$$
\begin{aligned}
B_{n,k,a}^2 &= \frac{1}{(n-1)^2 p^2} \mathsf{Var}\left(\sum_{i=a}^{a+k-1} \bar{\mathbb{1}}_{i-1} \right) \\
&\leq \frac{1}{(n-1)^2 p^2} \left(\sum_{i=a}^{a+k-1} \mathbb{E}[\bar{\mathbb{1}}_{i-1}^2] + \sum_{i=a}^{a+k-2} 2\mathbb{E}[\bar{\mathbb{1}}_{i-1} \bar{\mathbb{1}}_i] + ... + \sum_{i=a}^{a+k-1+\phi} 2\mathbb{E}[\bar{\mathbb{1}}_{i-1} \bar{\mathbb{1}}_{i-1+\phi}] \right) \\
&\leq \frac{Cp^2 B^2}{(n-1)^2 p^2} \cdot (k + (k-1) + ... + (k - \phi)) \\
&\leq \frac{(\phi+1) CkB^2}{(n-1)^2}
\end{aligned}
$$

where C is some constant bounding the number of terms in each cross-product expectation $2\mathbb{E}[\bar{\mathbb{1}}_{i-1} \bar{\mathbb{1}}_i], ..., 2\mathbb{E}[\bar{\mathbb{1}}_{i-1} \bar{\mathbb{1}}_{i-1+\phi}]$; and $\phi + 1$ is a constant as well.

Pick $\gamma = 0, \delta = 1$, then $\Delta_n = B^3/(n-1)^3$, $K_n = (\phi+1)CB^2/(n-1)^2$, and $L_n = \mathsf{Var}(\hat{\tau}_m)/(n-1)$.

We check that all the five conditions from Lemma EC.1 are satisfied.

1. $\mathbb{E}\left|X_{n,i}\right|^3 \leq \Delta_n = B^3/(n-1)^3$, because all the potential outcomes are bounded by B, so that $X_{n,i} \leq B/(n-1)$.
2. $B_{n,k,a}^2/k \leq K_n = (\phi+1)CB^2/(n-1)^2$.
3. $B_n^2/(n-1) \geq L_n = \mathsf{Var}(\hat{\tau}_m)/(n-1)$.
4. $K_n/L_n = (\phi+1)CB^2/(n-1)\mathsf{Var}(\hat{\tau}_m) = O(1)$, where the last equality is due to Assumption 5.
5. $\Delta_n/L_n^{3/2} = B^3/(n-1)^{3/2}\mathsf{Var}(\hat{\tau}_m)^{3/2} = O(1)$, where the last equality is due to Assumption 5.

Due to Lemma EC.1,

$$
\frac{\hat{\tau}_p - \tau_p}{\sqrt{\mathsf{Var}(\hat{\tau}_p)}} \xrightarrow{D} \mathcal{N}(0,1).
$$

□