

Additive rules for the quasi-linear bargaining problem

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Abstract

We study the class of additive rules for the quasi-linear bargaining problem introduced by Green. We provide a characterization of the class of all rules that are efficient, translation invariant, additive, and continuous. We present several subfamilies of rules: the one parameter family of t -Shapley rules, the weighted coalitional rules, the simplicial cone rules, and the Steiner point rules. We discuss additional properties that solutions in these families possess. We discuss the relation of these solutions to the general class. The Steiner point rules satisfy more of our properties than any other class of rules. We also show that if there are at least three agents, any rule in the class we characterize violates the axiom of no advantageous reallocation.

1 Introduction

This paper is concerned with decisions that affect a group of n players. These players' preferences depend upon a decision (x) and their receipt or payment of a divisible transferable resource (t), which we can call money. In the domain of problems we study, preferences are quasi-linear in money and are

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completely general with respect to the decision: hence the term quasi-linear bargaining problems. Preferences can be represented by $U(x, t) = u(x) + t$.

Our approach is normative and welfarist. We seek rules that choose good decisions and equitable vectors of monetary transfers among the players. These rules are allowed to depend on the way in which decisions affect players, but not on the nature of the decisions themselves. Any two problems which give rise to the same set of feasible utility allocations, before any transfers of money are made, should lead to the same outcome. Thus we envision the role of monetary transfers as capturing the compensation that players should make among themselves – such compensation being due to the fact there can be desirable decisions for some players that are inefficient for the group as a whole.

This model is introduced by Green [10, 11] and studied further by Moulin [13, 14, 15], Chun [3, 4, 5], and Chambers [2]. In this literature, several axioms are standard and will be accepted throughout our work. First, the chosen decision should be efficient. Given the form of the utility functions, the sum of players' willingnesses to pay should be maximized. Second, because there is a one-parameter family of equivalent utility representations for each player, we do not want the selection of a particular numerical representation to affect the outcome. This condition is expressed as the translation invariance of the solution with respect to the set of feasible utility allocations. Third, the solution should not be excessively sensitive to errors of measurement or errors in judgment. Defining a natural topology on problems, we thus require the solution to be continuous in this topology.

There is an unmanageably large collection of solutions satisfying these three conditions. The key additional property we study is motivated by the idea that if a problem can be decomposed into two sub-problems which do not interact at all, then one should arrive at the same outcome whether the original problem is solved as given or the two sub-problems are solved independently. This property amounts to the additivity of the solution. Solutions that are not additive will be subject to complex agenda-setting manipulations and will exhibit other pathologies and inconsistencies.

Green [11] obtains a characterization of additive solutions for the two-player case under a further condition that he calls recursive invariance (to be explained below). Chambers [2] drops the recursive invariance condition, but retains focus on the two-player case. In this paper we develop a characterization for the general n -player case. That is, we extend the results to any number of players and we drop the recursive invariance condition. Al-

though the set of solutions we obtain is very large, it has a mathematical characterization that can enable further analysis and refinement.

We then undertake several such refinements by studying several subfamilies of these general solutions. The first are called the t -Shapley rules, where t is a real valued parameter. These rules correspond to the two agent rules in Green. However, they form only a small subfamily of rules in the general case. The second subfamily we study is called the weighted coalitional rules. These are closely related to the t -Shapley rules, and in fact, the t -Shapley rules are linear combinations of weighted coalitional rules. However, in the case of more than two players, some weighted coalitional rules exhibit pathological properties. By laying bare the geometrical foundation of this family of solutions, the characterization we provide should enable the study of further requirements and desirable properties in the general case. Building off of the weighted coalitional rules, we study another family of rules that satisfies a property we call recursive invariance. This property requires that if a rule selects an alternative for a problem, then whenever this alternative is added to the problem, the rule should re-select it. The family we introduce that satisfies this is termed the simplicial cone rules. Unfortunately, this family also exhibits pathologies. They violate a very basic property stating that if a problem consists of only efficient alternatives, then one of the efficient alternatives should be selected. In fact, they violate various dummy properties as well. Therefore, building on ideas from the mathematical literature on selections, we identify a particular family of rules that satisfies all of the properties we have introduced—we term these rules the Steiner point rules (after the mathematical concept of the Steiner point).

Lastly, we consider a property that states that if the agents make transfers contingent on the selected alternative, no agents benefit. This axiom is called no advantageous reallocation. No advantageous reallocation is introduced and studied by Moulin [13] in the related non-welfarist model. We establish that if there are at least three agents, then no rule satisfying our primary axioms also satisfies no advantageous reallocation.

The structure of the paper is organized as follows. We introduce our primary representation theorem, and then demonstrate several families of rules. Each such family is demonstrated to violate some appealing property until we come to the family of Steiner point rules. After discussing the Steiner point rules and the properties they satisfy, we move on to show that there is no rule falling satisfying the axioms of our representation theorem that also satisfies no advantageous reallocation. Section 2 contains our

general representation theorem. Section 3 introduces the t -Shapley rules. Section 4 discusses the weighted coalitional rules. Section 5 is devoted to the study of no advantageous reallocation. Finally, Section 6 concludes.

2 A general representation for n -agents

Let N be a finite set of **agents**. Say that a subset $B \subset \mathbb{R}^N$ is **bounded above** if there exists some $x \in \mathbb{R}^N$ such that $B \subset \{y : y \leq x\}$.¹ A **problem** is a nonempty subset of \mathbb{R}^N which is closed, convex, comprehensive, and bounded above. By \mathcal{B} , we mean the set of all problems.

Let $\bar{x} : \mathcal{B} \rightarrow \mathbb{R}$ be defined as $\bar{x}(B) \equiv \max_{x \in B} \sum_N x_i$. We say x is **feasible** for a problem B if $\sum_N x_i \leq \bar{x}(B)$. Our interest is in providing a method for solving problems. To this end, a **rule** is a function $f : \mathcal{B} \rightarrow \mathbb{R}^N$ such that for all $B \in \mathcal{B}$, $f(B)$ is feasible for B . A rule associates with any given problem a unique feasible solution. In particular, it allows us to make recommendations across problems.

Let H be a function defined on the set of problems which maps to the set of hyperplanes of \mathbb{R}^N . Specifically, let $H(B)$ be defined as $H(B) \equiv \{x \in \mathbb{R}^N : \sum_N x_i = \bar{x}(B)\}$. Thus, $H(B)$ is the set of efficient points that the agents can achieve by making transfers.

For all sets A , $\mathcal{K}(A)$ is the convex and comprehensive hull of A .²

For two problems B, B' , define the sum $B + B' \equiv \{x + x' : x \in B, x' \in B'\}$.³

We posit the following axioms. Our first axiom states that for all problems, all solutions should be efficient.

Efficiency: For all $B \in \mathcal{B}$, $f(B) \in H(B)$.

Our next axiom specifies a robustness of the rule to the underlying utility specification. Formally, any two problems $B, B' \in \mathcal{B}$ such that $B' = B + x$ for some $x \in \mathbb{R}^2$ can be viewed as arising from the same underlying preferences. Hence, a rule should recommend the same social alternative and transfers in the new problem as in the old problem. But the utility value induced by

¹For $x, y \in \mathbb{R}^N$, $y \leq x$ means for all $i \in N$, $y_i \leq x_i$.

²The **comprehensive hull** of a set A is the set $\{x \in \mathbb{R}^N : \text{there exists } y \in A \text{ such that } y \geq x\}$. Here, $y \geq x$ means $y_i \geq x_i$ for all $i \in N$.

³The operator ‘+’ is sometimes referred to as the **Minkowski sum**.

the recommended alternative for the new problem is simply the old utility value, translated by x .

Translation invariance: For all $B \in \mathcal{B}$ and all $x \in \mathbb{R}^N$, $f(B + x) = f(B) + x$.⁴

The next axiom we discuss specifies that a rule should be robust to tying together certain types of problems. Specifically, suppose we have given two problems, $B_1, B_2 \in \mathcal{B}$ that are “unrelated.” Formally, we might think of the implicit underlying preferences of agents to be additively separable across two independent decisions. In such a situation, it is meaningful to solve the two problems separately. By solving the two problems separately, the vector of overall utilities that each agent receives is $f(B_1) + f(B_2)$. It is equally as valid to tie the two problems together, resulting in the problem $B_1 + B_2$. Generally speaking, there is no reason to expect that $f(B_1 + B_2) = f(B_1) + f(B_2)$. However, if it is the case that $f(B_1 + B_2) \neq f(B_1) + f(B_2)$, then there is at least one agent who benefits either from solving the problems separately, or from tying them together. Hence, there will typically be conflicts of interest. In order to rule out such conflicts, we postulate a simple additivity condition. The condition rules out the type of “agenda manipulation” just discussed.

Additivity: For all $B_1, B_2 \in \mathcal{B}$, $f(B_1 + B_2) = f(B_1) + f(B_2)$.

Additivity can also be interpreted as a requirement that a rule should be independent of the sequencing of problems, when problems are unrelated.

The next property states that if two problems are “close,” then their solutions should be “close.” In order to define this, we first define the **Hausdorff extended metric** on the space \mathcal{C} of closed subsets of \mathbb{R}^N .⁵ Let $d : \mathbb{R}^N \times \mathbb{R}^N$ be the Euclidean metric. Define the **distance** $d^* : \mathbb{R}^N \times \mathcal{C} \rightarrow \mathbb{R}_+$ as

$$d^*(x, B) \equiv \inf_{y \in B} d(x, y).$$

⁴We abuse notation and write $f(B) + x$ to mean $f(B) + \{x\}$.

⁵For d to be an **extended metric**, the following must be true:

- i*) For all $B, B' \in \mathcal{C}$, $d(B, B') \geq 0$ with equality if and only if $B = B'$
- ii*) For all $B, B' \in \mathcal{C}$, $d(B, B') = d(B', B)$
- iii*) For all $A, B, C \in \mathcal{C}$, $d(A, C) \leq d(A, B) + d(B, C)$.

The function d is a **metric** if it only takes real values.

Finally, the Hausdorff extended metric, $d_{\text{Haus}} : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}_+ \cup \{\infty\}$, is defined as

$$d_{\text{Haus}}(B, B') \equiv \max \left\{ \sup_{x \in B'} d^*(x, B), \sup_{x \in B} d^*(x, B') \right\}.$$

It can be verified that, restricted to the class of problems, d_{Haus} is actually a metric.

Continuity: There exists $M > 0$ such that for all $B_1, B_2 \in \mathcal{B}$, $d(f(B_1), f(B_2)) \leq M d_{\text{Haus}}(B_1, B_2)$.

Continuity tells us that the Euclidean distance between the solutions of two problems is uniformly bounded by some scale of the distance of the two problems. This type of continuity is thus in the sense of Lipschitz. While this is a stronger form of continuity than usually discussed in economic models, we do not know of any weaker form that will allow us to obtain a tractable characterization.

The main theorem is a result characterizing all rules satisfying efficiency, translation invariance, additivity, and continuity. The basic idea is to identify problems with their support functions, and then provide an integral representation of rules as additive functions on the set of such support functions.

Here, S_+^N refers to the intersection of the positive orthant with the unit sphere in \mathbb{R}^N .

The main representation theorem follows. Measures are always assumed to be countably additive.

Theorem 1: A rule f satisfies efficiency, translation invariance, additivity, and continuity if and only if there exists a nonnegative measure μ on the Borel subsets of S_+^N and an integrable function $h : S_+^N \rightarrow \mathbb{R}^N$ such that $f(B) \equiv \int_{S_+^N} h(u) (\sup_{x \in B} x \cdot u) d\mu(u)$, where h and μ satisfy the following restrictions:

$$\text{i) For } \mu\text{-almost every } u, \quad \sum_{i \in N} h_i(u) = \begin{cases} 0 & \text{if } u \neq \left(\frac{1}{\sqrt{|N|}}, \dots, \frac{1}{\sqrt{|N|}} \right) \\ \sqrt{|N|}/\mu \left(\left\{ \left(\frac{1}{\sqrt{|N|}}, \dots, \frac{1}{\sqrt{|N|}} \right) \right\} \right) & \text{otherwise} \end{cases}$$

ii) for all $i \in N$, $\int_{S_+^N} h_i(u) u_i d\mu(u) = 1$

iii) for all i, j such that $i \neq j$, $\int_{S_+^N} h_i(u) u_j d\mu(u) = 0$.

Proof. It is straightforward to establish that any rule f with a representation as in the theorem satisfies the axioms. Therefore, we will prove the converse statement only.

Step 1: Embedding problems into the space of support functions

Define a function $\sigma : \mathcal{B} \rightarrow C(S_+^N)$ which maps each problem into its support function, defined as $\sigma(B)(x) \equiv \sup_{y \in B} x \cdot y$. The function σ is one-to-one. Hence, on $\sigma(\mathcal{B})$, we may define $T : \sigma(\mathcal{B}) \rightarrow \mathbb{R}^N$ as $T(\sigma(B)) = f(B)$. It is easy to verify that T is positively linearly homogeneous, additive, and Lipschitz continuous in the sup-norm topology (the last statement follows from the well-known fact that $d_{\text{Haus}}(B, B') = d_{\text{sup}}(\sigma(B), \sigma(B'))$, when the support function is defined on the unit sphere).

Step 2: Defining a functional on the class of support functions

Write $T = (T_i)_{i \in N}$. Each T_i is then positively linearly homogeneous, additive, and Lipschitz continuous with Lipschitz constant M . Extend T_i to the linear hull of $\sigma(\mathcal{B})$, i.e. $\sigma(\mathcal{B}) - \sigma(\mathcal{B}) \equiv \{f - g : f \in \sigma(\mathcal{B}), g \in \sigma(\mathcal{B})\}$. Call the extension T_i^* . This extension is itself Lipschitz continuous; that is, let $g - g', h - h' \in \sigma(\mathcal{B}) - \sigma(\mathcal{B})$. Then $d(T^*(g - g'), T^*(h - h')) = d(T(g) - T(g'), T(h) - T(h'))$. Moreover, $d(T(g) - T(g'), T(h) - T(h')) = d(T(g) + T(h'), T(h) + T(g'))$. But since T is additive, we conclude $d(T(g) + T(h'), T(h) + T(g')) = d(T(g + h'), T(g' + h))$. By Lipschitz continuity of T , $d(T(g + h'), T(g' + h)) \leq M d_{\text{sup}}(g + h', g' + h)$. But the latter is equal to $M d_{\text{sup}}(g - g', h - h')$. Hence $d(T^*(g - g'), T^*(h - h')) \leq M d_{\text{sup}}(g - g', h - h')$, so that T^* is Lipschitz continuous. This establishes that T^* is also continuous.

By efficiency, for all $g \in \sigma(\mathcal{B})$, $\sum_{i \in N} T_i(g) = \sqrt{|N|} g \left(\frac{1}{\sqrt{|N|}}, \dots, \frac{1}{\sqrt{|N|}} \right)$. Fix $j \in N$. For all $i \neq j$, we may extend T_i^* to all of $C(S_+^N)$ so that the extension is continuous, using an appropriate version of the Hahn-Banach Theorem (e.g. Dunford and Schwartz [8], II.3.11). Call this extension T_i^{**} . For j , define $T_j^{**}(g) = \sqrt{|N|} g \left(\frac{1}{\sqrt{|N|}}, \dots, \frac{1}{\sqrt{|N|}} \right) - \sum_{i \neq j} T_i^{**}(g)$. Clearly,

T_j^{**} is continuous and is an extension of T_j^* , and for all $g \in C(S_+^N)$, $\sum_{i \in N} T_i^{**}(g) = \sqrt{|N|}g\left(\frac{1}{\sqrt{|N|}}, \dots, \frac{1}{\sqrt{|N|}}\right)$.

Step 3: Uncovering the integral representation agent-by-agent

Each T_i^{**} has an integral representation, by the Riesz representation theorem (for example, Aliprantis and Border [1], Theorem 13.14). Thus, $T_i^{**}(g) = \int_{S_+^N} g(x) d\mu_i(x)$. The (possibly signed) measures μ_i are each countably additive and of bounded variation, since S_+^N is compact and Hausdorff.

Step 4: Synthesizing the agents' measures to obtain one measure

Define the measure $\mu = \sum_{i \in N} |\mu_i|$.⁶ Each μ_i is then absolutely continuous with respect to μ , and so the Radon-Nikodym theorem guarantees the existence of measurable functions $h_i : S_+^N \rightarrow \mathbb{R}$ so that for all measurable g , $T_i^{**}(g) = \int_{S_+^N} h_i(u) g(u) d\mu(u)$. Thus, we may write $T^{**}(g) = \int_{S_+^N} h(u) g(u) d\mu(u)$, where $h : S_+^N \rightarrow \mathbb{R}^N$. Further, $\sum_{i \in N} T_i^{**}(g) = \int_{S_+^N} \sum_{i \in N} h_i(u) g(u) d\mu(u)$, which we know is equal to $\sqrt{|N|}g\left(\frac{1}{\sqrt{|N|}}, \dots, \frac{1}{\sqrt{|N|}}\right)$. This establishes that $\sum_{i \in N} h_i(u) = 0$ μ -almost everywhere, except at $x = \left(\frac{1}{\sqrt{|N|}}, \dots, \frac{1}{\sqrt{|N|}}\right)$, in which case $\sum_{i \in N} h_i\left(\frac{1}{\sqrt{|N|}}, \dots, \frac{1}{\sqrt{|N|}}\right) = \frac{\sqrt{|N|}}{\mu\left(\left\{\left(\frac{1}{\sqrt{|N|}}, \dots, \frac{1}{\sqrt{|N|}}\right)\right\}\right)}$.

Step 5: Translating the representation back to the space of problems

Translating back into the original framework, this tells us that $f(B) = \int_{S_+^N} h(x) (\sup_{y \in B} x \cdot y) d\mu(x)$, where μ is a positive, countably additive, mea-

⁶Here, $|\mu_i|$ denotes another measure called the **absolute value of μ_i** . When μ_i is countably additive and of bounded variation (as we know it is), then $|\mu_i|$ is also countably additive, and in particular, $\mu_i(A) \neq 0$ implies $|\mu_i|(A) > 0$. See Aliprantis and Border [1], Corollary 9.35 and Theorem 9.55.

sure, and h is a measurable function from S_+^N into \mathbb{R}^N , which satisfies

$$\sum_{i \in N} h_i(x) = \begin{cases} 0 & \text{if } x \neq \left(\frac{1}{\sqrt{|N|}}, \dots, \frac{1}{\sqrt{|N|}} \right) \\ \frac{\sqrt{|N|}}{\mu\left(\left\{ \frac{1}{\sqrt{|N|}}, \dots, \frac{1}{\sqrt{|N|}} \right\}\right)} & \text{for } x = \left(\frac{1}{\sqrt{|N|}}, \dots, \frac{1}{\sqrt{|N|}} \right) \end{cases}$$

μ -almost surely.

Step 6: Uncovering the implications of translation invariance

Under additivity, translation invariance is equivalent to the statement that for each unit vector e_i , $f(\mathcal{K}(\{e_i\})) = e_i$. The support function of $\mathcal{K}(\{e_i\})$ is given by $\sigma(\mathcal{K}(\{e_i\}))(u) = u_i$. Thus, one obtains $\int_{S_+^N} h(u) u_i d\mu(u) = e_i$, equivalent to the statement that for all i , $\int_{S_+^N} h_i(u) u_i d\mu(u) = 1$ and for all $i \neq j$, $\int_{S_+^N} h_i(u) u_j d\mu(u) = 0$. ■

A few remarks are in order. Firstly, representations similar to that appearing in Theorem 1 are discussed in the mathematics literature (for example, see [18], Section 7). Mathematicians are often interested in defining “selectors”—these are functions which carry any convex, compact set into selections from that set. A linear Lipschitz selector is a selector that is additive under Minkowski addition, and that is Lipschitz continuous. Perhaps the most well-known linear Lipschitz selector is the “Steiner point,” axiomatized by Schneider [20, 21]. Formally, our work differs from the work in the mathematics literature in two important respects. Firstly, we do not discuss compact sets; but comprehensive sets. Secondly, and more importantly, we do not study selectors. We study rules, and a rule in our context additionally satisfies the efficiency and translation invariance axioms, which restrict them even further. In the mathematics literature, it is known that any additive and Lipschitz function can be represented as integration with respect to a measure, but not much else is known (it is not even known what conditions are necessary and sufficient on such a measure for the function to be a selector).

Theorem 1 tells us that any solution satisfying the four axioms can be represented by a function h and a measure μ . It is worth discussing these objects. First, the function h maps from the nonnegative part of the unit sphere in Euclidean N -space. Elements of the unit sphere can be interpreted as a list of “weights,” one for each agent. For any problem, these weights are

used to compute the maximal “weighted utility” that can be achieved within the problem before making transfers. This suggests a “weighted utilitarian” notion.

The function h specifies another vector in Euclidean N -space for each such list of weights. This vector can be interpreted as a fixed list of relative utility values. It is scaled by the maximal weighted utility achieved from the list of weights. Thus, this can be viewed as a “payoff vector,” where the payoff is scaled by the maximal weighted utility. The payoff vectors are then aggregated over, according to a measure μ .

There are many degrees of freedom in this definition. In particular, we have many degrees of freedom in choosing h and μ . We are allowed to renormalize h as long as the renormalization is accompanied by a counterbalancing renormalization of μ . Thus, there is no sense in which these parameters are “unique.”

However, the representation does satisfy an interesting uniqueness property. Specifically, for all $i \in N$, each $h_i : S_+^N \rightarrow \mathbb{R}$ induces a measure over the Borel subsets of S_+^N through μ . This measure is defined as $\mu_i(E) \equiv \int_E h_i(u) d\mu(u)$ (this is the same μ_i that appears in the proof) Thus, any two representations satisfying properties i)-iii) listed in the statement of the theorem must possess the same list of induced measures. The reasoning behind this statement is simple: it is a well known fact that the set of differences of support functions (in the proof, $\sigma(\mathcal{B}) - \sigma(\mathcal{B})$) is sup-norm dense in the space of continuous functions, $C(S_+^N)$ (see, for example, Schneider [21], Theorem 1.7.9). Therefore, the continuous extension T_i^{**} provided in the proof for each agent must be completely determined by its restriction to the dense set $\sigma(\mathcal{B}) - \sigma(\mathcal{B})$. In other words, this extension is *unique*, and so its representing measure is unique.

There is also another way that we can imagine renormalizing solutions. The fact that lists of weights lie in S_+^N is useful for the proof, but has no economic content, and moreover is not necessary. Thus, one is also free to scale any $u \in S_+^N$ by some $\alpha > 0$ as long as h is then equivalently scaled by $1/\alpha$. The maximal weighted utility is scaled up by α while the vector the weights map to is scaled by α , having no aggregate effect. Such “renormalizations” will sometimes make the nature of the problem more transparent. In such environments, integration would no longer be performed over S_+^N , but over whatever lists of weights were deemed relevant. Clearly, when considering “renormalized” lists of weights, we never need to consider situations where two lists of weights are simply scale translations of each other.

3 Recursive invariance and the t -Shapley rules

The following property was introduced by Chun [4] (he calls it *trivial independence*). We use the terminology introduced by Green [11]. The property states that if a solution is recommended for a particular problem, and this solution is added to the original utility possibility set, then reapplying the rule to this new problem results in the solution for the original problem.

Recursive invariance: For all $B \in \mathcal{B}$, $f(\mathcal{K}(B \cup \{f(B)\})) = f(B)$.

The axiom was used by Green [11] in the class of two-agent problems. Upon adding recursive invariance and a mild symmetry axiom to our four main axioms, he established a characterization of a one-parameter family of rules. In this section, we discuss a natural extension of this class to the many-agent case. These rules identify any problem with a transferable utility game, and then recommend the Shapley value [22] of this associated game for the problem. We will also establish, by means of an example, that the t -Shapley rules are not the appropriate generalization of the family of rules characterized by Green.

Fix a parameter $t > 0$. For a given problem $B \in \mathcal{B}$, define the TU-game associated with a bargaining problem as follows:

For all $S \subset N$, $v_B(S) = \max_{x \in B} \sum_{i \in S} x_i - t(\bar{x}(B) - \sum_{i \in N} x_i)$. The quantity $v_B(S)$ is the maximal amount that coalition S can obtain, when being forced to pay some “tax” at rate t on the degree of inefficiency of the selected alternative. Given that we have defined a game in transferable utility, we can compute the Shapley value of the game. This corresponding value will be called the t -Shapley solution for the problem B .

The t -Shapley rules enjoy many properties. In particular, they are anonymous. Here, we simply show how to express the t -Shapley rules in the representation derived above. As in the previous section, we are allowed to “renormalize” the lists of weights in S_+^N . We shall do this in order to keep the analysis simple and clean.

Writing out explicitly the definition of the Shapley value, we may, for all $i \in N$, and all $B \in \mathcal{B}$, compute:

$$f_i^t(B) = \sum_{\{S \subset N: i \notin S\}} [v_B(S \cup \{i\}) - v_B(S)] \frac{|S|!(|N \setminus S| - 1)!}{|N|!}.$$

Rewriting $\nu_B(S) = \max_{x \in B} (1+t) \sum_{i \in S} x_i + t \sum_{i \in N \setminus S} x_i - t\bar{x}(B)$, we conclude

$$f_i^t(B) = \sum_{\{S \subset N: i \notin S\}} \left[\begin{array}{c} \max_{x \in B} \left[(1+t) \sum_{j \in S \cup \{i\}} x_j + t \sum_{j \in N \setminus (S \cup \{i\})} x_j \right] \\ - \max_{x \in B} \left[(1+t) \sum_{j \in S} x_j + t \sum_{j \in N \setminus S} x_j \right] \end{array} \right] \frac{|S|!(|N \setminus S| - 1)!}{|N|!}.$$

Thus, for all $S \subset N$, define $u_i^S = 1+t$ if $i \in S$, and $u_i^S = t$ if $i \notin S$.⁷ Rearranging the preceding obtains:

$$\begin{aligned} f_i^t(B) &= \sum_{\{S \subset N: i \in S\}} \left(\sup_{x \in B} u^S \cdot x \right) \frac{(|S| - 1)! |N \setminus S|!}{|N|!} \\ &\quad - \sum_{\{S \subset N: i \notin S\}} \left(\sup_{x \in B} u^S \cdot x \right) \frac{|S|!(|N \setminus S| - 1)!}{|N|!} \end{aligned}$$

Now, write $h_i(u^S) = \frac{1}{|S|}$ if $i \in S$ and $-\frac{1}{|N \setminus S|}$ if $i \notin S$. For all S , define $\mu(\{u^S\}) = \frac{|S|!(|N \setminus S|!)}{|N|!}$. Then we conclude

$$f_i^t(B) = \sum_{S \subset N} h_i(u^S) \left(\sup_{x \in B} u^S \cdot x \right) \mu(\{u^S\}).$$

This gives us exactly the type of representation obtained in Theorem 1. One can easily verify that all of the conditions are satisfied.

The t -Shapley rules are intuitively appealing and enjoy many normative properties. In particular, they satisfy all of the axioms that we used in Theorem 1. In the two-agent case, they also satisfy recursive invariance. Unfortunately, however, recursive invariance of the t -Shapley rules does not hold in the general many-agent case. We offer a three-agent example, showing that the simplest of t -Shapley rules, for which $t = 0$, is not recursively invariant.

Example: Define $B \equiv \mathcal{K}(\{(0, 0, 0), (6, -100, -100)\})$; *i.e.* the convex and comprehensive hull of the origin and the point $(6, -100, -100)$. We claim that $f^0(B) \neq f^0(\mathcal{K}(B \cup \{f^0(B)\}))$. The transferable utility

⁷Hence, the element in S_+^N to which this vector corresponds is the unit vector $\frac{u}{\|u\|}$.

game v_B is defined as follows:

$$\begin{aligned}
v_B(\{1\}) &= 6 \\
v_B(\{2\}) &= 0 \\
v_B(\{3\}) &= 0 \\
v_B(\{1, 2\}) &= 0 \\
v_B(\{1, 3\}) &= 0 \\
v_B(\{2, 3\}) &= 0 \\
v_B(\{1, 2, 3\}) &= 0.
\end{aligned}$$

It is simple to verify that the Shapley value of this game is $(2, -1, -1)$, so that $f^0(B) = (2, -1, -1)$. Now, consider the problem $\mathcal{K}(B \cup \{f^0(B)\})$. Consider the transferable utility game $v' \equiv v_{\mathcal{K}(B \cup \{f^0(B)\})}$ associated with this problem. It is defined as follows:

$$\begin{aligned}
v'(\{1\}) &= 6 \\
v'(\{2\}) &= 0 \\
v'(\{3\}) &= 0 \\
v'(\{1, 2\}) &= 1 \\
v'(\{1, 3\}) &= 1 \\
v'(\{2, 3\}) &= 0 \\
v'(\{1, 2, 3\}) &= 0.
\end{aligned}$$

It is simple to verify that the Shapley value of this game is $(8/3, -7/6, -7/6)$, so that $f^0(\mathcal{K}(B \cup \{f^0(B)\})) = (8/3, -7/6, -7/6) \neq (2, -1, -1) = f^0(B)$. This contradicts recursive invariance.

4 The weighted coalitional rules

We here introduce another family of rules. They are motivated by the following observation: Suppose a rule satisfies our primary axioms. Note that the measure μ associated with such a rule must have a support consisting of at least $|N|$ vectors. Otherwise, translation invariance cannot be satisfied.

To this end, suppose that the measure μ associated with this rule has a support of exactly $|N|$ vectors. Write the support of μ as $\{P_1, \dots, P_N\}$.

Let P be the $|N| \times |N|$ matrix whose rows are the P_i 's. We index rows by subscript and columns by superscript. We claim that for all $B \in \mathcal{B}$, $f(B) = P^{-1} [\sup_{x \in B} P_i \cdot x]_{i \in N}$.

Proposition: Suppose f satisfies the axioms listed in Theorem 1, and let μ be the measure associated with f . Suppose that the support of μ is $\{P_1, \dots, P_N\}$. Then for all $B \in \mathcal{B}$, $f(B) = P^{-1} [\sup_{x \in B} P_i \cdot x]_{i \in N}$, where P is the matrix whose rows are P_i 's.

Proof. Let f satisfy the hypothesis of the proposition. The rule f can then be written so that for all $B \in \mathcal{B}$, $f(B) = \sum_{i \in N} h(P_i) \mu(\{P_i\}) (\sup_{x \in B} P_i \cdot x)$. In particular, for all $j \in N$ and all $B \in \mathcal{B}$, $f_j(B) = \sum_{i \in N} h_j(P_i) \mu(\{P_i\}) (\sup_{x \in B} P_i \cdot x)$.

Let $x \in \mathbb{R}^N$ be arbitrary. By translation invariance, we establish that $x = f(\mathcal{K}(\{x\})) = \sum_{i \in N} h(P_i) (P_i \cdot x) \mu(\{P_i\})$. Define the $N \times N$ matrix Q as $Q_j^i = h_j(P_i) \mu(\{P_i\})$. The preceding expressions then read $f_j(B) = \sum_{i \in N} Q_j^i [\sup_{x \in B} P_i \cdot x]_{i \in N}$, or $f(B) = Q [\sup_{x \in B} P_i \cdot x]_{i \in N}$. We claim that $Q = P^{-1}$. By *ii*) of Theorem 1, for all $j \in N$, $\sum_{i \in N} Q_j^i P_i^j = 1$. Thus, $Q_j \cdot P^j = 1$. By *iii*) of Theorem 1, if $j \neq k$, $\sum_{i \in N} Q_j^i P_i^k = 0$. Thus, $Q_j \cdot P^k = 0$. These two statements imply that $QP = I$. Since P and Q are each $|N| \times |N|$ matrices, we conclude that $Q = P^{-1}$. Hence $f(B) = P^{-1} [\sup_{x \in B} P_i \cdot x]_{i \in N}$. ■

Thus, let $\left\{ P_1, P_2, \dots, P_{|N|-1}, \left(\frac{1}{\sqrt{|N|}} \right)_{i \in N} \right\}$ be a set of linearly independent vectors in S_+^N . Label $P_N = \left(\frac{1}{\sqrt{|N|}} \right)_{i \in N}$. As the set $\{P_1, \dots, P_N\}$ is linearly independent, we can construct an invertible matrix P so that the rows of P are exactly P_i 's. The **weighted coalitional rule according to \mathbf{P}** is defined as $f(B) = P^{-1} [\sup_{x \in B} P_k \cdot x]_{k=1}^n$. It is trivial to verify that the weighted coalitional rules are efficient, translation invariant, additive, and continuous. They are also recursively invariant.

Note that the Proposition establishes that for any set of $|N|$ linearly independent vectors, there is a *unique* rule whose measure μ has this set as its support. The unique such rule is the weighted coalitional rule according to any matrix whose rows are the elements in the support of μ . Moreover, the weighted coalitional rules are those rules whose support is *minimal*.

The weighted coalitional rules have a simple geometric interpretation, which leads to an interpretation in terms of weighted utilitarianism. Given

a matrix P , the weighted coalitional rule according to P works as follows. Given is a problem $B \in \mathcal{B}$. Fix a row of P , say P_k ; this row gives a list of “weights,” one for each agent in society. The maximal social weighted utility according to weights P_k that can be achieved by society before making transfers is simply $[\sup_{x \in B} P_k \cdot x]$. For each row of P , there is a maximal weighted utility of this form (for the row of equal coordinates, we actually get a maximal aggregate non-weighted utility). The vector $[\sup_{x \in B} P_k \cdot x]_{k=1}^n$ gives this profile of maximal weighted social utilities. Hence, the vector $P^{-1} [\sup_{x \in B} P_k \cdot x]_{k=1}^n$ gives the unique vector in \mathbb{R}^N that achieves the same weighted social utilities as the maximal weighted social utilities attainable with problem B . Geometrically, this vector is the unique intersection of the tangent hyperplanes to B in the directions P_k .

5 The simplicial cone rules

The weighted coalitional rules allow us to introduce many more rules, some of which are recursively invariant. Here, we discuss one such family, which we term the simplicial cone rules. The simplicial cone rules are recursively invariant; however, we will see that they are pathological in at least one sense. We first formally describe the simplicial cone rules, then explain the terminology.

To define the simplicial cone rules for a set N of agents, first fix a collection of vectors $\{u_i\}_{i \in N} \subset S_+^N \setminus \left\{ \left(\frac{1}{\sqrt{|N|}} \right)_{i \in N} \right\}$ satisfying the following three properties:

- i) $\{u_i\}_{i \in N}$ are linearly independent
- ii) for all $j \in N$, $\left\{ \{u_i\}_{i \in N \setminus \{j\}}, \left(\frac{1}{\sqrt{|N|}} \right)_{i \in N} \right\}$ are linearly independent
- iii) There exists $\alpha \in \mathbb{R}_+^N$ such that $\left(\frac{1}{\sqrt{|N|}} \right)_{i \in N} = \sum_{i \in N} \alpha_i u_i$.

Next, fix a list of weights $\lambda \in \mathbb{R}_+^N$ such that $\sum_{i \in N} \lambda_i = 1$. For each $j \in N$, we may consider the weighted coalitional rule generated by $\{u_i\}_{i \in N \setminus \{j\}}$.

This is the weighted coalitional rule according to $\left\{ \{u_i\}_{i \in N \setminus \{j\}}, \left(\frac{1}{\sqrt{|N|}} \right)_{i \in N} \right\}$. Call this weighted coalitional rule f^j . We now define a rule $f = \sum_{j \in N} \lambda_j f^j$. A rule that can be expressed in this fashion will be termed a **simplicial cone rule**. Note that if only one λ_i is nonzero, then we obtain a weighted

coalitional rule; indeed, the simplicial cone rules generalize the weighted coalitional rules.

We call these rules simplicial cone rules for the following geometric reason. Imagine a pointed cone with a simplicial base, say, \mathcal{C} . We may without loss of generality suppose that all of the coordinates of the base sum to the same constant. Next, fix some selection from this base. We assume that the cone has the property that it contains all of \mathbb{R}_+^N and that it does not intersect $\mathbb{R}_+^N \setminus \{0\}$. Now, for any given problem B , it is clear that there exists some x for which $B \subset \mathcal{C} + x$. This is always true; what is also always true about a simplicial cone is that there exists a unique infimal such x , say x^* . Consider then the intersection of $\mathcal{C} + x^*$ with $H(B)$; this intersection is a simplex (as the cone has a simplex as base). Moreover, for any given problem, this simplex has the same “shape,” it is only translated or scaled to different degrees. Therefore, we may meaningfully scale and translate the selection made from the base appropriately. This scaled selection is then the solution chosen for the problem B .

Theorem 2: The simplicial cone rules are recursively invariant.

Proof. Given is a list of linearly independent vectors $\{u_i\}_{i \in N} \subset S_+^N \setminus \left\{ \left(\frac{1}{\sqrt{|N|}} \right)_{i \in N} \right\}$ and a list of weights $\lambda \in \mathbb{R}_+^N$ such that $\sum_{i \in N} \lambda_i = 1$, defining a rule f . As f is a convex combination of weighted coalitional rules that are generated in various combinations by the vectors $\{u_i\}_{i \in N}$, it is enough to verify that for all $B \in \mathcal{B}$ and all $i \in N$, $\sup_{x \in B} u_i \cdot x \geq u_i \cdot f(B)$. By verifying this inequality, we establish that $\sup_{x \in B} u_i \cdot x = \sup_{x \in \mathcal{K}(B \cup \{f(B)\})} u_i \cdot x$, so that for all weighted coalitional rules f^i under consideration, $f^i(B) = f^i(\mathcal{K}(B \cup \{f(B)\}))$. This, in turn, is enough to verify that $f(B) = f(\mathcal{K}(B \cup \{f(B)\}))$.

Thus, $f(B) = \sum_{i \in N} \lambda_i f^i(B)$, where f^i is the weighted coalitional rule corresponding to the weights $\left\{ \{u_j\}_{j \in N \setminus \{i\}}, \left(\frac{1}{\sqrt{|N|}} \right)_{j \in N} \right\}$. Clearly, by definition of weighted coalitional rule, for all $j \in N$ and all $i \neq j$, $\sup_{x \in B} u_j \cdot x = u_j \cdot f^i(B)$ (as each f^i is a weighted coalitional rule that is defined by a collection of vectors, one of which is u_j , so that $f^i(B)$ lies on the tangent hyperplane to B in direction u_j).

Therefore, it is enough to show that for all $j \in N$, if $\lambda_j > 0$, then $\sup_{x \in B} u_j \cdot x \geq u_j \cdot f^j(B)$. To this end, suppose, by means of contradic-

tion, that $u_j \cdot f^j(B) > \sup_{x \in B} u_j \cdot x$. By definition of a simplicial cone rule, there exists $\alpha \in \mathbb{R}_+^N$ such that $\left(\frac{1}{\sqrt{|N|}}\right)_{i \in N} = \sum_{i \in N} \alpha_i u_i$. Recall that $\sigma : \mathcal{B} \rightarrow C(S_+^N)$ is the support function mapping. Therefore, $\sigma(B) \left(\left(\frac{1}{\sqrt{|N|}}\right)_{i \in N} \right) = \sigma(B) \left(\sum_{i \in N} \alpha_i u_i \right)$. As support functions are sublinear and positively homogeneous (see Rockafellar [19], Theorem 13.2), $\sigma(B) \left(\sum_{i \in N} \alpha_i u_i \right) \leq \sum_{i \in N} \alpha_i \sigma(B)(u_i)$. As $\sup_{x \in B} u_j \cdot x < u_j \cdot f^j(B)$, conclude that $\sigma(B)(u_j) < \sigma(\mathcal{K}(\{f^j(B)\}))(u_j)$. Moreover, for all $i \neq j$, $u_i \cdot f^j(B) = \sigma(B)(u_i)$, so that $\sigma(B)(u_i) = \sigma(\mathcal{K}(\{f^j(B)\}))(u_i)$. Therefore, $\sum_{i \in N} \alpha_i \sigma(B)(u_i) < \sum_{i \in N} \alpha_i \sigma(\mathcal{K}(\{f^j(B)\}))(u_i)$. Now, $\sigma(\mathcal{K}(\{f^j(B)\}))$ is linear, as it is the support function of a singleton. Therefore, $\sum_{i \in N} \alpha_i \sigma(\mathcal{K}(\{f^j(B)\}))(u_i) = \sigma(\mathcal{K}(\{f^j(B)\})) \left(\sum_{i \in N} \alpha_i u_i \right)$. Thus, $\sigma(\mathcal{K}(\{f^j(B)\})) \left(\left(\frac{1}{\sqrt{|N|}}\right)_{i \in N} \right) > \sigma(B) \left(\left(\frac{1}{\sqrt{|N|}}\right)_{i \in N} \right)$, an obvious contradiction.

Hence, $\sup_{x \in B} u_j \cdot x \geq u_j \cdot f^j(B)$, verifying that the simplicial cone rules are recursively invariant. ■

It is easy to see that there exist anonymous simplicial cone rules. One simply needs to take a list of vectors $\{u_i\}_{i \in N}$ that are permutations of each other, and set the list of weights $\lambda = \left(\frac{1}{|N|}\right)_{i \in N}$. A canonical example is a rule for which each u_i is the indicator vector of agent i , $u_i = 1_i$.

Therefore, we have demonstrated the existence of rules satisfying our primary axioms, *and* which are both recursively invariant and anonymous. However, the simplicial cone rules suffer from a very problematic drawback.

Imagine a problem in which every possible decision that can be made is efficient. In such a problem, no agent ever needs to be compensated for forgoing inefficient decisions that may be beneficial to him. A society facing this problem has no use for monetary transfers; indeed, such a society should simply choose one of the efficient decisions. The following axiom formalizes this notion.

Selection: Let $B \in \mathcal{B}$ such that $B = \mathcal{K}(B \cap H(B))$. Then $f(B) \in B \cap H(B)$.

Every simplicial cone rule violates selection.

Theorem 3: There does not exist a simplicial cone rule that satisfies selection.

Proof. Suppose, by means of contradiction, that there exists a simplicial cone rule f that satisfies selection. The rule f is a convex combination of weighted coalitional rules based on the vectors $\{u_i\}_{i \in N}$. At least one of the rules f^i of which f is a convex combination has a strictly positive weight; without loss of generality, let us suppose that $f = \sum_{i \in N} \lambda_i f^i$ where $\lambda_1 > 0$. For all $i \in N$, let $U^i \in \mathbb{R}^{N \times N}$ be the matrix whose rows are indexed by $\left\{ \{u_j\}_{j \in N \setminus \{i\}}, \left(\frac{1}{\sqrt{|N|}} \right)_{j \in N} \right\}$. By condition *ii*) in the definition of simplicial cone rule, the matrix U^i is invertible, for all $i \in N$. Let $1^{n-1}\alpha \in \mathbb{R}^N$ denote a vector whose first $n-1$ rows are ones, and whose last row is some scalar α . For all α , define $B^\alpha \equiv \mathcal{K} \left(\left\{ [U^i]^{-1} (1^{n-1}\alpha) \right\}_{i \in N \setminus \{1\}} \right)$.

We claim that for all α , $B^\alpha = \mathcal{K} (B^\alpha \cap H(B^\alpha))$. But this is simple; clearly, $\left(\frac{1}{\sqrt{|N|}} \right)_{j \in N} \cdot [U^i]^{-1} (1^{n-1}\alpha) = \alpha$ for all $i \in N \setminus \{1\}$, so that for all $i \in N \setminus \{1\}$, $[U^i]^{-1} (1^{n-1}\alpha)$ is efficient. Thus, $B^\alpha \cap H(B^\alpha) = \text{conv} \left(\left\{ [U^i]^{-1} (1^{n-1}\alpha) \right\}_{i \in N \setminus \{1\}} \right)$. Moreover, $u^1 \cdot [U^i]^{-1} (1^{n-1}\alpha) = 1$ for all $i \in N \setminus \{1\}$, as each U^i contains a row which is u^1 . Therefore, if f satisfies selection, it must be the case that $u^1 \cdot f(B^\alpha) = 1$ for all α .

Clearly, for all $i \in N \setminus \{1\}$, $u^1 \cdot f^i(B^\alpha) = 1$; this follows as $u^1 \cdot [U^i]^{-1} (1^{n-1}\alpha) = 1$ for all $i \in N \setminus \{1\}$ implies $\sup_{x \in B^\alpha} u^1 \cdot x = 1$ and by definition of f^i (it is a weighted coalitional rule according to a collection of vectors, one of which is u^1). Therefore, as $\lambda_1 > 0$, we only need to establish that $u^1 \cdot f^1(B^\alpha) \neq 1$ for some α . Suppose, by means of contradiction, that $u^1 \cdot f^1(B^\alpha) = 1$ for all α . For all $i \in N \setminus \{1\}$, $u^i \cdot f^1(B^\alpha) = 1$. Letting $U \in \mathbb{R}^N \times \mathbb{R}^N$ be the matrix whose rows are $\{u_i\}_{i \in N}$, we therefore establish that $f^1(B^\alpha) = U^{-1} 1^N$ for all α . Hence, $f^1(B^\alpha)$ is independent of α , which is impossible, since efficiency requires that $\left(\frac{1}{\sqrt{|N|}} \right)_{j \in N} \cdot f^1(B^\alpha) = \alpha$. Hence there exists α for which $f^1(B^\alpha) \neq 1$, so that selection is violated. ■

In the next section, we introduce a family that remedies the above situation.

6 The Steiner point rules

This family of rules is based on a concept from the mathematics literature—the concept of a “Steiner point.” Every element in this family will satisfy all of the conditions listed in our main theorem, as well as recursive invariance and selection. Unfortunately, we have no characterization of this family (as we had no characterization of the preceding families). This family ends our search for rules satisfying appealing properties.

The Steiner point seems to have first appeared in the economics literature in Green’s [10] original work on bargaining with transfers. More recently, it is a focal point of several decision theoretic works (for example, see Hayashi [12] and Stinchcombe [23]). Here, we define it formally, as well as a family of generalized Steiner points, also introduced by Green.

Denote by \mathcal{K}^N the set of **convex bodies** in \mathbb{R}^N . Convex bodies are simply compact, convex subsets, in this case, of \mathbb{R}^N . The **Steiner point** is a function $St : \mathcal{K}^N \rightarrow \mathbb{R}^N$ defined by

$$St(K) \equiv \int_{S^N} \arg \max_{x \in K} (u \cdot x) d\lambda(u),$$

where here λ refers to the normalized Lebesgue measure on the unit sphere (normalized so that $\lambda(S^N) = 1$). The expression $h(u) = \arg \max_{x \in K} (u \cdot x)$ is not generally a function; typically it is a correspondence. However; the set of points for which h is multi-valued is of Lebesgue measure zero, so that the Steiner point is well-defined. If one is bothered by the integration of a correspondence, one may simply take any arbitrary measurable selection of h . An alternative representation of the Steiner point, which is often more convenient to work with, is

$$St(K) = |N| \int_{S^N} \left(\max_{x \in K} u \cdot x \right) u d\lambda(u).$$

The equivalence between these two representations is well-known (see, for example, Przeslawski [18]).

The Steiner point is a selector—in other words, the Steiner point of a convex body K is an element of K . Moreover, it is additive (under Minkowski addition) and Lipschitz continuous (in the Hausdorff topology). Indeed; the Steiner point is characterized as the unique function which is both additive and continuous, and invariant under certain “similarity” transformations.

One may define many natural generalizations of the Steiner selector; based on other probability measures over the unit sphere (indeed, these selectors play a key role in green [10]). However, at this stage, we are primarily interested in demonstrating the existence of rules satisfying efficiency, additivity, continuity, translation invariance, recursive invariance, anonymity, and selection. For our purposes, the standard Steiner selector is enough.

Our program here is as follows. For any given problem, we will identify a unique convex body with that problem. The mapping from problems to convex bodies will be additive and Lipschitz continuous in the Hausdorff topology. Then, given this convex body, we will select its Steiner point. The Steiner point will be the solution for the original problem. It will be clear that our method of mapping problems to convex bodies allows selection to be satisfied.

Thus, let $m \in (\sqrt{|N| - 1}, \sqrt{|N|})$. Define

$$\psi(m) \equiv \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = m \right\} \cap S^N.$$

It is clear that $\psi(m) \subset S_+^N$. Note that $\psi(m)$ is an $|N| - 1$ -dimensional hypersphere lying strictly in \mathbb{R}_+^N (so that in the case of three agents, for example, it is a circle). It lies in the hyperplane whose unit normal is the ray of equal coordinates.

Given any $B \in \mathcal{B}$, we use the vectors in $\psi(m)$ to define a convex body which is contained in $H(B)$. Recall that $\sigma(B)$ denotes the support function associated with B . Let

$$W(B) \equiv H(B) \cap \bigcap_{u \in \psi(m)} \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i \leq \sigma(B)(u) \right\}.$$

Note that

$$\bigcap_{u \in \psi(m)} \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i \leq \sigma(B)(u) \right\}$$

is a cone containing B (whose point is not necessarily at the origin). It is the intersection of the half-spaces generated by the hyperplanes tangent to B for all directions in $\psi(m)$. The convex body $W(B)$ is the intersection of the efficient hyperplane for B with this cone.

We define the Steiner point rule, f^{St} , as $f^{St}(B) \equiv St(W(B))$.

The Steiner point rules can be given a representation as in Theorem 1, so that it is easily verified that they satisfy efficiency, translation invariance, additivity, and continuity.

Specifically, let μ be a measure whose support is $\psi(m) \cup \left\{ \left(\frac{1}{\sqrt{|N|}} \right)_{i \in N} \right\}$. The measure μ restricted to $\psi(m)$ is a scaled Lebesgue measure (so that $\mu(\psi(m)) = |N| - 1$) and assigns $\mu \left(\left\{ \left(\frac{1}{\sqrt{|N|}} \right)_{i \in N} \right\} \right) = \sqrt{|N|}$. Let $h \left(\left(\frac{1}{\sqrt{|N|}} \right)_{i \in N} \right) = \left(\frac{1}{|N|} \right)_{i \in N}$, and for all $u \in \psi(m)$, let $h(u) = v$, where v is the orthogonal projection of u onto the hyperplane $\{x \in \mathbb{R}^N : \sum_{i \in N} x_i = 0\}$, scaled so that $v \cdot u = 1$. It is not difficult to establish that the pair (h, μ) satisfies all of the conditions listed in Theorem 1. That this representation results in the Steiner point rule is easily verified, by proving the coincidence of these rules with the Steiner point rules on the class of problems B for which $\bar{x}(B) = 0$.

Such a representation of the Steiner point rules allows us to establish that these rules satisfy all of the axioms listed in Theorem 1. Moreover, the original representation allows us to easily verify that both recursive invariance and selection are satisfied. Recursive invariance is verified because $St(W(B)) \in W(B)$, and by the trivial observation that $W(B) \subset W(\mathcal{K}(B \cup f^{St}(B))) \subset W(\mathcal{K}(W(B))) = W(B)$, so that $W(\mathcal{K}(B \cup f^{St}(B))) = W(B)$. Selection is verified as for any problem B for which $B = \mathcal{K}(B \cap H(B))$, it is the case that $W(B) = B \cap H(B)$, from which we conclude that $f^{St}(B) \in B \cap H(B)$.

7 On the possibility of advantageous transfers

An **advantageous reallocation** for a coalition $M \subset N$ exists for problem $B \in \mathcal{B}$ if there exists $B' \in \mathcal{B}$ such that

$$\left\{ \left(\sum_{i \in M} x_i, x_{-M} \right) : x \in B \right\} = \left\{ \left(\sum_{i \in M} x_i, x_{-M} \right) : x \in B' \right\}$$

and for all $i \in M$, $f_i(B') > f_i(B)$, with at least one inequality strict. An advantageous reallocation exists if it is possible for a group of agents to get together, and change the set of alternatives by promising ex-ante to make

contingent monetary transfers among themselves upon the realization of a particular social alternative. In particular, our (strong) definition allows groups of agents to significantly expand the underlying bargaining set. Our definition appears to allow groups of agents to significantly expand the underlying problem. Of course our theory is welfarist. We only recognize the possibility that there exists an underlying set of alternatives and a preference profile generating B that generates B' when groups specify ex-ante which contingent monetary transfers will be made.

No advantageous reallocation For all $B \in \mathcal{B}$ and all $M \subset N$, there does not exist an advantageous reallocation of B for M .

The main result of this section is that for any rule satisfying our main axioms, there exists a problem B which gives some coalition M an advantageous reallocation. This is surprising, as our main axioms are satisfied by many rules.

We first begin with a simple lemma that discusses an implication of our primary axioms in two-agent environments. It states that, restricted to the class of problems for which there are exactly two decisions, each of which are efficient, the rule always recommends some weighted combination of the two decisions, where the weights are independent of the problem in question.

Lemma 1: Let f satisfy efficiency, translation invariance, additivity, and continuity. Suppose that $|N| = 2$, where $N = \{i, j\}$. Then there exists $\lambda \in \mathbb{R}$ such that the following is true: For all $x, y \in \mathbb{R}^N$ such that $x_i + x_j = y_i + y_j$ and $x_i \leq y_i$, $f(\mathcal{K}(\{x, y\})) = \lambda x + (1 - \lambda)y$.

Proof. We offer a proof that relies on an application of our general representation theorem, although the lemma can also be derived independently.

By the general representation theorem, there exists some $h : S_+^N \rightarrow \mathbb{R}^N$, as well as a measure μ defined on the Borel subsets of S_+^N which parametrize the rule. Define

$$\lambda \equiv \int_{\{u \in S_+^N : u_i < u_j\}} h_i(u) (u_i - u_j) d\mu(u).$$

We will show that for all $x, y \in \mathbb{R}^N$ such that $x_i + x_j = y_i + y_j$ and $x_i \leq y_j$, $f(\mathcal{K}(\{x, y\})) = \lambda x + (1 - \lambda)y$.

To this end, by translation invariance of f , it is enough to prove the statement for those x, y for which $x_i + x_j = y_i + y_j = 0$. Let $(x, -x), (y, -y) \in \mathbb{R}^N$, and suppose that $x \leq y$. By the representation of f ,

$$f(\mathcal{K}(\{x, y\})) = \int_{S_+^N} h(u) (\max\{u_i x - u_j x, u_i y - u_j y\}) d\mu(u).$$

For (u_i, u_j) such that $u_i < u_j$, $\max\{u_i x - u_j x, u_i y - u_j y\} = u_i x - u_j x$, and for (u_i, u_j) such that $u_j < u_i$, $\max\{u_i x - u_j x, u_i y - u_j y\} = u_i y - u_j y$. For $u_i = u_j$, $\max\{u_i x - u_j x, u_i y - u_j y\} = 0$. Therefore,

$$\begin{aligned} & f(\mathcal{K}(\{x, y\})) \\ &= \int_{\{u \in S_+^N : u_i < u_j\}} h(u) (u_i x - u_j x) d\mu(u) \\ & \quad + \int_{\{u \in S_+^N : u_j < u_i\}} h(u) (u_i y - u_j y) d\mu(u). \end{aligned}$$

Factoring out x and y from the integrals obtains

$$\begin{aligned} &= x \int_{\{u \in S_+^N : u_i < u_j\}} h(u) (u_i - u_j) d\mu(u) \\ & \quad + y \int_{\{u \in S_+^N : u_j < u_i\}} h(u) (u_i - u_j) d\mu(u). \end{aligned}$$

As for all $u \neq \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $h_i(u) + h_j(u) = 0$, we conclude

$$\begin{aligned} &= x \left(\int_{\{u \in S_+^N : u_i < u_j\}} h_i(u) (u_i - u_j) d\mu(u), - \int_{\{u \in S_+^N : u_i < u_j\}} h_i(u) (u_i - u_j) d\mu(u) \right) \\ & \quad + y \left(\int_{\{u \in S_+^N : u_j < u_i\}} h_i(u) (u_i - u_j) d\mu(u), - \int_{\{u \in S_+^N : u_j < u_i\}} h_i(u) (u_i - u_j) d\mu(u) \right). \end{aligned}$$

Hence,

$$\begin{aligned} &= (x, -x) \int_{\{u \in S_+^N : u_i < u_j\}} h_i(u) (u_i - u_j) d\mu(u) \\ & \quad + (y, -y) \int_{\{u \in S_+^N : u_j < u_i\}} h_i(u) (u_i - u_j) d\mu(u). \end{aligned}$$

Lastly, we verify that $\int_{\{u \in S_+^N : u_j < u_i\}} h_i(u) (u_i - u_j) d\mu(u) = 1 - \lambda$. To this end, we establish that

$$\begin{aligned} & \int_{\{u \in S_+^N : u_i < u_j\}} h_i(u) (u_i - u_j) d\mu(u) \\ & + \int_{\{u \in S_+^N : u_j < u_i\}} h_i(u) (u_i - u_j) d\mu(u) = 1. \end{aligned}$$

The following equality is trivial:

$$\begin{aligned} & \int_{\{u \in S_+^N : u_i < u_j\}} h_i(u) (u_i - u_j) d\mu(u) \\ & + \int_{\{u \in S_+^N : u_j < u_i\}} h_i(u) (u_i - u_j) d\mu(u) \\ & = \int_{S_+^N \setminus \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}} h_i(u) (u_i - u_j) d\mu(u), \end{aligned}$$

Moreover, if $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $u_i - u_j = 0$. Thus the preceding expression is equal to $\int_{S_+^N} h_i(u) (u_i - u_j) d\mu(u)$. Separating, we establish

$$\begin{aligned} & \int_{S_+^N} h_i(u) (u_i - u_j) d\mu(u) \\ & = \int_{S_+^N} h_i(u) u_i d\mu(u) \\ & \quad - \int_{S_+^N} h_i(u) u_j d\mu(u). \end{aligned}$$

By conditions *ii*) and *iii*) in Theorem 1, this quantity is therefore equal to 1, so that $\int_{\{u \in S_+^N : u_j < u_i\}} h_i(u) (u_i - u_j) d\mu(u) = 1 - \lambda$.

Therefore, $f(\mathcal{K}(\{x, y\})) = \lambda(x, -x) + (1 - \lambda)(y, -y)$. ■

Theorem: Suppose that $|N| \geq 3$. There does not exist a rule satisfying efficiency, translation invariance, continuity, additivity, and no advantageous reallocation.

Proof. Step 1: The rule chooses aggregate welfare levels for each group of agents independently

First, we claim that for all coalitions $M \subset N$, and all $B, B' \in \mathcal{B}$ such that

$$\left\{ \left(\sum_{i \in M} x_i, x_{-M} \right) : x \in B \right\} = \left\{ \left(\sum_{i \in M} x_i, x_{-M} \right) : x \in B' \right\},$$

$\sum_{i \in M} f_i(B) = \sum_{i \in M} f_i(B')$. The argument is due to Moulin [13]. Suppose, by means of contradiction, that there exists $M \subset N$, and $B, B' \in \mathcal{B}$ where

$$\left\{ \left(\sum_{i \in M} x_i, x_{-M} \right) : x \in B \right\} = \left\{ \left(\sum_{i \in M} x_i, x_{-M} \right) : x \in B' \right\}$$

and $\sum_{i \in M} f_i(B) < \sum_{i \in M} f_i(B')$. Let $z \in \mathbb{R}^N$ be defined as

$$z_i \equiv \begin{cases} f_i(B) - f_i(B') + \frac{(\sum_{i \in M} f_i(B') - \sum_{i \in M} f_i(B))}{|M|} & \text{if } i \in M \\ 0 & \text{if } i \notin M \end{cases}.$$

Define $B'' \equiv B' + z$. By translation invariance, $f(B'') = f(B') + z$, so that for all $i \in M$, $f_i(B'') = f_i(B') + z_i = f_i(B) + \frac{(\sum_{i \in M} f_i(B') - \sum_{i \in M} f_i(B))}{|M|} > f_i(B)$. Next,

$$\begin{aligned} \left\{ \left(\sum_{i \in M} x_i, x_{-M} \right) : x \in B'' \right\} &= \left\{ \left(\sum_{i \in M} x_i + z_i, x_{-M} \right) : x \in B' \right\} \\ &= \left\{ \left(\sum_{i \in M} x_i, x_{-M} \right) : x \in B' \right\} = \left\{ \left(\sum_{i \in M} x_i, x_{-M} \right) : x \in B \right\}. \end{aligned}$$

Hence, we have constructed B'' which gives an advantageous transfer for M for the problem B .

Next, for all $M \subset N$, and all $B, B' \in \mathcal{B}$ such that

$$\left\{ \left(\sum_{i \in M} x_i, x_{-M} \right) : x \in B \right\} = \left\{ \left(\sum_{i \in M} x_i, x_{-M} \right) : x \in B' \right\},$$

we claim that for all $i \notin M$, $f_i(B) = f_i(B')$. This follows trivially from the statement in the preceding paragraph, and by applying the no-advantageous reallocation requirement to the problems B, B' and the coalition $M \cup \{i\}$.

Step 2: Constructing an induced rule for a partition of the agents into groups

Step 1 will be used in order to construct an “induced rule” which is defined on groups of the original agents. To this end, without loss of generality, label $N = \{1, \dots, n\}$. Let $m < n$. We may partition N into m groups $\{N_j\}_{j=1}^m$, so that for all $j < m$, $N_j = \{j\}$, and $N_m = \{m, \dots, n\}$. Label the partition $\mathcal{P} = \{N_j\}_{j=1}^m$. We show how to construct an induced rule on the partition, meaning that the agents are $\{N_j\}_{j=1}^m$.

To this end, let $\mathcal{B}^{\mathcal{P}}$ be the collection of closed, convex, comprehensive sets in $\mathbb{R}^{\mathcal{P}}$ that are bounded above. We claim that for all $B \in \mathcal{B}^{\mathcal{P}}$, there exists $B'(B) \in \mathcal{B}$ such that $B = \left\{ \left(\sum_{i \in N_j} x_i \right)_{j=1}^m : x \in B' \right\}$. We define a function which carries elements of $\mathbb{R}^{\mathcal{P}}$ into elements of \mathbb{R}^N . Thus, define $X : \mathbb{R}^{\mathcal{P}} \rightarrow \mathbb{R}^N$ by

$$X_i(x) = \begin{cases} x_{N_i} & \text{if } i \leq m \\ 0 & \text{otherwise} \end{cases}.$$

For all $B \in \mathcal{B}^{\mathcal{P}}$, define $B'(B)$ as the comprehensive hull of $\{X(x) : x \in B\}$.

First, it is clear that $\{X(x) : x \in B\}$ is closed, convex, and bounded above. Therefore, $B'(B) \in \mathcal{B}$. Moreover, we claim that $B = \left\{ \left(\sum_{i \in N_j} x_i \right)_{j=1}^m : x \in B' \right\}$. Thus, for all $x \in B$, $\left(\sum_{i \in N_j} X_i(x) \right)_{j=1}^m = x$. By definition of $B'(B)$, for all $x' \in B'(B)$, there exists $x \in B$ such that $x' \leq X(x)$. Hence $\left(\sum_{i \in N_j} X_i(x') \right)_{j=1}^m \leq x$, so that $\left(\sum_{i \in N_j} X_i(x') \right)_{j=1}^m \in B$.

We define an induced rule $f^{\mathcal{P}} : \mathcal{B}^{\mathcal{P}} \rightarrow \mathbb{R}^{\mathcal{P}}$ by $f^{\mathcal{P}}(B) = \left(\sum_{i \in N_j} f_i(B'(B)) \right)_{j=1}^m$. It is easy to see that for all $B, B^* \in \mathcal{B}^{\mathcal{P}}$, $B'(B + B^*) = B'(B) + B'(B^*)$, so that the rule $f^{\mathcal{P}}$ is additive. One can also similarly check its translation invariance. The efficiency and continuity of $f^{\mathcal{P}}$ follow immediately from the efficiency and continuity of f . Lastly, no advantageous reallocation is also trivially satisfied by $f^{\mathcal{P}}$.

Step 3: Construction of two problems leading to a contradiction

By Step 2, it is without loss of generality to assume that $|N| = 3$. We will establish that no three-agent rule can satisfy all of the axioms. Without loss of generality, label $N = \{1, 2, 3\}$.

By Step 2, f can be used to construct a collection of induced two-agent rules. In particular, for each agent $i \in N$, let $\mathcal{P}^i = \{\{i\}, \{j, k\}\}$ be a partition of N into a one-agent group containing agent i and a two-agent group containing the remaining agents. This induces a two-agent rule $f^{\mathcal{P}^i}$ as in Step 2, which satisfies all of the axioms. In particular, the Lemma establishes that for each such rule, there exists a corresponding $\lambda(i)$ associated with $\{i\} \in \mathcal{P}^i$.

We construct two problems in \mathcal{B} , each of which induces a two-agent problem that is the convex, comprehensive hull of two points. To this end, define

$$B \equiv \left\{ x \in \mathbb{R}^N : x \leq 1 \text{ and } \sum_{i \in N} x_i \leq 2 \right\}.$$

Clearly, this is a well-defined problem. For each partition \mathcal{P}^i , B induces a problem $B^i \in \mathcal{B}^{\mathcal{P}^i}$, where

$$B^i = \left\{ (x, y) \in \mathbb{R}^{\mathcal{P}^i} : x \leq 1, y \leq 2, x + y \leq 2 \right\}.$$

By the Lemma, $f_{\{i\}}^{\mathcal{P}^i}(B^i) = \lambda(i)$. By Step 1, we conclude $f_i(B) = \lambda(i)$. Hence $\lambda(1) + \lambda(2) + \lambda(3) = 2$. Thus, there exists some i such that $\lambda(i) > 0$. Without loss of generality, we suppose that $\lambda(1) > 0$.

Let $B^* \in \mathcal{B}$ be defined as

$$B^* \equiv \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i \leq 2, x_2 \leq 1, x_3 \leq 1 \right\} \\ \cap \left\{ x \in \mathbb{R}^N : x_1 + x_3 \leq 2, x_1 + x_2 \leq 2, x_1 \leq 2 \right\}.$$

In particular, for $i = 2, 3$, $B^{*i} = B^i$. Moreover,

$$B^{*1} = \left\{ x \in \mathbb{R}^{P^1} : x_{\{1\}} + x_{\{2,3\}} \leq 2, x_{\{1\}} \leq 2, x_{\{2,3\}} \leq 2 \right\}.$$

By the Lemma, $f_{\{1\}}^{\mathcal{P}^1}(B^{*1}) = 2\lambda(1)$, and by Step 1, $f_1(B^*) = 2\lambda(1)$. For $i = 1, 2$, $f_{\{i\}}^{\mathcal{P}^i}(B^{*i}) = \lambda(i)$, so that $f_i(B^*) = \lambda(i)$. Conclude

$$\begin{aligned} & f_1(B^*) + f_2(B^*) + f_3(B^*) \\ &= 2\lambda(1) + \lambda(2) + \lambda(3) \\ &= \lambda(1) + [\lambda(1) + \lambda(2) + \lambda(3)] \\ &= \lambda(1) + 2 > 2. \end{aligned}$$

Therefore, $f(B^*)$ is infeasible for B^* , a contradiction. ■

8 Conclusion

Our model is one in which there is no aggregate deficit or surplus of the privately consumed good. However, it is quite simple to extend our model to allow such possibilities. In such an extended environment, a rule would be a function $f : \mathcal{B} \times \mathbb{R} \rightarrow \mathbb{R}^N$. Thus, a “problem” would consist of a set of utilities induced by a collection of social alternatives, $B \in \mathcal{B}$, together with an aggregate subsidy t toward the agents, which could be positive or negative. Call this domain of “extended problems” \mathcal{B}' . Say that $x \in \mathbb{R}^N$ is feasible for (B, t) if $\sum_N x_i \leq \bar{x}(B) + t$. Our axioms would then have to be extended appropriately. The following definition of additivity is natural: For all $(B, t), (B', t') \in \mathcal{B}'$, $f(B, t) + f(B', t') = f(B + B', t + t')$. For any rule satisfying this definition, we obtain $f(B, t) = f(B, 0) + f(\mathcal{K}(\{0\}), t)$. Defining $g : \mathcal{B} \rightarrow \mathbb{R}$ as $g(B) = f(B, 0)$ and $h : \mathbb{R} \rightarrow \mathbb{R}$ as $h(t) = f(\mathcal{K}(\{0\}), t)$, we conclude that both g and h are additive and continuous. Hence, there exists some $\lambda \in \mathbb{R}^N$ such that $\sum_N \lambda_i = 1$, so that for all $(B, t) \in \mathcal{B}'$, $f(B, t) = g(B) + \lambda t$, where g satisfies our original axioms. Thus, extending our model in this fashion simply requires that any aggregate surplus be divided at some fixed rate among the agents.

There are several obvious directions for future research. We hope to obtain a more “geometric” characterization of the family of rules satisfying our axioms. The literature on Lipschitz selectors suggests that such a characterization may be possible. In addition, in the two agent case, it is known that any rule satisfying our axioms can be identified with a signed measure over the weighted coalitional rules, where the solution for any problem is simply calculated as the expectation of the solutions recommended by the weighted coalitional rules [2]. This result breaks down in the many-agent case, as the Steiner point rules demonstrate.

Moreover, there are several important properties that we hope to investigate in more detail. One such property is recursive invariance, an axiom that Chun [4] and Green [11] make heavy use of. Indeed, Chun suggests that any obvious rule should satisfy recursive invariance (he calls it trivial independence). Recursive invariance states that, upon solving a problem, if we are to add another social alternative to the problem whose welfare levels coincide with the solution of the original problem, then the solution should not respond. A characterization of all rules in our family together with recursive invariance is provided in Green [11]. However, there is no obvious generalization of this two-agent family to the many agent case, and so such

a characterization seems out of reach at this point.

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