Economics Letters 16 (1984) 31–36
North-Holland

PARTICIPATION CONSTRAINTS IN THE VICKREY AUCTION *

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Received 28 December 1983

Economic agents are characterized by two privately observable parameters: their willingness to pay for an item being auctioned, and their reservation utility level which must be exceeded, in expectation, to induce them to participate in this auction. This creates a situation in which the distribution of willingnesses to pay among the bidders is endogenous. For the case in which the parameters are jointly uniformly distributed, the existence of an equilibrium is proven and characterized.

1. Introduction

Models of self-selection in economics, of which the theory of auctions is a well-studied case, typically assume that the private information of each agent can be described by a one-dimensional parameter. For example, the preferences of an individual engaged in an auction for an indivisible object are describable by his willingness to pay for that item. Agents who actively enter bids in this auction choose them based on their true willingness to pay and on the bidding strategies of the other participants. Throughout auction theory it has been natural to assume that no agent could be forced to participate in the auction. If all agents' opportunity cost of participating were the same, this constraint could be translated into the statement that the expected utility obtained in the equilibrium of the auction exceeds the value of this common opportunity.

* Support from NSF and Commissariat au Plan are gratefully acknowledged.

The structure of auctions described above yields a particularly simple system. Because the private piece of information affects individuals' welfare monotonically, irrespective of the outcome of the auction, the system of participation constraints is automatically satisfied if the individual with the lowest possible willingness to pay can be induced to participate. More generally, the monopolist or central agent who designs the auction mechanism can set up entry fees in such a way as to eliminate the lower tail of the distribution of willingnesses to pay. The remaining participants will have their willingnesses to pay distributed according to the original distribution in the population, conditioned by exceeding this lower bound.

When the private information of individuals is more complex than a single parameter can capture, the distribution of agents' characteristics conditioned on their participation in the auction must be determined endogenously. In this paper we present a model of the Vickrey auction in which each agent is informed of two parameters. One, θ, is the value of the item to him. The other, λ, is the opportunity cost of participating in this auction. For each θ there will be a critical value of λ above which the agent will not participate. The participation constraint is binding at many points in the parameter space, instead of only one. We show that there exists an equilibrium relationship $\lambda^*(\theta)$ such that agents participate in the auction only if $\lambda < \lambda^*(\theta)$ and we characterize this relationship as the solution to a contraction mapping.

2. The bidding model

Consider a population of N buyers. Each buyer is characterized by a willingness to pay θ and a level of outside opportunity λ which is his expected utility if he does not participate in the auction.

The population results from independent random drawings in a distribution $F(\theta,\lambda)$ on $[0,1]\times[0,1]$. Let f_θ and f_λ (F_θ and F_λ) denote the marginal densities (and distributions) of θ and λ.

An agent with a willingness to pay θ participates in the auction only if his λ is less that his expected utility in the auction. His expected utility in the auction depends on the characteristics of the other participants.

Assume, provisionally, that an equilibrium exists. The proof of existence is given below for the case when F is uniform. Let $\lambda^*(\theta)$ denote the equilibrium expected utility of an agent whose willingness to pay for the item is θ. The densitiy of θ given participation is

$$g_{\lambda^*}(\theta) = \Pr\left[\theta \leqslant \tilde{\theta} \leqslant \theta + \mathrm{d}\theta | \lambda \leqslant \lambda^*(\theta)\right]$$

Let $G_{\lambda^*}(\theta)$ be the corresponding distribution function. Assuming that agents are risk neutral, the expected utility of an agent θ is obtained as follows:

Let m denote the maximal bid of the other agents. The value of participating in the auction, for a bidder whose true evaluation is θ, is

$$\lambda^*(\theta) = \int_0^\theta (\theta - m)\mathrm{d}G_{\lambda^*}^{n-1}(m). \tag{1}$$

Now let us assume that $F(\theta, \lambda)$ is uniform on $[0,1]^2$

$$\begin{aligned} G_{\lambda^*}(m) &= 1 - \int_m^1 \lambda^*(\theta)\mathrm{d}\theta \qquad \text{if} \quad m > 0, \\ &= \int_0^1 (1 - \lambda^*(\theta))\mathrm{d}\theta \quad \text{if} \quad m = 0. \end{aligned}$$

Hence, the equation defining $\lambda^*(\theta)$ is

$$\begin{aligned} \lambda^*(\theta) = &\int_0^\theta (\theta - m)(n-1)\left[1 - \int_m^1 \lambda^*(\tau)\mathrm{d}\tau\right]^{n-2} \lambda^*(m)\mathrm{d}m \\ &+ \theta\left[\int_0^1 (1 - \lambda^*(\tau))\mathrm{d}\tau\right]^{n-1}. \end{aligned} \tag{2}$$

Lemma 1. *If it exists,* $\lambda^*(\cdot)$ *is* C^∞ *and*

(*i*) $\dfrac{\mathrm{d}\lambda^*}{\mathrm{d}\theta} \geqslant 0, \quad \dfrac{\mathrm{d}^2\lambda^*}{\mathrm{d}\theta^2} \geqslant 0,$

(*ii*) $0 \leqslant \lambda^*(\theta) \leqslant \theta,$

(*iii*) $\lambda^*(0)=0, \quad \left.\dfrac{\mathrm{d}^k\lambda}{\mathrm{d}\theta^k}\right|_{\theta=0_+}=0 \quad \forall k,$

(*iv*) $\left.\dfrac{\mathrm{d}\lambda^*}{\mathrm{d}\theta}\right|_{\theta=1_-}=1.$

Proof. From (1), $\lambda^*(\cdot)$ is increasing and therefore almost everywhere differentiable. It is then integrable. From (2) it is continuous, as the integral of an almost everywhere differentiable function. But again from (2) it is differentiable as the integral of a continuous function. Repeating this argument we have that λ^* is C^∞.

Lemma 2. If it exists, $\lambda^*(\cdot)$ is not analytic.

Proof. From Lemma 1(iii), all derivatives at 0 are zero. If $\lambda^*(\cdot)$ were analytical, it would be zero. But $\lambda^* \equiv 0$ is not a solution to (2).

Let $\mathscr{C}$ be the space of continuous functions on [0,1] which are bounded above by the identity. Endow $\mathscr{C}$ with the sup-topology whose norm is denoted $\|\cdot\|$. $\mathscr{C}$ is a closed subset of a complete metric space and hence is a complete metric space.

Lemma 3. The function $F(\lambda(\cdot))$ from $\mathscr{C}$ into $\mathscr{C}$ defined by

$$F(\lambda(\cdot))=(n-1)\int_0^\theta(\theta-m)\left[1-\int_m^1\lambda(\tau)\mathrm{d}\tau\right]^{n-2}\lambda(m)\mathrm{d}m$$
$$+\theta\left[\int_0^1(1-\lambda(\theta))\mathrm{d}\theta\right]^{n-1}$$

is a contraction.

Proof. Note that

$$\int_m^1\lambda(\tau)\mathrm{d}\tau\leqslant\int_0^1\tau\mathrm{d}\tau\leqslant\tfrac{1}{2}, \tag{3}$$

$$(a-b)^n=(a-b)\left(a^{n-1}+ba^{n-2}+\ldots+b^{n-2}a+b^{n-1}\right)$$
$$\leqslant(a-b)\frac{n}{2^n}\quad\text{if}\quad a\leqslant\tfrac{1}{2},\quad b\leqslant\tfrac{1}{2},$$

$$(a-b)^n < (a-b)\tfrac{1}{2} \quad \text{if} \quad n > 1. \tag{4}$$

From the definition of F we have

$$\begin{aligned}
&F(\lambda^{t+1}(\theta)) - F(\lambda^t(\theta)) \\
&= (n-1)\int_0^\theta (\theta - m)\left\{\left[1 - \int_m^1 \lambda^{t+1}(\tau)\mathrm{d}\tau\right]^{n-2} \lambda^{t+1}(m)\right. \\
&\left. - \left[1 - \int_m^1 \lambda^t(\tau)\mathrm{d}\tau\right]^{n-2} \lambda^t(m)\right\}\mathrm{d}m \\
&+ \theta\left[\left[\int_0^1 (1 - \lambda^{t+1}(\tau))\mathrm{d}\tau\right]^{n-1} - \left[\int_0^1 (1 - \lambda^t(\tau))\mathrm{d}\tau\right]^{n-1}\right].
\end{aligned}$$

Integrating by parts we get

$$\begin{aligned}
&F(\lambda^{t+1}(\theta)) - F(\lambda^t(\theta)) \\
&= \int_0^\theta \left\{\left[1 - \int_m^1 \lambda^{t+1}(\tau)\mathrm{d}\tau\right]^{n-1} - \left[1 - \int_m^1 \lambda^t(\tau)\mathrm{d}\tau\right]^{n-1}\right\}\mathrm{d}m.
\end{aligned}$$

Using (3) and (4)

$$\begin{aligned}
|F(\lambda^{t+1}(\theta)) - F(\lambda^t(\theta))| &< \tfrac{1}{2}\int_0^\theta \left|\int_m^1 (\lambda^t(\tau)) - \lambda^{t+1}(\tau)\right| \mathrm{d}\tau \mathrm{d}m \\
&< \tfrac{1}{2}\|\lambda^{t+1}(\cdot) - \lambda^t(\cdot)\| \int_0^\theta \mathrm{d}m,
\end{aligned}$$

$$\|F(\lambda^{t+1}(\cdot)) - F(\lambda^t(\cdot))\| < \tfrac{1}{2}\|\lambda^{t+1}(\cdot) - \lambda^t(\cdot)\|.$$

This completes the proof that F is a contraction.

Therefore, according to the Banach fix point theorem [Schwarz (1970)] there exists a unique equilibrium function $\lambda^*(\theta)$. By virtue of Lemma 1, we know that it has the typical shape drawn in fig. 1.

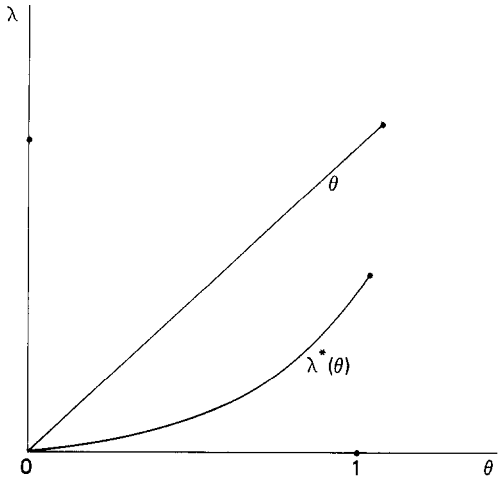

Fig. 1.

Reference

Schwarz, L., 1970, Analyse: Topologie générale et analyse functionelle (Hermann, Paris).