

**VARIANCE-MINIMIZING MONETARY POLICIES
WITH LAGGED PRICE ADJUSTMENT
AND RATIONAL EXPECTATIONS***

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This paper considers a macroeconomic model with rational expectations in which prices are incompletely flexible. Markets therefore fail to clear. In such a model monetary policy is not neutral. The variance of real and nominal quantities and interest rates is sensitive to the parameters of the feedback rule that determines the money supply. The monetary policy that achieves the goal of minimizing the steady-state variance of real output is characterized. We also examine monetary policies that are restricted in their generality and derive 'second-best' variance-minimizing feedback rules.

1. Introduction

Aggregative macroeconomic models incorporating rational expectations and market-clearing have led to the conclusion that systematic monetary policy will be entirely ineffective. This striking proposition has been the subject of much discussion in the literature. Although some elements of disagreement persist, it is generally perceived that violations of either the rational expectations assumption or the assumption that all markets are competitive, with perfectly flexible prices and wages, will cause this proposition to be falsified.¹

In this paper we ask what should be the form of the optimal money supply rule when prices are determined by a lagged adjustment to the

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¹Some references to the proposition and the resulting debate are Lucas (1972), Sargent-Wallace (1975, 1976), Barro (1976), Phelps-Taylor (1977), Fischer (1977), and McCallum (1977, 1978, 1979), Woglom (1979), Frydman (1979). See Fischer (1979) for a recent review on the policy debate, and Buiter (1980) and Schiller (1978) for evaluations of macroeconomic models incorporating rational expectations. Aspects of the debate in relation to the assumption of continuous market clearing will be more closely examined below.

imbalance between demand and supply in the market for goods. All of the other assumptions of the rational expectations models are maintained. The objective is to minimize the asymptotic variance of output because fluctuations in output or employment are usually taken to be the aim of stabilization policy. We also provide theoretical bounds on the percentage reduction in this variance that is attainable from pursuing the optimal monetary policy, as opposed to certain other 'benchmark' policies.

It is also of interest that the effect of a policy designed to stabilize output will, in our model, destabilize the price level. Rational expectations models, ours included, have primarily been formulated under the assumption that the structural equations are linear in anticipated future values of the endogenous variables. The effect of uncertainty about the future on today's decisions has been largely suppressed. By inducing a higher level of variability in the price level (and interest rates) one might expect that the mean *level* of output would be lower when its variance is reduced. Although we do not pursue that line of research here, it is important to note that the impotence of monetary policy to affect the asymptotic means of endogenous variables is a byproduct of the form of the structural equations assumed, as well as of the rational expectations hypothesis. The relevant tradeoff in our, linear, model is between the asymptotic *variances* of the endogenous variables. Therefore, in a model where uncertainty about the future is explicitly recognized, the relevant tradeoff will be between increasing the steady-state level of output and decreasing its variability.

In the next section the model is described, its rational expectations equilibrium is derived, and the asymptotic variance-covariance matrix is computed. The conditions on policy parameters that yield a stable dynamic structure are also analyzed. Section 3 uses the results derived in section 2 to discuss the money supply rule that would yield the minimum variance. This rule requires observations of both supply and demand, which may not be equal, and hence it may not actually be a feasible policy. Therefore, in section 4, simpler rules that require less information than the fully optimal one are discussed. The percentage reduction in the variance of output that can be achieved by the latter policies is computed, assuming particular values for the structural parameters. The stability of the system under some special rules is discussed. Stability can be problematic for realistic parameter values. Concluding comments are offered in section 5.

We assume that actual output will be the minimum of supply and demand, and we attempt to minimize its variance. Therefore, in our appendix, we provide the approximation for the variance of the minimum of two jointly normally distributed random variables.

2. The model and its solution

To make our analysis readily comparable with that of other papers in this

area we choose the following specification of the structural equations:

$$y_t = u_{1t}, \quad (1)$$

$$e_t = b(r_t - ({}_{t-1}p_{t+1}^* - {}_{t-1}p_t^*)) + u_{2t}, \quad (2)$$

$$m_t = p_t + c_1 e_t + c_2 r_t + u_{3t}, \quad (3)$$

$$p_t = p_{t-1} + d(e_{t-1} - y_{t-1}) + u_{4t}, \quad (4)$$

where

y_t = aggregate supply,
 e_t = aggregate demand,
 m_t = demand for money,
 r_t = nominal interest rate,
 p_t = price level,

all in period t , and ${}_{t-1}p_{t+1}^*$ and ${}_{t-1}p_t^*$ are the expected price levels as forecasted at $t-1$ to hold in periods $t+1$ and t , respectively. Predetermined variables include e_{t-1} , y_{t-1} , p_{t-1} and r_{t-1} . At date t , the expectations held at date $t-1$ are also viewed as predetermined; but in a dynamic stochastic equilibrium they are endogenous, being the true mathematical expectations of these variables conditioned on the observations at $t-1$. The model is described in deviation form, so that endogenous variables are discrepancies from the long-term mean values.

Eq. (1) above is an aggregate supply curve which for simplicity is taken to be vertical.² We are therefore abstracting from such ideas as money in the production function which have been discussed in monetary growth theory. Given that the model is in deviation form and that our interest is in stabilization policy, this simplification does not seem too bad. Eqs. (2) and (3) describe the aggregate demand curve and the money market equilibrium condition, respectively. Again the capacity effect has been abstracted from eq. (2), while (3) takes a standard form. All three equations are a simplification of the well-known framework of Sargent–Wallace (1975). Eq. (4), on the other hand, is the essential difference to the earlier models in this area, and it is a description of sluggish price adjustment.

Rule (4) can be rationalized in various ways, and it needs a detailed discussion. Traditionally (4) is said to represent the law of supply and demand, but in the context of growth theory Fischer (1972) argued that the auctioneer should take into account inflationary expectations, so that ${}_{t-1}p_t^*$

²It may be noted here that our later results would hold with the Lucas-type supply function, but we have not introduced it due to difficulties of interpreting it in connection with eq. (4). Indeed, the Lucas function is often not thought of as a behavioral equation, but rather a reduced form from a model of the labor market.

$-p_{t-1}$ should be added to the right-hand side. Similarly, with different formulations Barro (1972), Fischer (1972) and Mussa (1978) arrive at exactly the same conclusion from an aggregation over optimal price-setting behavior of individual agents. Mimicking these suggestions in broad terms we obtain the idea that 'desired' or 'optimal' price level, p_t^d , should obey the specification

$$p_t^d = \gamma(e_{t-1} - y_{t-1}) + {}_{t-1}p_t^* \quad (4')$$

Eq. (4') is indeed a standard formulation and it has been often used recently. If all prices are adjusted to their desired level at the beginning of each period, i.e., $p_t = p_t^d$, and expectations are rational, (4') has the very strong implication that we are back to the situation of continuous market clearing. To obtain disequilibrium behavior we add to (4') the postulate that actual prices adjust only partially to the desired level, so that

$$p_t - p_{t-1} = \beta(p_t^d - p_{t-1}), \quad 0 \leq \beta \leq 1. \quad (4'')$$

From (4') and (4'') we obtain the specification

$$p_t = p_{t-1} + \gamma\beta(e_t - y_t) + \beta({}_{t-1}p_t^* - p_{t-1}) + u_{4t}, \quad (4''')$$

where we have also added the error term. It is easy to see that (4) is obtained from (4''') by solving recursively, and, as long as³ $\beta \neq 1$, we have the value $d = \gamma\beta(1 - \beta)^{-1}$ for the coefficient of excess demand in (4). [See Honkapohja (1979) for an analysis of neutrality where $p_t^d = \gamma(e_t - y_t)$, in which case multiple solutions result. The non-uniqueness would raise difficult conceptual issues for policy optimization, and we wish to avoid them here.]

At this point it is worth pointing out that neutrality requires *both* aggregate demand and supply to be independent of monetary policy parameters. The usual demonstration that the forecast error $p_t - {}_{t-1}p_t^*$ be so is insufficient.^{4,5} These conclusions follow directly the fact that, in the

³It might be worth pointing out that with the lagging operation the case $\beta = 1$ of perfect synchronization of price adjustments leads to inconsistencies. In that event one may replace ${}_{t-1}p_t^* - p_{t-1}$ by the expected full employment rate of inflation ${}_{t-1}\bar{p}_t^* - \bar{p}_{t-1}$, where the bar refers to the general equilibrium solution of Sargent-Wallace (1975). Then $\beta = 1$ results in neutrality of money. We owe this result to Ben McCallum who called our attention to the fact that not all lagged price adjustment rules result in non-neutrality.

⁴This is the flaw in McCallum's (1977, 1978) analyses of the neutrality proposition under certain types of price stickiness. Indeed, it is evident that in those models aggregate supply is independent of the policy rule, but aggregate demand is not, which McCallum (1978, p. 425) also admits. See also McCallum (1980).

⁵In a somewhat different model Flemming (1979) has analyzed the effect of varying wage flexibility on the variance of output under certain fixed money supply rules. His principal result states that sometimes an increase in wage flexibility can lead to higher variance of employment.

presence of discrepancies between aggregate demand and supply, considerations of aggregate demand become relevant. Indeed, one can think of different targets for stabilization policy when such discrepancies are present. Modern so-called disequilibrium theory [see Barro–Grossman (1976) or Malinvaud (1977)] suggests immediately $\min(e_t, y_t)$ as the relevant output variable, so that minimization of its variance is a natural choice as the objective of stabilization policy.⁶

To close the system we need a description of the money supply rule. We assume a non-stochastic rule allowing dependence in principal on all of the relevant predetermined variables,

$$m_t = \alpha_e e_{t-1} + \alpha_y y_{t-1} + \alpha_p p_{t-1} + \alpha_r r_{t-1}. \quad (5)$$

At date t the short-run equilibrium is determined by solving (1)–(5) for the endogenous variables e_t , y_t , p_t , r_t and m_t .

Our goal is to choose the α 's so that the resulting stochastic process exhibits the minimum possible variance of $\min(e_t, y_t)$ asymptotically. Although this objective is compatible with the spirit of disequilibrium macroeconomics, our specification of the structural equations does not entirely carry through with this program.⁷ First, in the money demand equation the output variable should affect the true level of economic activity, namely $\min(e_t, y_t)$. Such a specification, however, would lead to enormous complications in the analysis since the reduced-form equations would be nonlinear in the disturbance terms, and hence the rational expectations could not be computed without a more precise description of the error structure. Our choice of e_t as the relevant output variable accords with common practice. [Using y_t would actually be simpler because one could determine the equilibrium r_t in the money market and then e_t from (2) recursively.] Second, it is to be observed that no spillover effects appear in the specification of aggregate demand and supply. The modern literature has stressed their importance; see e.g. Barro–Grossman (1976) and Malinvaud (1977). In some sense ours is only a one-sector model, so the only misspecification is that mentioned above. However, in a more complete analysis spillover effects should be taken into account. Unfortunately, they raise formidable mathematical problems in solving the model.

⁶Other possible objectives might be the minimization of $\text{var}(e_t - y_t)$ or $E(\min(e_t, y_t))^2$, and in the present framework the optimal rule remains the same, for reasons that become evident in the course of the analysis. The conclusions are therefore robust with respect to the choice of an output variable as the target of stabilization policy.

⁷It is also possible to interpret the model as a fairly standard IS-LM model, with eq. (7) specifying the full employment level of output, eq. (4) or its variants being a Phillips-curve relation from sluggish markup behavior, and rational expectations being incorporated. Indeed, the Sargent–Wallace (1975) model was designed to represent many features of the usual IS-LM framework. In what follows we stick to the 'disequilibrium' interpretation, but the reader can easily interpret the results according to the IS-LM frame of reference.

We now proceed to discuss the solution of (1)–(5) in a dynamic model with rational expectations linking the succession of short-run equilibrium. To solve the system at date t , notice that

$${}_{t-1}p_t^* \equiv Ep_t = p_{t-1} + d(e_{t-1} - y_{t-1}). \quad (6)$$

Therefore, to solve (1)–(5) we need only determine the rational expectation ${}_{t-1}p_{t+1}^*$. Leading (6) and taking expectations conditional on data at $t-1$ we have

$${}_{t-1}p_{t+1}^* - {}_{t-1}p_t^* = d({}_{t-1}e_t^* - {}_{t-1}y_t^*). \quad (7)$$

Notice that

$$y_t = u_{1t}, \quad (8)$$

and hence

$${}_{t-1}y_t^* = 0.$$

The problem amounts to finding ${}_{t-1}e_t^*$. Rewrite (2) as

$$e_t = b(r_t - d_{t-1}e_t^*) + u_{2t}. \quad (9)$$

Equate (3) and (5), solve for r_t in terms of the predetermined variables, p_t, e_t , and the disturbances. Eliminating p_t from the resulting expression using (4) yields

$$r_t = \frac{-c_1}{c_2} e_t + \frac{\alpha_e - d}{c_2} e_{t-1} + \frac{\alpha_y + d}{c_2} y_{t-1} + \frac{\alpha_p - 1}{c_2} p_{t-1} + \frac{\alpha_r}{c_2} r_{t-1} - \frac{u_{4t}}{c_2} - \frac{u_{3t}}{c_2}. \quad (10)$$

Substituting (10) into (9) and taking expectations provides us with the required reduced-form expression for ${}_{t-1}e_t$,

$${}_{t-1}e_t^* = \frac{1}{(bc_1/c_2) + bd} \left\{ b \frac{\alpha_e - d}{c_2} e_{t-1} + b \frac{\alpha_y + d}{c_2} y_{t-1} + b \frac{\alpha_p - 1}{c_2} p_{t-1} + b \frac{\alpha_r}{c_2} r_{t-1} \right\}. \quad (11)$$

Substituting (11) into (9) we obtain an equation relating e_t, r_t , the disturbances and the predetermined variables. Together with (10) this forms a

pair of simultaneous equations determining e_t and r_t . The solution of these equations yields

$$e_t = \frac{\alpha_e - d}{c_2 + bc_1 - bd} e_{t-1} + \frac{\alpha_y + d}{c_2 + bc_1 - bd} y_{t-1} + \frac{\alpha_p - 1}{c_2 + bc_1 - bd} p_{t-1} + \frac{\alpha_r}{c_2 + bc_1 - bd} r_{t-1} + u_{2t} - \frac{b}{c_2} u_{3t} - \frac{b}{c_2} u_{4t}, \quad (12)$$

$$r_t = \frac{\alpha_e - d}{c_2 + bc_1 - bd} K e_{t-1} + \frac{\alpha_y + d}{c_2 + bc_1 - bd} y_{t-1} + \frac{\alpha_p - 1}{c_2 + bc_1 - bd} K p_{t-1} + \frac{\alpha_r}{c_2 + bc_1 - bd} K r_{t-1} - \frac{c_1}{c_2} u_{2t} + \left(\frac{bc_1}{c_2^2} - \frac{1}{c_2} \right) u_{3t} + \left(\frac{bc_1}{c_2^2} - \frac{1}{c_2} \right) u_{4t}, \quad (13)$$

where

$$K = \frac{c_2 + bc_1 - bd - c_1}{c_2}.$$

The reduced form of the system is, therefore, (4), (8), (12) and (13), or, in matrix notation

$$x_t = Ax_{t-1} + Bu_t, \quad (14)$$

where

$$x_t = (e_t, y_t, p_t, r_t)^T, \quad u_t = (u_{1t}, u_{2t}, u_{3t}, u_{4t})^T,$$

and

$$A = \begin{pmatrix} \varepsilon_e & \varepsilon_y & \varepsilon_p & \varepsilon_r \\ 0 & 0 & 0 & 0 \\ d & -d & 1 & 0 \\ \varepsilon_e K & \varepsilon_y K & \varepsilon_p K & \varepsilon_r K \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 1 & -b/c_2 & -b/c_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -c_1/c_2 & (bc_1/c_2^2 - 1/c_2) & (bc_1/c_2^2 - 1/c_2) \end{pmatrix},$$

the ε 's being given by the coefficients of the corresponding variables in (12).

The policy variables (α 's) enter the equilibrium vector autoregression through the ε 's in the A matrix only. Because the relationship between the α 's

and the ε 's is linear and of full rank, we can consider the latter as the policy instruments directly.

The goal of our policy optimization is taken to be the minimization of the variance of $\min(e_t, y_t)$ in the steady state. Because y_t and steady-state correlation between y_t and e_t cannot be affected by any policy rule this amounts to minimizing the variance of e_t [compare (A.7) in the appendix]. We therefore turn to an analysis of how $\text{var} e_t$ depends on the ε 's. Then we will utilize the joint distribution of e_t and y_t to discuss the extent to which one can expect the variability of $\min(e_t, y_t)$ to be mitigated by monetary policies of some special types.

Let Ω be the variance-covariance matrix of x_t in the steady state, and let V be the variance-covariance matrix of Bu_t . It follows from the definition of a steady state that

$$\Omega - A\Omega A^T = V.$$

We will assume that the vector u_t is independently identically distributed over time, with a diagonal variance-covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & \sigma_{44} \end{pmatrix}.$$

Since $V = B\Sigma B^T$ we have

$$V = \begin{pmatrix} \sigma_{22} + \left(\frac{b}{c_2}\right)^2 \sigma_{33} + \left(\frac{b}{c_2}\right)^2 \sigma_{44} & 0 & \frac{-b}{c_2} \sigma_{44} & -\frac{c_1}{c_2} \sigma_{22} & -\left(\frac{c_1 b/2 - b/c_2}{c_2}\right) (\sigma_{33} + \sigma_{44}) \\ & 0 & 0 & 0 & \\ & & \sigma_{44} & \frac{bc_1/c_2 - 1}{c_2} \sigma_{44} & \\ & & & & \left(\frac{c_1}{c_2}\right)^2 \sigma_{22} + \left(\frac{bc_1/c_2 - 1}{c_2}\right)^2 (\sigma_{33} + \sigma_{44}) \end{pmatrix}.$$

(15)

The computation of $\Omega - A\Omega A^T$ is tedious but straightforward, yielding a matrix Z given by

$$\left(\begin{array}{cccc} (1 - \varepsilon_e^2)\omega_{11} - 2\varepsilon_e\varepsilon_y\omega_{12} & \omega_{12} & -d\varepsilon_e\omega_{11} - d(\varepsilon_y - \varepsilon_e)\omega_{12} & \omega_{14} - K(z_{11} - \omega_{11}) \\ -2\varepsilon_e\varepsilon_p\omega_{13} - 2\varepsilon_e\varepsilon_r\omega_{14} & & (1 - \varepsilon_e - d\varepsilon_p)\omega_{13} + d\varepsilon_r\omega_{14} & \\ -\varepsilon_y^2\omega_{22} - 2\varepsilon_y\varepsilon_p\omega_{23} & & -d\varepsilon_y\omega_{22} - (\varepsilon_y - d\varepsilon_p)\omega_{23} & \\ -2\varepsilon_y\varepsilon_r\omega_{24} - \varepsilon_p^2\omega_{33} & & +d\varepsilon_r\omega_{24} - \varepsilon_p\omega_{33} - \varepsilon_r\omega_{34} & \\ -2\varepsilon_p\varepsilon_r\omega_{34} - \varepsilon_r^2\omega_{44} & & & \\ & \omega_{22} & \omega_{23} & \omega_{24} \\ & & -d^2\omega_{11} - 2d\omega_{13} + 2d^2\omega_{12} & \omega_{34} + K(z_{13} - \omega_{13}) \\ & & -d^2\omega_{22} + 2d\omega_{23} & \\ & & & \omega_{44} - K^2(z_{11} - \omega_{11}) \end{array} \right) \quad (16)$$

where the z 's are respectively the indicated elements of this matrix, which are not repeated for the sake of brevity. (We have written numerical subscripts on the ω 's instead of identifying them with the name of the variables to which they correspond, using the obvious notation $e \leftrightarrow 1, y \leftrightarrow 2, p \leftrightarrow 3, r \leftrightarrow 4$.)

The ten unknown ω 's can be solved by identifying the entries in the upper right-hand parts of the two matrices — (15) and (16). Fortunately at least some simplification is possible because the simple structure of the second row and column of (15) allows us to identify $\omega_{12}, \omega_{22}, \omega_{23}, \omega_{24}$ with the corresponding elements of (16).

This leaves us with six linear equations for the remaining six ω 's. In matrix notation we can write these as

$$Mw = c, \quad (17)$$

where

$$w = (\omega_{11}, \omega_{13}, \omega_{14}, \omega_{33}, \omega_{34}, \omega_{44})^T,$$

$$M = \begin{pmatrix} 1 - \varepsilon_e^2 & -2\varepsilon_e\varepsilon_p & -2\varepsilon_e\varepsilon_r & -\varepsilon_p^2 & -2\varepsilon_p\varepsilon_r & -\varepsilon_r^2 \\ -d\varepsilon_e & 1 - \varepsilon_e - d\varepsilon_p & -d\varepsilon_r & -\varepsilon_p & -\varepsilon_r & 0 \\ -K\varepsilon_e^2 & -2K\varepsilon_e\varepsilon_p & 1 - 2K\varepsilon_e\varepsilon_r & -K\varepsilon_p^2 & -2K\varepsilon_p\varepsilon_r & -K\varepsilon_r^2 \\ -d^2 & -2d & 0 & 0 & 0 & 0 \\ -Kd\varepsilon_e & -K\varepsilon_e - Kd\varepsilon_p & -Kd\varepsilon_r & -K\varepsilon_p & 1 - K\varepsilon_r & 0 \\ -K^2\varepsilon_e^2 & -2K^2\varepsilon_e\varepsilon_p & -2K^2\varepsilon_e\varepsilon_r & -K^2\varepsilon_p^2 & -2K^2\varepsilon_p\varepsilon_r & 1 - K^2\varepsilon_r^2 \end{pmatrix},$$

$$c = \begin{pmatrix} v_{11} + 2\varepsilon_e \varepsilon_y v_{12} + \varepsilon_y^2 v_{22} + 2\varepsilon_y \varepsilon_p v_{23} + 2\varepsilon_y \varepsilon_r v_{24} \\ v_{13} + d(\varepsilon_y - \varepsilon_e) v_{12} - d\varepsilon_y v_{22} + (\varepsilon_y - d\varepsilon_p) v_{23} - d\varepsilon_r v_{24} \\ v_{14} + 2K\varepsilon_e \varepsilon_y v_{12} + K\varepsilon_y^2 v_{22} + 2K\varepsilon_y \varepsilon_p v_{23} + 2K\varepsilon_y \varepsilon_r v_{24} \\ v_{33} - 2d^2 v_{12} + d^2 v_{22} - 2dv_{23} \\ v_{34} + dK(\varepsilon_y - \varepsilon_e) v_{12} - dK\varepsilon_y v_{22} + K(\varepsilon_y - d\varepsilon_p) v_{23} - dK\varepsilon_r v_{24} \\ v_{44} + 2K^2 \varepsilon_e \varepsilon_y v_{12} + K^2 \varepsilon_y^2 v_{22} + 2K^2 \varepsilon_y \varepsilon_p v_{23} + 2K^2 \varepsilon_y \varepsilon_r v_{24} \end{pmatrix}$$

We are interested on the solution to (17) only in so far as it concerns the variance, ω_{11} , of e . Therefore solving (17) by Cramer's rule, we need $\det M$ and $\det M_1$, where M_1 is M with the first column replaced by c .

Again, extensive but straightforward computation yields

$$\begin{aligned} \det M &= -2d\varepsilon_p - d^2\varepsilon_p^2 + d^3\varepsilon_p^3 - 3Kd^2\varepsilon_p^2\varepsilon_r \\ &\quad + 2K^2d\varepsilon_p\varepsilon_r^2 + 4Kd\varepsilon_e\varepsilon_p\varepsilon_r - 3d^2\varepsilon_e\varepsilon_p^2 + 2d\varepsilon_e^2\varepsilon_p. \end{aligned} \quad (18)$$

The numerator, M_1 , is quite complex in general. To simplify the analysis we assume that the price adjustment process is deterministic and that monetary policy is invariant to lagged values of the nominal interest rate,⁸

$$\sigma_{44} = 0, \quad \varepsilon_r = 0.$$

Then we obtain

$$\begin{aligned} \det M_1 &= -2d\varepsilon_p(\sigma_{22} + (b/c_2)^2\sigma_{33}) \\ &\quad + \sigma_{11}(-2d\varepsilon_y^2\varepsilon_p + 2d^2\varepsilon_y\varepsilon_p^2 - d^2(d\varepsilon_p^3 - \varepsilon_p^2 - \varepsilon_e\varepsilon_p^2)), \end{aligned} \quad (19)$$

so that, asymptotically,

$$\text{var } e = \frac{(\sigma_{22} + (b/c_2)^2\sigma_{33}) + \sigma_{11}(\varepsilon_y^2 - d\varepsilon_y\varepsilon_p + (d\varepsilon_p/2)(d\varepsilon_p - 1 - \varepsilon_e))}{1 - (-d/2)\varepsilon_p + (d^2/2)\varepsilon_p^2 + \varepsilon_e^2 - (3/2)d\varepsilon_e\varepsilon_p}. \quad (20)$$

The stability conditions of this vector autoregressive process play an important role in analyzing the minimum of (20). The system is stable if and

⁸At the optimum it turns out that the interest rate should not affect monetary policy, so this is not really a restriction.

only if all the characteristic roots of A have modulus less than unity. Under the condition that $\varepsilon_p = 0$ this means

- (i) $d\varepsilon_p < 0$,
 - (ii) $1 - \varepsilon_e + d\varepsilon_p > 0$,
 - (iii) $2 + 2\varepsilon_e - d\varepsilon_p > 0$.
- (21)

3. Analysis of variance minimizing policy

Consider the denominator of (20). For each fixed value of ε_p it is concave in ε_e . The stability conditions imply that ε_e is bounded between

$$d\varepsilon_p + \varepsilon_e \geq d\varepsilon_p/2 - 1,$$

and at these limiting values, the denominator is equal to zero. Therefore it is within the interval $[0, 1]$ whenever the stability conditions are satisfied.

The policy parameters enter the numerator of (20) in the coefficient of σ_{11} ,

$$\varepsilon_y^2 - d\varepsilon_y\varepsilon_p + (d\varepsilon_p/2)(d\varepsilon_p - 1 - \varepsilon_e). \tag{22}$$

We now show that this is non-negative whenever the stability conditions are satisfied.

Note that for any fixed ε_p the minimum of $\varepsilon_y^2 - d\varepsilon_y\varepsilon_p$ with respect to ε_y is $-d^2\varepsilon_p^2/4$. This can always be attained because ε_y does not enter into the stability conditions (21) and therefore is unconstrained. To analyze the final term in (22), note that the first and third stability conditions require $d\varepsilon_p/2 < 0$ and $d\varepsilon_p - 1 - \varepsilon_e < d\varepsilon_p/2 < 0$. Therefore the final term in (22) is positive and bounded below by $d^2\varepsilon_p^2/4$. Hence (22) is necessarily non-negative.

Therefore one sees that the expression for the variances can never be below $(\sigma_{22} + (b/c_2)\sigma_{33})$ which accords with intuition since this is the variance of the reduced-form expression for e even in the short-run. The conditions under which this can be attained necessitate setting (22) equal to zero, violating (21.ii) in the sense that the restrictions are in the borderline case where the steady state no longer exists. In the borderline p_t becomes a random walk but other variables remain stable. This theoretical lower bound cannot be attained, but it can be approached.

There are two ways in which (22) can be set equal to zero. Either

$$d\varepsilon_p/2 - 1 - \varepsilon_e = 0 \quad \text{and} \quad \varepsilon_y = d\varepsilon_p/2, \tag{23}$$

or

$$\varepsilon_y = d\varepsilon_p = 0. \tag{24}$$

Under (23), the denominator will not be maximized. Therefore the unique variance minimizing policy is to set $\varepsilon_y = \varepsilon_p = 0$ [to satisfy (23)] and $\varepsilon_e = 0$ [to maximize the denominator of (20)].

This policy is a very natural one. Using the definitions of the ε 's in terms of the parameters of the monetary feedback rules, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, we see that the policy is

$$m_t = de_{t-1} - dy_{t-1} + p_{t-1} \equiv p_t. \quad (25)$$

Thus the variance minimizing policy is to set m_t in advance at the level which would keep the real supply of money in period t constant. Because there is no variance in the price-setting equation, this clearly stabilizes any systematic dynamic fluctuations in the real side of the system leaving only the variance attributable to random factors within period t ($\sigma_{22} + (b/c_2)^2 \sigma_{33}$). Sensitivity of m_t to the interest rate is unnecessary.

4. Properties of some other money supply rules

Having discovered the theoretical minimum variance and the first-best rule we shall now consider some other money supply rules: both second-best rules and interesting fixed rules. However, the treatment is far from exhaustive and confined to some computationally not too cumbersome cases.

4.1. Rules with $\alpha_1 = \alpha_2 = 0$

This additional constraint states that the monetary authority does not use any information about lagged values of aggregate demand and supply. This case arises naturally. Aggregate demand and supply may not be separately observable, especially in a disequilibrium setting.

It is in principle possible to work out the second-best rule in all different cases, but the details are very messy. To analyze some aspects set $\alpha_1 = \alpha_2 = 0$, so $\varepsilon_e = -d/\Delta$, $\varepsilon_y = d/\Delta$, where $\Delta = c_2 + bc_1 - bd$. The general expression for the variance of aggregate demand is

$$\omega_{11} = \frac{[\sigma_{22} + (b/c_2)^2 \sigma_{33}] + \{(d\varepsilon_p/2)[d\varepsilon_p - (1 + d/\Delta)] + (d/\Delta)^2\} \sigma_{11}}{1 - \{(d\varepsilon_p/2)[d\varepsilon_p - (1 - 3(d/\Delta))] + (d/\Delta)^2\}}. \quad (26)$$

The stability conditions stipulate $-(1 + d/\Delta) < d\varepsilon_p < \min\{0, 2(1 - d/\Delta)\}$, so that $-1 < d/\Delta < 3$ is needed for the sake of consistency.

Let us now adopt the sign conventions of Sargent–Wallace (1975). They postulate $c_1 > 0$, $c_2 < 0$, $b < 0$. In that case the sign of Δ is uncertain and depends on the magnitude of d . Recall that $d = \gamma\beta(1 - \beta)^{-1}$, where γ is the degree of responsiveness of desired prices to excess demand and measures the

flexibility of actual prices. Clearly $d \geq 0$, and keeping γ constant, it can be seen that $\lim_{\beta \rightarrow 0^+} d = 0$ and $\lim_{\beta \rightarrow 1^-} d = \infty$, so d can take widely different values. Consequently $\Delta < 0$ and $\Delta > 0$ for small and large values of d , respectively. By inspecting (26) it can be seen that in the coefficient of σ_{11} the part involving $d\varepsilon_p$ is always positive, and the whole term is minimized by setting $d\varepsilon_p$ equal to the upper bound. On the other hand, the term involving $d\varepsilon_p$ in the denominator may be decreasing or increasing in $d\varepsilon_p$, but it is certainly decreasing when $d/\Delta < \frac{1}{3}$. Therefore $\varepsilon_p \approx 0$ is certainly the second-best rule, when $-1 < d/\Delta < \frac{1}{3}$.

When $\varepsilon_p \approx 0$ we get

$$\omega_{11} \approx \frac{[\sigma_{22} + (b/c_2)^2 \sigma_{33}] + \sigma_{11}(d/\Delta)^2}{1 - (d/\Delta)^2} \tag{27}$$

Table 1
Efficacy of the rule $\alpha_1 = \alpha_2 = 0, \alpha_3 \approx 1$, compared to first-best rule.

		Percentage increase in var(min(y, e))
$d=0.05$	$\omega_{11} = 1.3478$	4.4
0.10	1.7227	21
0.15	2.3751	50
0.20	8.2267	310

The efficacy of the rule depends on the structural parameters, especially the value of d . As our illustration let us use the following numerical values $\sigma_{11} = \sigma_{22} = \sigma_{33} = 1, b = -0.1, c_1 = 0.5, c_2 = -0.2$. Then table 1 gives the increase in the variance of output for different low values of d . In the table d takes values which imply $\Delta < 0$. When $\Delta > 0$, the feasibility of the policy $\varepsilon_p \approx 0$ necessitates $d/\Delta < 1$, as is evident from the stability conditions. Therefore we require $(1+b)d < c_2 + bc_1 < 0$ or $b < -1$, i.e., aggregate demand should be elastic with respect to the real rate of interest. This is a strong requirement, and if $b > -1$, the second-best policy can not be $\varepsilon_p \approx 0$. Therefore for the high values of d which imply $\Delta > 0$, the second-best policy is likely to involve $\alpha_3 < 1$, i.e., less than the full accommodation of the first-best rule.

4.2. Rules with $\alpha_3 = 0$

Next, let us consider policies in which the monetary authority doesn't use

information about the price level in its money supply rule. As in the overall optimal rule the best value for ε_y is $d\varepsilon_p/2$ so that in this case

$$\omega_{11} = \frac{[\sigma_{22} + (b/c_2)^2 \sigma_{33}] + \sigma_{11} [d\varepsilon_e/2\Delta + d/2\Delta + (d/2\Delta)^2]}{1 - [\varepsilon_e^2 + (3d/2\Delta)\varepsilon_e + d/2\Delta + \frac{1}{2}(d/\Delta)^2]}, \tag{28}$$

where $1 - d/\Delta > \varepsilon_e > -1 - d/2\Delta$ by stability. The second-best rule cannot be at an endpoint of the range, since there the denominator vanishes. On the other hand, to minimize the numerator ε_e should be made as small as possible, and therefore an interior minimum exists. Again the details are cumbersome, though by stability $d/\Delta > 0$, and it follows that the second-best rule will involve $\varepsilon_e < 0$, i.e., $\alpha_3 < d$.⁹ This means that the rule should not fully accommodate demand shocks in their effect to real balances. In the special case $\sigma_{11} = 0$, that is no supply shocks, the solution would be easy.

4.3. Constant nominal money supply

One intuitively interesting rule is to hold the nominal supply of money constant. This can perhaps be considered to be an analogue of Milton Friedman’s constant growth rate rule for a non-growing and non-inflationary economy. By setting $\alpha_1 = \alpha_2 = \alpha_3 = 0$, we obtain the formula

$$\omega_{11} = [\sigma_{22} + (b/c_2)^2 \sigma_{33}] [1 + d/(2\Delta - d)] + \sigma_{11} [(4(d/\Delta) + 1)/(2\Delta - d)] d, \tag{29}$$

for the steady-state variance of aggregate demand. The stability conditions necessitate $\Delta > 0$ and $2\Delta > d$. The former necessitates that d be large enough to ensure $\Delta > 0$, while the latter can be rewritten in the form $(b + \frac{1}{2})d < c_2 + bc$.

Table 2
Performance of the constant money supply rule,
compared to first-best rule.

$d =$	$\omega_{11} =$	Percentage increase in $\text{var}(\min(e, y))$
6	281.55	2569
10	100.90	926
20	70.43	649
50	60.17	556
100	57.46	531
500	55.46	513

⁹The proof involves calculating $d\omega_{11}/d\varepsilon_e$ which is essentially quadratic with all coefficients positive, so that, by Descartes’ theorem, its zeros are negative.

Since $c_2 + bc_1 < 0$ and $d \geq 0$, the requirement $b < -\frac{1}{2}$ is necessary for stability. Therefore enough price flexibility and high enough elasticity of aggregate demand with respect to the real rate of interest are needed to ensure stability of the constant money supply rule. In the stable case the performance of the rule evidently depends on the values of the structural parameters, in particular on d . Let us choose $\sigma_{11} = \sigma_{22} = \sigma_{33} = 1$, $b = -0.6$, $c_1 = 0.5$, $c_2 = -0.2$ as the numerical parameter values in order to evaluate for different values of d the increase in variance of output resulting from the use of the constant money supply rule, in relation to the optimal rule. Table 2 suggests that the constant money supply rule does not perform very well. The high percentages themselves are not very interesting. More importantly we note the changes in the last column, indicating that increasing the value of d beyond a certain point will not improve the performance. Moreover the increase in the variance is largely due to the term involving σ_{11} . In any event it is worth examining the limiting behavior of the coefficients of σ_{11} and $[\sigma_{22} + (b/c_2)^2 \sigma_{33}]$. For these we get

$$1 + d(2A - d)^{-1} \rightarrow 1 + (1 + 2b)^{-1} \quad \text{as } d \rightarrow \infty,$$

and

$$d(2A - d)^{-1}(4(d/A) + 1) \rightarrow (1 + 2b)^{-1}(4|b| + 1) \quad \text{as } d \rightarrow \infty.$$

Therefore the higher the elasticity of aggregate demand with respect to the real rate of interest the better is the performance of the constant money supply rule.

5. Concluding remarks

The results in the paper can be briefly summarized as follows. First, if prices adjust slowly and discrepancies between aggregate demand and supply can appear, the demonstration of monetary neutrality requires that both aggregate demand and supply be independent of systematic policy rules. For many price adjustment rules and specifications of the model neutrality does not hold, but for certain specifications neutrality prevails. Second, if one postulates a vertical (or Lucas-type) aggregate supply function, then the problem of stabilizing output usually leads to minimizing the variance of aggregate demand. Under our lagged price adjustment rules the optimal policy requires that real balances be held constant. If the monetary authority has no information of aggregate demand and supply, the coefficient of lagged prices in the second-best money supply rule should still be approximately unity when prices are very sluggish. With more responsiveness this policy is

infeasible and less accommodation is needed. If no price variable in it is permitted the resulting second-best rule stipulates less accommodation than in the first-best rule with respect to aggregate demand and supply. Finally, stability can be problematic for some second-best and 'benchmark' rules.

While the model allows for discrepancies in aggregate demand and supply and uses their minimum as the target variable, the paper does not allow for the full complexities of a disequilibrium model in the sense that behavior does not fully reflect past and expected disequilibria. At present, formidable formal complications do not permit such an analysis. Moreover, the results known so far suggest that the neutrality proposition is sensitive to the specification of the model, especially of the price adjustment rule. Therefore, more theoretical and empirical work is needed before the issues can have a final resolution.

Appendix: Approximations for $E[\min(X, Y)]$ and $\text{var}[\min(X, Y)]$

In the main text the minimum of aggregate demand and supply was used as the target variable of policy. Since the formula is not readily available we derive the approximations in some detail.

The problem is to find approximations for $E[\min(X, Y)]$ and $\text{var}[\min(X, Y)]$, when (X, Y) is multnormally distributed with zero mean and a given covariance matrix,

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}.$$

Let $Z = \min(X, Y)$. Then

$$P\{Z \leq z\} = P\{X \leq z, Y \leq z\} + P\{X \leq z, Y > z\} + P\{Y \leq z, X > z\}.$$

Utilizing the facts $P\{X \leq z, Y > z\} = P\{X \leq z, -Y \leq -z\}$ and $P\{Y \leq z, X > z\} = P\{-X \leq -z, Y \leq z\}$, we get

$$P\{Z \leq z\} = F_{X,Y}(Z, Z) + F_{X,-Y}(Z, -Z) + F_{-X,Y}(-Z, Z), \quad (\text{A.1})$$

where $F_{X,Y}()$ etc. indicate the relevant cumulative distribution functions. If $(X, Y) \sim N(0, \Sigma)$, then $(x, -Y)$ and $(-X, Y)$ are both $N(0, \tilde{\Sigma})$, where

$$\tilde{\Sigma} = \begin{bmatrix} \sigma_1^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_2^2 \end{bmatrix}.$$

From (A.1) it is possible in principle to evaluate $E(Z)$ and $\text{var}(Z)$. However,

closed-form expressions are unobtainable. To develop an approximation we use the power series [see Cramer (1946, p. 270)]

$$\int_{-\infty}^x \int_{-\infty}^y f(u, v) du dv = \sum_{v=0}^{\infty} \frac{\Phi^{(v)}(x/\sigma_1)\Phi^{(v)}(y/\sigma_2)}{v!} \rho^v, \tag{A.2}$$

where $f(u, v)$ is the multivariate normal density function with zero mean and

$$\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

covariance matrix, and $\Phi(\cdot)$ is the cumulative distribution function of the standard univariate normal distribution. From (A.2) we get a series form for the cumulative distribution function of Z ,

$$\begin{aligned} P\{Z \leq z\} &= \sum_{v=0}^{\infty} \frac{\Phi^{(v)}(z/\sigma_1)\Phi^{(v)}(z/\sigma_2)}{v!} \rho^v \\ &+ \sum_{v=0}^{\infty} \frac{\Phi^{(v)}(z/\sigma_1)\Phi^{(v)}(-z/\sigma_2)}{v!} (-\rho)^v \\ &+ \sum_{v=0}^{\infty} \frac{\Phi^{(v)}(-z/\sigma_1)\Phi^{(v)}(z/\sigma_2)}{v!} (-\rho)^v. \end{aligned} \tag{A.3}$$

Since $\Phi^{(v)}(-z) = \Phi^{(v)}(z)$ for $v = 1, 3, 5, \dots$, and $\Phi^{(v)}(-z) = -\Phi^{(v)}(z)$ for $v = 2, 4, 6, \dots$, differentiating the right-hand side of (A.3) term by term, the density function, $g(z)$, of Z takes the form

$$\begin{aligned} g(z) &= \sum_{v=0}^{\infty} ((-\rho)^v/v!) [(1/\sigma_1) \Phi^{(v)}(-z/\sigma_2) \Phi^{(v+1)}(z/\sigma_1) \\ &+ (1/\sigma_2) \Phi^{(v)}(-z/\sigma_1) \Phi^{(v+1)}(z/\sigma_2)]. \end{aligned} \tag{A.4}$$

Using (A.4) the integration $zg(z)dz$ yields term-by-term the following results:

$v=0$:

$$-(\sigma_1/\sigma_2 + \sigma_2/\sigma_1) \int_{-\infty}^{\infty} \Phi'(z/\sigma_1) \Phi'(z/\sigma_2) dz,$$

$v=1$:

$$\rho \int_{-\infty}^{\infty} \Phi'(z/\sigma_1) \Phi'(z/\sigma_2) dz, \tag{A.5}$$

$v=2$:

$$\rho^2 \sigma_1 \sigma_2 / (2(\sigma_1^2 + \sigma_2^2)^{3/2} \sqrt{2\pi}).$$

Eliminating the remaining integrals in (A.5) one obtains the final approximation

$$E(Z) \approx \frac{\sigma_1 \sigma_2}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \left[\rho - (\sigma_1/\sigma_2 + \sigma_2/\sigma_1) + \frac{\rho^2}{2(\sigma_1^2 + \sigma_2^2)} \right],$$

in which the last term in the square brackets is likely to be very small.

Next we derive $E(Z^2)$. For $v=0$ in the power series (A.3) we get

$$\sigma_1^2 + \sigma_2^2 - \sigma_1 \int_{-\infty}^{\infty} \Phi'(z/\sigma_1) \Phi(z/\sigma_2) dz - \sigma_2 \int_{-\infty}^{\infty} \Phi'(z/\sigma_2) \Phi(z/\sigma_1) dz, \quad (\text{A.6})$$

while by antisymmetry of the integrands all other terms vanish. To eliminate the integrals in (A.6) we utilize the power series [see e.g. van der Waerden (1969, p. 11)]

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \left[x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{2!2^2 \cdot 5} - \frac{x^7}{3!2^3 \cdot 7} + \dots \right]. \quad (\text{A.7})$$

In the ensuing integrations the part of (A.7) in the square brackets vanishes, as upon integrating term-by-term each term is as multiples of odd moments of the standard univariate normal distribution. Thus we get

$$E(Z^2) \approx \frac{1}{2}(\sigma_1^2 + \sigma_2^2).$$

By taking only the linear approximation of $E(Z)$ we finally have

$$\text{var}(Z) \approx \frac{\pi-1}{2\pi} (\sigma_1^2 + \sigma_2^2) + \frac{\sigma_{12}}{2\pi} - \frac{\sigma_{12}^2}{2\pi[\sigma_1^2 + \sigma_2^2]}, \quad (\text{A.8})$$

where we have resorted from the correlation coefficient back to the covariance $\sigma_{12} = \rho\sigma_1\sigma_2$.

References

- Barro, R.J., 1972, A theory of monopolistic price adjustment, *Review of Economic Studies* 39, 17-26.
- Barro, R.J., 1976, Rational expectations and the role of monetary policy, *Journal of Monetary Economics* 2, 1-33.
- Barro, R.J. and H.I. Grossman, 1976, *Money, employment and inflation* (Cambridge University Press, Cambridge).
- Buiter, W., 1980, The macroeconomics of Dr. Pangloss: A critical survey of the new classical macroeconomics, *Economic Journal* 90, 34-50.
- Cramer, H., 1946, *Mathematical methods of statistics* (Princeton University Press, Princeton, NJ).

- Fischer, S., 1972, Keynes–Wicksell and neoclassical models of money and growth, *American Economic Review* 62, 880–890.
- Fischer, S., 1977, Long term contracts, rational expectations, and the optimal money supply rule, *Journal of Political Economy* 85, 191–206.
- Fischer, S., 1979, On activist monetary policy with rational expectations, NBER working paper no. 341 (National Bureau of Economic Research, New York).
- Flemming, J., 1979, Wage flexibility and employment stability: The Keynesian property of a neo-classical model, Unpublished paper.
- Frydman, R. 1979, Sluggish price adjustments and the effectiveness of monetary policy under rational expectations, Discussion paper no. 79–07 (Columbia University, New York).
- Honkapohja, S., 1979, Monetary non-neutrality with anticipatory price setting and rational expectations, *Economics Letters* 4, 23–27; also Corrigendum, forthcoming.
- Lucas, R.F., 1972, Expectations and the neutrality of money, *Journal of Economic Theory* 4, 103–114.
- McCallum, B., 1977, Price-level stickiness and the feasibility of monetary policy with rational expectations, *Journal of Political Economy* 85, 627–634.
- McCallum, B., 1978, Price level adjustments and the rational expectations approach to macroeconomic stabilization policy, *Journal of Money, Credit and Banking* 10, 418–436.
- McCallum, B., 1979, Monetarism, rational expectations, oligopolistic pricing and the MPS econometric model, *Journal of Political Economy* 87, 57–73.
- McCallum, B., 1980, Hahn's theoretical viewpoint on unemployment: A comment, *Economica* 42, 301–305.
- Malinvaud, E., 1977, *The theory of unemployment reconsidered* (Basil Blackwell, Oxford).
- Mussa, M., 1978, Sticky prices and disequilibrium adjustment in a rational model of the inflationary process, Center for Mathematical Studies in Business and Economics report no. 7844 (University of Chicago, Chicago, IL).
- Phelps, E.S. and J.B. Taylor, 1977, Stabilizing powers of monetary policy under rational expectations, *Journal of Political Economy* 85, 163–190.
- Sargent, T.J. and N. Wallace, 1975, Rational expectations, the optimal monetary instrument, and the optimal money supply rule, *Journal of Political Economy* 83, 241–254.
- Sargent, T.J. and N. Wallace, 1976, Rational expectations and the theory of economic policy, *Journal of Monetary Economics* 2, 169–184.
- Schiller, R.J., 1978, Rational expectations and the dynamic structure of macro-economic models: A critical review, *Journal of Monetary Economics* 4, 1–44.
- van der Waerden, B.L., 1967, *Mathematical statistics* (Springer-Verlag, New York).
- Woglom, G., 1979, Rational expectations and monetary policy in a simple macro-economic model, *Quarterly Journal of Economics* 93, 91–105.