

APPROXIMATING THE EFFICIENCY GAIN OF TAX REFORMS

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Proper analysis of tax reform requires evaluation of the welfare effects induced by a change from one tax system to another. We present two methods for estimating these changes using only local information pertaining to an initial equilibrium with distortive taxes. It is shown that these methods provide very accurate approximations to the true gains even when large tax changes are involved. Concentrating on a model with capital and labor income taxes, we show that other approximations whose reference point is a nondistortive equilibrium are considerably less precise. Some concluding remarks are made on the potential of these methods for optimization purposes.

1. Introduction

Tax reform is concerned with modifications of an existing tax system. On the other hand, commonly used measures of the welfare effects of tax changes are based on small changes from an initial non-distorted equilibrium. This practice neglects three principal aspects of most tax reforms.

First, formulas for the efficiency loss of a tax (deadweight burden) implicitly assume that induced revenue changes are offset by lump-sum transfers. Since these are rarely available in practice, the way in which compensating revenue measures are taken may have further welfare implications.

Second, the distortions generated by the existing tax structure cannot be disregarded when analyzing alternatives. Assuming the equality of private

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and social costs, the standard formula [Harberger (1954)] provides an adequate measure of the differential welfare loss of taxation. However, when these costs diverge, the effect of tax changes on the quantities consumed must be weighted by the difference between them. The full effect of a tax change will comprise both types of effects.

Finally, when relatively large reforms are contemplated, conventional linear approximations may be subject to error. Global, rather than local, information is required for a more accurate estimate. In the absence of functional forms valid for unobserved ranges of the data, such approximations are inevitable.

This paper investigates the nature of the appropriate approximations to welfare losses in the presence of the problems mentioned above.

Section 2 covers the theory of welfare loss approximations. The exact measure of excess burden is discussed first. Then it is applied to the Harberger formula, at an equilibrium without taxes. With initial taxes, two more complex approximation methods are presented and analyzed.

Section 3 applies these approximations to the welfare analysis of changing interest income taxes and offsetting the revenue lost by increasing wage taxes. The accuracy of these approximations is contrasted with earlier methods. Section 4 discusses the possibility of using these approximations to find globally optimal tax structures.

2. Excess burden of taxation

2.1. Definition of excess burden

The efficiency analysis of taxation considers the consumption sector to be represented by a single utility-maximizing individual who takes prices as given. He chooses a vector of consumption levels x_i , $i=1, \dots, N$ for N commodities, whose unit production costs are assumed to be constant at p_i , $i=1, \dots, N$. Prices faced by the consumer, q_i , $i=1, \dots, N$ are determined by specific taxes t_i , $i=1, \dots, N$ according to

$$q_i = p_i + t_i, \quad i = 1, \dots, N. \quad (1)$$

If the consumer's income is I and he maximizes $U(x)$, the optimal quantities demanded are functions of $q = (q_1, \dots, q_N)$ and I . Alternatively, we consider the dual to this problem:

$$\min_x q \cdot x,$$

subject to

$$U(x) \geq u.$$

The solution to this problem gives

$$x_i = x_i(q, u), \quad i = 1, \dots, N, \quad (2)$$

which is the system of compensated demand functions. Let $x(q, u) = (x_1(q, u), \dots, x_n(q, u))$. Further; the value of this problem denoted by $E(q, u) \equiv q \cdot x(q, u)$ is the *expenditure function* which gives the minimum level of income required to attain the indicated utility at the consumer prices q .

A tax system $t = (t_1, \dots, t_n)$ generates revenues $T(q, p, u) = t \cdot x(q, u) = (q - p)x(q, u)$ when the consumer faces prices q and has utility u .

The deadweight loss of a tax system t at producer prices p and utility u , denoted by $L(q, p, u)$, is defined [see Diamond and McFadden (1974)] to be the difference in the income level necessary to maintain this utility and that required to sustain it in the absence of taxes, minus the revenues generated:

$$L(q, p, u) \equiv E(q, u) - E(p, u) - T(q, p, u). \quad (3)$$

It is customary to set u such that

$$E(p, u) = I \quad (4)$$

that is, to choose the utility level associated with the pretax situation. The interpretation is that tax revenues are returned to the individual by lump-sum transfers. The deadweight loss represents the *additional* compensation, beyond this, that would be necessary for achieving the utility level u .

In general, therefore, it is necessary to have exact knowledge of the expenditure function in order to calculate L . By duality, this amounts to requiring a complete specification of the utility function.

2.2. Harberger's formula

In practice, knowledge of the expenditure function is not available. Accordingly, using estimates only of the slopes of compensated demand functions, Hicks (1939), Harberger (1954), and Hotelling (1938) suggested the well-known approximation formula

$$\hat{L}(p + t, p, u) = -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N t_i t_j S_{ij}, \quad (5)$$

where

$$S_{ij} = S_{ij}(q, u) \equiv -\frac{\partial x_i(q, u)}{\partial q_j}, \quad i, j = 1, 2, \dots, N. \quad (6)$$

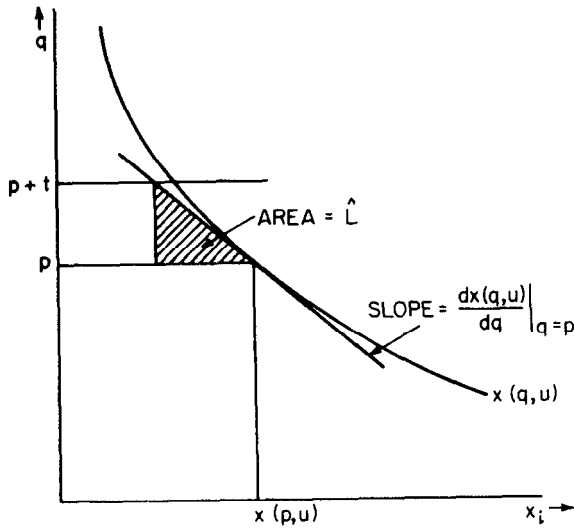


Fig. 1

That is, S_{ij} are the slopes of the compensated demand functions.

In the case of a single taxed commodity model, this gives the standard 'triangle' measure of excess burden.

As is well known, this is an accurate approximation only if t is small so that all S_{ij} 's can be regarded as constant. The curvature properties of $x(q, u)$ would have to be known for an exact measure, when this linear approximation would be inaccurate.

2.3. Differential welfare loss with initial taxes: Expansion in p

More generally, when $t \neq 0$ at the initial situation, as is typically the case in tax reform issues, Harberger's formula must be modified to include the induced effects on existing revenues. If a tax change from t to $t + \Delta t$ is contemplated, the change in the deadweight burden is

$$L(p + t + \Delta t, p, u) - L(p + t, p, u), \quad (7)$$

which can be approximated by the first two terms of its Taylor series expansion in q

$$\Delta t \frac{\partial L(q, p, u)}{\partial q} + \frac{1}{2} (\Delta t)^2 \frac{\partial^2 L(q, p, u)}{\partial q^2} (\Delta t). \quad (8)$$

Using the definition of L and the fact that derivatives of the expenditure function are equal to the corresponding compensated demands ($\partial E(q, u)/\partial q = x(q, u)$), these first two terms can be written as

$$-\sum_i \Delta t_i \sum_j t_j S_{ij} - \frac{1}{2} \sum_i \sum_j \Delta t_i \Delta t_j \left(S_{ij} + \sum_{k=1}^N t_k \frac{\partial S_{ij}}{\partial q_k} \right) + \dots \quad (9)$$

Clearly when $t=0$, this reduces to Harberger's formula. When $t \neq 0$, two additional effects must be considered.

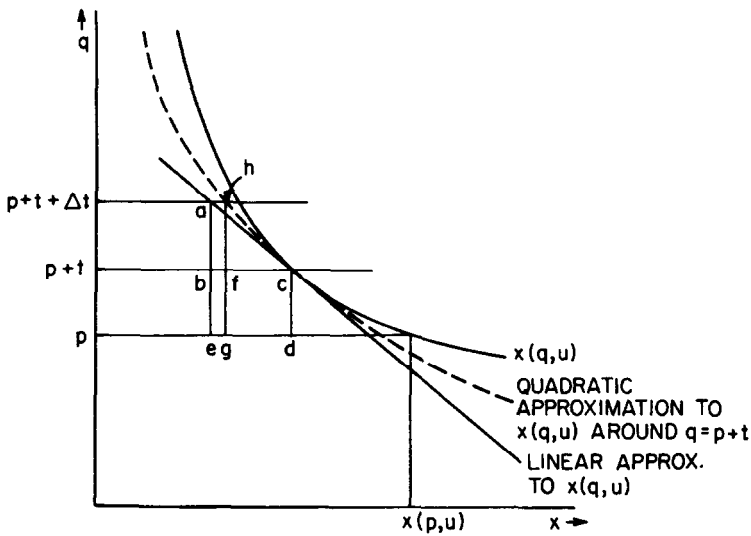


Fig. 2

The first represents the fact that tax revenues collected at t will be changed by the introduction of the new taxes Δt . In the one-commodity case illustrated above, this is represented by the rectangle $bcde$, which is the 'first-order' welfare reflecting the divergence between private and social costs in the initial situation.

The second-order effect has two parts. One is the same in form as that in the case of no initial taxes. It is an approximation to a change in consumer's surplus. It uses an estimate of the change in quantity from a first-order expansion of the compensated demand, multiplied by the change in price. The familiar triangle abc results.

The remaining part of the second-order effect is the second new term arising when taxes are present in the initial situation. It is an attempt to

approximate the change in tax revenue more precisely by using a quadratic expansion of compensated demand to estimate the change in quantity. It modifies the first-order effect to $fc dg$. Thus the change in welfare up to the second order is approximated by the figure composed of abc and $fc dg$.

Several observations should be made about this expansion. It is not the same estimate as one would obtain by using a second-order approximation to demand for both the first- and second-order effects – a procedure which would give rise to the area between the curve hc and gd . (Note that because the curve between h and c is a known quadratic function this area could be computed exactly by integration and need not be approximated itself.) This does not imply that the approximation (9) is necessarily a less accurate estimate of the true welfare loss than is the second-order approximation discussed above.

Unlike in the case of $t=0$, information about the slopes of compensated demands is inadequate for a valid second-order approximation. If Δt is sufficiently small that the second-order terms can be neglected, information about slopes is sufficient and the welfare loss is measured by the change in the revenue at the initial system due to the induced change in quantities, $\sum_i t_i \sum_j S_{ij} \Delta t_j$. If a more precise approximation is desired, information concerning the second derivatives of the compensated demand functions and the existing tax system is necessary.

The approximation formula (9) uses the derivatives evaluated at the taxed equilibrium and expands along a path of tax rates from t to $t + \Delta t$. An alternative approximation method suggests itself on the basis of the Harberger analysis of the untaxed equilibrium. That is, the welfare losses at t and $t + \Delta t$ can be calculated vis-a-vis the initial untaxed situation. The difference between these is a measure of the welfare change in moving from one to the other.

Using (9) at $t=0$ we find that

$$\hat{L}(p+t, p, u) = -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N t_i t_j S_{ij} \quad (10)$$

and

$$\hat{L}(p+t+\Delta t, p, u) = -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N (t_i + \Delta t_i)(t_j + \Delta t_j) S_{ij}. \quad (11)$$

The difference is

$$-\sum_{i=1}^N \Delta t_i \sum_{j=1}^N t_j S_{ij} - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \Delta t_i \Delta t_j S_{ij}. \quad (12)$$

This is to be compared with (9). These two estimates of the welfare loss differ in two ways. First, the terms in $t_k(\partial S_{ij}/\partial q_k)$ are not present when the 'indirect' path of approximation is used. Second, derivatives are evaluated at tax rates of zero rather than at t . Notice that if $t=0$, then the two estimates coincide since neither of these effects would be operative. We will show that it may be important to use the more direct approximation method, for obtaining an estimate of the true welfare loss, in order to avoid the biases we have discussed.

2.4. Differential welfare loss with initial taxes: Expansion in log p

An alternative form of approximation can be obtained by writing the welfare loss as a function of the logarithm of prices

$$\bar{L}(\log q, \log p, u) \equiv L(q, p, u). \tag{13}$$

The second-order approximation to the differential welfare loss, $\bar{L}(\log q + \Delta t, \log p, u) - \bar{L}(\log q, \log p, u)$, from an initial taxed equilibrium $q = p + t$, is given by

$$-\sum_i \left(\log \frac{q_i + \Delta t_i}{q_i} \right) \sum_j t_j x_j \varepsilon_{ji} - \frac{1}{2} \sum_i \sum_j \left(\log \frac{q_i + \Delta t_i}{q_i} \right) \left(\log \frac{q_j + \Delta t_j}{q_j} \right) \left\{ q_j x_j \varepsilon_{ji} + \sum_k t_k x_k \left[\frac{\partial \varepsilon_{ji}}{\partial \log q_j} + \varepsilon_{ki} \varepsilon_{kj} \right] \right\} \tag{14}$$

where

$$\varepsilon_{ji} = - \frac{d \log x_j(q, u)}{d \log q_i}.$$

The correspondence between the terms in (14) and those of (9) is apparent. As approximations they differ in that (9) assumes that the slopes of the compensated demand functions are linear in prices over the range of the tax change, while (14) assumes that the elasticities are linear in the logarithm of prices.

When there are no taxes in the initial equilibrium, (14) becomes

$$-\frac{1}{2} \sum_i \sum_j \left(\log \frac{p_i + \Delta t_i}{p_i} \right) \left(\log \frac{p_j + \Delta t_j}{p_j} \right) p_j x_j \varepsilon_{ji}. \tag{15}$$

Let us give a geometric interpretation of this measure. First note that this is *not* the area under a constant elasticity approximation to the compensated demand curve.

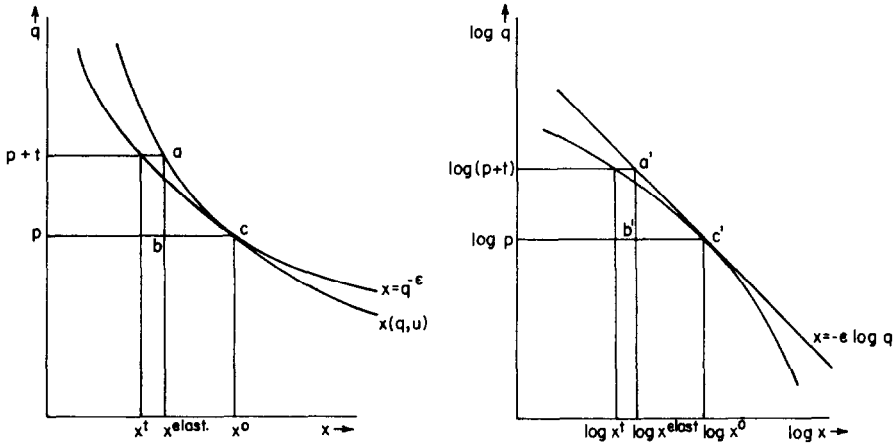


Fig. 3

In the one-commodity case shown in fig. 3, this area, abc , would be

$$\int_p^{p+t} z^{-\epsilon} dz - t(p+t)^{-\epsilon}, \tag{16}$$

which is clearly different from (14),

$$-\frac{1}{2} \left(\log \frac{p+t}{p} \right)^2 p x \epsilon \tag{17}$$

in general.

The correct interpretation is to consider the area under the compensated demand curve when plotted on double-log graph paper, $a'b'c'$, and then to convert this into a measure in units of numeraire using the local approximation at the initial equilibrium around c' . A unit of differential area at c' corresponds to $1/px$ units of differential area at c . Therefore the welfare loss measured by $a'b'c'$, which has an area of $\frac{1}{2}\epsilon(\log(p+t/p))^2$ on double-log graph paper is precisely (17). Thus even if the true compensated demand function had constant price elasticity throughout its range, (16) would not be exact since it measures areas at a constant rate instead of performing the true required integration as in (16). Therefore, the question of whether (14) or (15) is a more accurate approximation to the true welfare loss can only be decided empirically.

A further contrast between expanding in $\log p$ and expanding in p arises when we compare the second-order approximation around $p+t$ to the difference in the two second-order approximations around p , one to $p+t$ and

one to $p+t+\Delta t$. If slopes are constant then the difference between the two methods disappears when the expansion is taken in p ; but if elasticities are constant, the two methods still differ by

$$-\frac{1}{2} \sum_i \sum_j \log\left(\frac{q_i + \Delta t_i}{q_i}\right) \log\left(\frac{q_j + \Delta t_j}{q_j}\right) \sum_k t_k x_k \varepsilon_{ki} \varepsilon_{kj}.$$

It is important to recognize this difference because, for practical reasons, it may not be possible to obtain reliable econometric estimates of rates of change of elasticities. The direct approximation of formula (14) which incorporates products of elasticities when the initial equilibrium has taxes should be used.

3. An example

3.1. *Capital and labour income taxation*

The welfare cost of capital income taxation is an old and important problem which can be approached through the methods shown above. It is relevant for the current debate concerning consumption vs. income taxation as well as tax reform in a variety of other regards. Feldstein (1978) has recently recognized that in analyzing the welfare effect of eliminating capital income taxation the methods by which tax revenues could be maintained must be considered as well. He therefore considered offsetting the revenue loss from capital income taxes with a higher tax rate on labor income. The net welfare change represents a composition of these two effects.

This was modeled in a two-period context in which labor supply in the first period is variable and the individual's second-period consumption is the post-tax value of his savings. Feldstein computed the deadweight burden for two tax systems, before and after the tax reform, and used the difference as a measure of the welfare effect of this policy.

Each of these deadweight burdens is approximated by using the Harberger formula (5) following the indirect method discussed in section 2.3 above. Implicit in the use of this formula is that the values of S_{ij} are to be computed at the no-tax point. Using plausible parameter values chosen to serve as a lower bound on the welfare loss, Feldstein estimated that it would amount to 1.87% of net wage income.¹

To evaluate the accuracy of this estimate, global information on the utility and expenditure functions would be necessary. An alternative would be to use the fitted functional form at the initial equilibrium for estimation of

¹Feldstein estimated the relevant slopes from available elasticity estimates using observed market data. These are not, therefore, the relevant slopes for the untaxed situation.

magnitudes used in the second-order approximation around that point. The path along which a linear extrapolation is taken is the line segment joining the price systems before and after the tax reform is made. More detailed information about the local behavior of the compensated demands would thus provide a substitute for a more roundabout procedure.²

To compare these approximations we develop an explicit model below and use the same parameter values in each of these calculations.

3.2. A logarithmic utility model

Individuals are assumed to maximize

$$U(c_1, c_2, 1-l) = \alpha \log c_1 + \beta \log c_2 + \gamma \log(1-l), \quad (18)$$

subject to

$$c_1 + \frac{c_2}{1+r} - wl = 0, \quad (19)$$

where

- c_1 = current consumption,
- c_2 = retirement consumption,
- $1-l$ = leisure in period 1,
- r = net (after-tax) interest rate,
- w = net (after-tax) wage rate, in units of c_1 .

The nonnegative constants α , β , γ can be chosen to satisfy $\alpha + \beta + \gamma = 1$ without loss of generality.

The associated expenditure function is

$$E(w, r, u) = A(1+r)^{-\beta} w^\gamma e^u - w \quad (20)$$

where $A = \alpha^{-\alpha} \beta^{-\beta} \gamma^{-\gamma}$.

The compensated demands can be obtained directly as the partial derivatives of $E(w, r, u)$ with respect to the prices of the goods in question. The price of second-period consumption is $1/(1+r)$ and the price of leisure is w . Thus the compensated demands are:

$$\begin{aligned} c_1 &= \alpha A (1+r)^{-\beta} w^\gamma e^u, \\ c_2 &= \beta A (1+r)^{-1-\beta} w^\gamma e^u, \\ 1-l &= \gamma A (1+r)^{-\beta} w^{\gamma-1} e^u, \end{aligned} \quad (21)$$

²In this case, slopes estimated from current market data *would* be the relevant ones to use, since S_{ij} and S_{ijk} are to be evaluated at the tax equilibrium.

where c_1 has been obtained by substitution into the budget equation

$$c_1 + \frac{c_2}{1+r} - wl = E(w, r, u). \quad (22)$$

There are two ad-valorem taxes in the system. A labor income tax at the rate τ , and a capital income tax at the rate t . Thus the net returns are related to the gross returns \bar{r} and \bar{w} by

$$\begin{aligned} r &= \bar{r}(1-t), \\ w &= \bar{w}(1-\tau). \end{aligned} \quad (23)$$

The present value of tax receipts, T , evaluated at the pretax rate of interest is given by

$$T = \tau \bar{w} l + \left(\frac{t \bar{r}}{1 + \bar{r}} \right) S, \quad (24)$$

where S is first-period savings in units of the consumption good, defined through the budget equation by

$$S \doteq wl - c_1. \quad (25)$$

3.3. *Exact welfare loss estimate*

We consider initial tax rates of 40% on both labor and capital income, $\tau = t = 0.4$. In the alternative situation, the labor income tax has been raised to keep total revenue constant, and the capital income tax has been eliminated.

We choose the following parameter values, which correspond to those of Feldstein (1976):³

$$\begin{aligned} \alpha &= 0.63, \\ \beta &= 0.07, \\ \gamma &= 0.30. \end{aligned} \quad (26)$$

These imply a savings rate of 10% out of earned income and a marginal propensity to spend on additional leisure of 0.3. The functional form chosen implies zero uncompensated elasticities of savings and of labor supply.

³This functional form implies that marginal and average propensities to save are equal. Feldstein used an average propensity of 0.1 (corresponding to $\beta = 0.07$) but a marginal savings rate of 0.2.

However, as noted by Feldstein, there may be substantial welfare effects because the compensated elasticities are not zero and because future consumption rather than its present value, savings, is the argument of the utility function.

We take a 25-year savings horizon and a pretax rate of interest of 12% per annum. Thus, $\bar{r}=17$. Labor units are chosen so that the initial net wage is unity, $w=1$, and therefore $\bar{w}=w/(1-\tau)=1/0.6$.

Given these parameters, tax revenue can be calculated to be 0.4924 (in units of c_1). Now we seek a value of the labor tax rate, τ , that will keep revenues constant in the absence of capital income taxation assuming that compensation is made on a lump-sum basis so that utility remains at its initial level. We thus solve

$$\begin{aligned} 0.4924 &= \tau' \bar{w} l(\bar{w}(1-\tau'), \bar{r}, u) \\ &= \tau' \bar{w} [1 - \gamma A(1 + \bar{r})^{-\beta} \bar{w}^{\gamma-1} (1-\tau')^{\gamma-1} e^u] \end{aligned} \quad (27)$$

for τ' , substituting from (17) for the quantity demanded, and we find $\tau' = 0.4205$.

We now utilize this example to compare the true loss with the second-order approximation discussed in section 2 and with the estimate derived from Feldstein's approximation method.

Using the expenditure function (16) and the welfare loss formula (3) we find that the true welfare change in moving between the equal tax revenue junctures $t=0.4$, $\tau=0.4$ and $t'=0$, $\tau'=0.4205$ is

$$\Delta L = -0.008535. \quad (28)$$

That is, the tax system without capital taxation induces a gain of 1.22% of initial net labor income (since $wl \equiv 0.7$) or approximately \$13 billion in the U.S. economy. This is equivalent to 1.73% of the present value of tax revenue, computed by using the pretax rate of interest. These are only 65% of Feldstein's estimates which are 1.87% of labor income and 2.58% of tax revenue.

3.4. Welfare loss approximations

3.4.1. Expansion in p . In principle, we would like to use our approximation formula at the initial consumption point, as developed in section 2. To do so we must translate the ad-valorem tax on capital and labor income to the corresponding specific taxes used in (9).

If t is the ad-valorem tax on capital income, the corresponding specific tax on second-period consumption is given by

$$t_1 = \frac{\bar{r}t}{(1 + \bar{r})(1 + \bar{r}(1 - t))}, \tag{29}$$

and, similarly, the specific tax on leisure consumed corresponding to the ad-valorem tax rate τ is given by

$$t_2 = -\tau\bar{w}. \tag{30}$$

The negative sign indicates that a wage tax is actually a subsidy to leisure.

In the initial situation these taxes are

$$t_1 = 0.0337, \quad t_2 = -0.6667. \tag{31}$$

In order to estimate the welfare loss, we need to know the specific tax on labor income that will be consistent with original revenue level. The exact calculation given above will not be available if the functional form of utility is unknown. The only information that can be used is local knowledge of derivatives of the compensated demands at the equilibrium point. Specifically, we will use the second differential of the tax revenue to solve for Δt_2 as a function of Δt_1 . This will provide a quadratic approximation to the true income-tax rate change.

It should be noted that second derivatives of the compensated demands are necessary in the calculation of the second-order approximation to ΔL and hence we should not ignore this information in our approximation of Δt_2 . It will be seen that the second-order approximation is closer to the true value of the revenue-preserving t_2 than the estimate using first-order terms alone. However, due to the large tax reform contemplated, local knowledge will provide only a coarse estimate.

For convenience in arranging matrices and vectors below, we identify second-period consumption, c_2 , with x_1 ; its specific tax rate being t_1 . Leisure, $1 - l$, is identified with x_2 ; its specific tax being t_2 .

Tax revenue is

$$R = t_1 c_2 - t_2 l = t_1 x_1 - t_2(1 - x_2). \tag{32}$$

Therefore the second differential of R is

$$\begin{aligned} dR = & dt_1(x_1 + t_1 S_{11} + t_2 S_{12}) + dt_2(-(1 - x_2) + t_1 S_{12} + t_2 S_{22}) \\ & + \frac{1}{2} \{ (dt_1)^2 (2S_{11} + t_1 S_{111} + t_2 S_{121}) + 2(dt_1)(dt_2) \\ & \times (2S_{12} + t_1 S_{112} + t_2 S_{122}) \\ & + (dt_2)^2 (2S_{22} + t_1 S_{122} + t_2 S_{222}) \}, \end{aligned} \tag{33}$$

where

$$S_{ijk} = \frac{\partial S_{ij}}{\partial t_k} = \frac{\partial^2 x_i}{\partial t_j \partial t_k}, \quad \text{for } i, j, k = 1, 2. \quad (34)$$

Symmetry of the substitution terms insures that these are independent of the order of the indices. Using the parameters of the utility function, the substitution matrix is

$$(S_{ij}) = \begin{bmatrix} -8.171 & 0.235 \\ 0.235 & -0.210 \end{bmatrix}. \quad (35)$$

The cross-partials of the compensated demands, S_{ijk} , are given by

$$\begin{aligned} S_{111} &= 176.53, \\ S_{112} &= -2.451 = S_{211}, \\ S_{212} &= -0.1647 = S_{122}, \\ S_{222} &= 0.3572, \end{aligned} \quad (36)$$

using the symmetry properties of the compensated demand functions.

Using (29), substituting $dt_1 = -0.0337$, since we consider eliminating capital income taxation entirely, we find that the change in the labor income-tax rate that will satisfy $dR = 0$ up to the second-order terms is

$$dt_2 = -0.0298.$$

That is, the specific wage tax falls to $t_2 = 0.6965$ which corresponds to an ad-valorem rate of 41.79%. This is to be contrasted with the exact revenue-preserving rate previously found to be 42.05%.

Recall the formula for the differential welfare loss, which can be written as

$$\begin{aligned} \Delta L &= -\sum_i \Delta t_i \sum_j t_j S_{ij} - \frac{1}{2} \sum_i \sum_j \Delta t_i \Delta t_j S_{ij} \\ &\quad - \frac{1}{2} \sum_i \sum_j \sum_k \Delta t_i \Delta t_j \cdot t_k S_{ijk}. \end{aligned}$$

We find that the terms of this expression are -0.01015 , $+0.00471$ and -0.00423 , respectively, so that

$$\Delta L = -0.00967, \quad \text{or approximately } \$15 \text{ billion.} \quad (37)$$

We can see that all these terms in the approximation contribute in a significant way to the overall result. In particular, those second-order terms involving the cross-partial of the compensated demand functions cannot be ignored in an approximation of this type, as discussed in section 2 on theoretical grounds.

Moreover, the approximation on the whole is quite accurate: -0.00967 vs. a true value of -0.008535 , a relative error of 14% of the true value. This should be contrasted to Feldstein's method which gives a value of \$21 billion, a relative error of 53%; and to the estimates to which Feldstein's criticism was addressed, using a lump-sum tax as a substitute, which estimate a \$60 billion efficiency gain.

3.4.2. *Expansion in log p.* We can alternatively use the approximation formula (14), which takes a simple form in the present example since the compensated demand functions have constant elasticity in prices:

$$\varepsilon_{11} = -(1 - \beta) = -0.93,$$

$$\varepsilon_{12} = \gamma = 0.30,$$

$$\varepsilon_{21} = \beta = 0.07,$$

$$\varepsilon_{22} = \gamma - 1 = -0.70.$$

We use the same tax rates as in the calculation above to facilitate comparison of the welfare loss formulas.⁴ In logarithmic terms:

$$\log\left(\frac{q_1 + \Delta t_1}{q_1}\right) = -0.47412,$$

$$\log\left(\frac{q_2 + \Delta t_2}{q_2}\right) = -0.030253.$$

Substituting into (14) we find an estimated differential welfare loss of -0.0091522 , which is closer to the true loss (-0.008535) than that calculated by the linear expansion (-0.00967). Although no definite conclusion can be drawn, it is probably better to expand the loss function in $\log p$, as above, rather than the linear expansion of the Harberger method in the previous section.

It must be emphasized that the relationship among the accuracies of the two approximations given above is not to be thought of as a general result

⁴If one were to redo the computation entirely in logarithmic terms, an alternative expansion of dR would have to be performed. But this would change the accuracy of the equal revenue constraint as well as the point at which the welfare loss is given.

in any way. It is simply a biproduct of the particular example at hand. To find bounds on the accuracy of either form of expansion one can utilize Taylor's theorem, together with an assumed upper bound of the absolute value of the sequence of derivatives of the true loss function. However, as the selection of such a bound is arbitrary in the absence of more detailed information, we have not presented such an error-bound calculation above. In any case, even though one form of expansion may have a lower error bound than the other, it is not necessarily more accurate in any particular circumstance. Therefore the calculations above must be regarded as illustrative only.

4. Approximating optimal tax calculations

In principle, the methods outlined above can be used to estimate the welfare effects of any tax changes. In particular, we may use the second-order approximations over a wide range of taxes to find the global optimum. This involves two steps. The revenue constraint is satisfied by solving the quadratic approximation (33) for one tax rate in terms of the other. The solution is then substituted into the second-order approximation to the welfare loss.

Unfortunately, our computations show that the slight errors introduced at the first stage are actually associated with sufficiently large discrepancies in the revenue constraint to mask the location of the optimal tax system. For example, using the expansion in p in both the welfare and revenue approximations, the estimated welfare gain continued to increase even as the interest income tax became negative. Because of the separability of utility in leisure and the two consumption goods, we know [see Diamond–Mirrlees (1971) and Atkinson–Stiglitz (1976)] that the true optimal tax configuration involves only a wage tax. As the capital tax was decreased, the estimated wage tax was insufficient to preserve the original level of revenues. The discrepancy widened as the tax rates were shifted farther from their original position, and this effect introduced a spurious computed welfare gain.

On the positive side, however, this method correctly identified the range in which the optimal tax lies. It indicates a sharp increase in welfare from decreasing the capital income tax to 20%, and a relatively flat welfare function below that point. In fact, 80% of the total potential welfare gain could be realized by this reduction, so the failure to find the true optimum may not be that serious.

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