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THE NATURE OF STOCHASTIC EQUILIBRIA[1]

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This paper formulates the notion of stochastic equilibria as invariant probability distributions consistent with the behavior patterns of individuals and the disequilibrium adjustment mechanism of the economy. Conditions for existence, uniqueness, and stability of such equilibria are examined.

1. INTRODUCTION

WE CONSIDER A CLASS of problems in this paper in which the economic environment is stochastic. We will be concerned primarily with developing an equilibrium concept for general equilibrium models of this type. However the essential ideas can be carried over directly to partial equilibrium applications. The choice of the specific general equilibrium model used results primarily from a desire to facilitate comparisons with earlier work on alternative equilibrium concepts for this model (see Hildenbrand [**9**] and Majumdar and Bhattacharya [**2** and **3**]).

Randomness can arise from several sources. We will be considering, for concreteness, a simple exchange economy in which the basic data are the preferences and endowments of the economic agents. Either of these can be random. Typically, randomness of endowments can be allowed for by creating contingent markets in which case the Arrow-Debreu deterministic equilibrium suffices. It is conceptually much more difficult to create markets contingent on tastes due to the difficulties of discovering the true taste pattern of an individual, difficulties which do not arise in the case of endowment vectors which can be observed directly. We will be considering an economy without markets for every future contingency and thus there will remain some randomness. This "residual" uncertainty in the economy necessitates equilibrium concepts other than the Arrow-Debreu system of market clearing prices.

2. NOTATION

We shall find it convenient, before proceeding further, to develop some notation. Let L be the set of positive reals and P the set of all *strictly positive* l-vectors, i.e., $x = (x_i)$ is in P if and only if x_i is in L for all i. Let $S = \{p = (p_i) \in P : \Sigma_{i=1}^{l} p_i = 1\}$. The topological closures of S and P are denoted by $\bar{S}$ and $\bar{P}$ respectively. $\mathscr{B}(R^l)$ is the *Borel σ-field* of R^l.[2] For any subset Y of R^l which belongs to $\mathscr{B}(R^l)$, the *Borel σ-field* of Y is denoted by $\mathscr{B}(Y)$. One has $\mathscr{B}(Y) = \{E \cap Y : E \in \mathscr{B}(R^l)\}$.

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[2] This is the σ-field generated by open sets of R^l. See Billingsley [**4**, p. 11, problem 6].

Clearly $\mathscr{B}(Y) \subset \mathscr{B}(R^l)$ for any Y in $\mathscr{B}(R^l)$. The set of all *probability measures* on $\mathscr{B}(Y)$ is denoted, for simplicity, by $M(Y)$. We usually endow $M(Y)$ with the topology of *weak convergence*. A sequence μ_n of elements of $M(Y)$ converges in this topology (briefly, *converges weakly*) to some μ in $M(Y)$ if for every bounded, real-valued continuous function f on Y, $\int_Y f\,d\mu_n$ converges to $\int_Y f\,d\mu$. The results on weak convergence to be used in Section 4 can be found in Billingsley [**3**]. For a random vector X, its mathematical expectation is denoted by EX.

3. THE CONCEPT OF EQUILIBRIUM

Typically, an equilibrium of a model is a state having the property that once *attained*, there would be no tendency to depart from that state in the absence of any change in the parameters of the model. The absence of any endogenous tendency to change seems to be the basic characteristic of "equilibrium" (see Frisch [**8**]; actually the use of the word goes much further back, at least to Alfred Marshall).

Let $\mathscr{E}$ be a *deterministic* exchange economy with N agents. Each agent is representable by his *demand function* reflecting his tastes and preferences and his *endowment*. To be precise, let $\mathscr{D}$ be the set of all functions f from $S \times L$ to $\bar{P}$ satisfying the following properties:

(i) f is continuous on $S \times L$;

(ii) $p \cdot f(p, w) = w$ for all p in S and w in L (where $\cdot$ denotes the scalar product);

(iii) if (p_n, w_n) in $S \times L$ converges to (p, w) in $(\bar{S} - S) \times L$, then $\Sigma_{i=1}^{l} |f_i(p_n, w_n)|$ diverges to $+\infty$ with increasing n.

We shall write, for a typical element Z in Euclidean space,

$$\|Z\| \equiv \sum |Z_i|.$$

The last condition (iii) implies that every commodity is desired. An agent is completely described by a pair $\alpha_i = \langle f_i, e_i \rangle$ in $\mathscr{D} \times P$. Given a price vector p in S, his *wealth* is $w_i = p \cdot e_i$ and his *planned demand* is given by $\xi(\alpha_i, p) \equiv f_i(p, p \cdot e_i)$. The excess demand $\zeta_{\mathscr{E}}(p)$ in the economy $\mathscr{E}$ corresponding to a price vector p in S is defined as

$$\zeta_{\mathscr{E}}(p) = \sum_{i=1}^{N} [f_i(p, p \cdot e_i) - e_i]. \tag{3.1}$$

An *equilibrium price vector* p^* is an element of S such that $\zeta_{\mathscr{E}}(p^*) = 0$. If the prevailing price vector in the market is, in fact, an equilibrium price vector, and all the agents make their plans taking it as given, then all the plans can be implemented. Given the tastes and endowments of the agents, there would be no mechanism to bring about any changes—since all the plans are consistent. If the announced price vector is not an equilibrium, then total excess demand is nonzero, implying that not all the plans can be carried out. The agents whose plans are not realized are expected to generate pressures for a price change. A familiar hypothesis is that if excess demand is positive (respectively, negative), prices would go up (respectively down).

All this is a familiar story. Extension of the concept of equilibrium to a model with random preferences raises interesting possibilities and conceptual problems. Suppose that the characteristics, i.e., the demand functions and the endowments of the agents are random. Formally, there is a probability space $(\Omega, \mathscr{F}, \mu)$, Ω being the set of all *states of the environment of the economy*. A *random agent* $\alpha(\ )$ *is a* ($\mathscr{F}$-measurable) function from Ω into $\mathscr{D} \times P$. A particular state $\omega \in \Omega$, completely specifies the preferences and endowments of the agents. The occurrence of ω is determined according to the probability law μ. *We shall always assume that the evolution of the environment is stochastically independent of the price systems prevailing in the market.* In other words, the probability $\mu(F)$ that the state which is going to occur belongs to F does not depend on p.

Given p in S, the total excess demand in the random economy $\mathscr{E}$, consisting of agents $\{\alpha_i(\ )\}$ is the random vector $\zeta_{\mathscr{E}}(\ ;p)$ defined as

$$(3.2)\qquad \zeta_{\mathscr{E}}(\omega\,;p) = \sum_{i=1}^{N} [\xi(\alpha_i(\omega);p) - e_i(\omega)]$$

where $\xi(\alpha_i(\ ), p)$ is the random vector of demand of agent $\alpha_i(\ ) \equiv \langle f_i(\ ); e_i(\ )\rangle$ corresponding to the price vector p in S. For any ω in Ω, one can define

$$(3.3)\qquad w(\mathscr{E}_\omega) = \{p \in S : \zeta_{\mathscr{E}}(\omega\,;p) = 0\}.$$

It is easy to construct examples in which there is *no* p in S such that p belongs to $w(\mathscr{E}_\omega)$ for a.e. ω. Consistent with the treatment of Arrow and Debreu is the statement that $w(\mathscr{E}_\omega)$ is the set of all equilibrium price vectors in the state ω. If the initial price vector p happens to belong to $w(\mathscr{E}_{\bar{\omega}})$ for some $\bar{\omega}$, then all markets will be cleared and all plans will be realized if $\bar{\omega}$ occurs. A random price equilibrium $p(\ )$ is a ($\mathscr{F}$-measurable) map from Ω into S such that for a.e. ω, $p(\omega)$ belongs to $w(\mathscr{E}_\omega)$. The properties of the sets $w(\mathscr{E}_\omega)$ in relation to changes in the size of the economy were studied in Bhattacharya and Majumdar [**3**]. A sequence $\mathscr{E}_n$ of economies was considered, consisting of N_n *stochastically independent* random agents, and N_n was assumed to increase with n in a specific manner. Conditions were given under which $\lim w(\mathscr{E}_{n_\omega})$ exists[3] for a.e. ω, and is independent of ω. Furthermore under appropriate assumptions the sequence $\sqrt{N_n}\,(p_n(\ ) - Ep_n(\ ))$ of suitably normalized random price equilibria was shown to converge *weakly* to a normal distribution. Many of the results can also be extended to allow for particular forms of stochastic dependence (such as *exchangeability* or *strong mixing*).

Consider now a central planning board which has to set the price without knowing what state is going to occur. From the point of view of this board, certain questions are rather natural. For example, if a particular price vector p is set, what can be said about the distribution of total excess demand? It can be shown that if the agents are stochastically independent, the error involved in approximating[4] the distribution of $\zeta_{\mathscr{E}}(\ , p)$ by a normal distribution (with suitable parameters)

[3] Convergence was established in Hausdorff metric.

[4] This approximation is uniform over all Borel-measurable convex subsets of R^l.

is inversely proportional to $N^{\frac{1}{2}}$. Furthermore, Hildenbrand [**9**] has given conditions under which there exists a price vector p^* in S such that the corresponding expected excess demand $E\zeta_{\mathscr{E}}(\ ,p^*) = 0$ [5] and per capita excess demand corresponding to p^* is small with high μ-probability if N is large. Note that if the initial price vector is p^*, then it is conceivable that whatever state occurs, there is a set of agents who are unable to carry out their plans (in fact, all the agents may be unable to do so)—after all, $\zeta_{\mathscr{E}}(\omega\,;p^*)$ need not be zero for any ω, and still $E\zeta_{\mathscr{E}}(\ \,;p^*)$ may be zero. Nothing in the model suggests why pressures will not be generated for bringing about a change in p^*, pointing out that plans of many agents are not being realized. Moreover, for a large economy, a small per-capita excess demand may be consistent with a major imbalance in absolute terms in some markets.

In this paper we develop an alternative definition of equilibrium which exists in economies in which stochastic independence of tastes and endowments fails and hence in which some of the previous results are not applicable. The definition we use is also related to the process of price adjustment in disequilibrium. Adjustments play no role in the results above.

The conceptual difficulty in extending the notion of an equilibrium to random economies may be briefly recapitulated: if one defines an equilibrium as a *deterministic* price vector with some "desirable" properties, then the plans (i.e., planned demand vectors) of the agents formulated by taking that price system as given may not in general be realized and consistent in any state of the environment. The market-clearing prices depend on the state of the environment, and there is no way of guaranteeing that markets will be cleared in the actual state that occurs. Hence, it becomes especially important to try to say something about possible adjustments when excess demand or supply shows up and this is the question that we turn to in the next section.

4. STOCHASTIC EQUILIBRIUM RELATIVE TO ADJUSTMENTS

Time is treated as a discrete variable $t = 0, 1, 2, \ldots$. As before, the same probability space $(\Omega, \mathscr{F}, \mu)$ represents the set of all possible states of the environment at any date. The sequence $\langle \omega_t \rangle$ of the states of the environment in different time periods $t = 0, 1, \ldots$ is taken to be *independent and identically distributed according to μ for all t*, i.e., independently of the past states of the environment the probability that the state obtaining at date t belongs to some F in $\mathscr{F}$ is given by $\mu(F)$. This probability does *not* depend on the price vectors (or their distributions) at date t.

The price adjustment process can be described as follows: the initial price is chosen according to some distribution π_0. Independent of the price vector, the state of the environment ω_0 is chosen by "Nature" according to the probability distribution μ. The excess demand corresponding to any price p is then $\zeta(\omega_0\,; p)$. There is an *adjustment function h* from $R^l \times S$ into S. It determines the price in period 1 uniquely, given the quantity of excess demands and the price at date 0; thus, $h[\zeta(\omega_0\,; p); p]$ is the price at date 1, if the price at date 0 is p. We are not

[5] To be sure, 0 represents the zero vector.

concerned in this paper with an explanatory theory for h itself. Presumably it is the result of a learning process performed by agents in the economy who are responsible for changing prices. In a more general system, which allows for inventory accumulation, the price may depend on the level of inventories held. Note that the form of the function used is general enough to allow dependence between price changes on one market and inventory levels of closely substitutable goods.

But it must be pointed out that there are two very severe restrictions on the structure of the model that are implicit in the use of such an adjustment hypothesis. First, h itself does not change. That is, the learning process or other optimization that gave rise to h is complete and no experience of the economy will alter its conclusions. Second, h depends only on the current excess demand and not on any historical data—the strength of this condition is self-evident. At date 1 (irrespective of the price vector), the state ω_1 occurs (according to the distribution μ) generating an excess demand vector corresponding to that price, and so on. In general, the process is described by the following:

$$(4.1) \qquad p_{t+1}(\ ) = h[\zeta(\ ; p_t(\ )); p_t(\ )].$$

The initial distribution π_0 is simply an element of $M(S)$. The distribution π will be called a *stochastic equilibrium* (relative to h) if p_t has distribution π implies p_{t+1} has distribution π.

For p in S let $\theta(\ ; p)$ be the probability distribution of excess demand (given the price vector p). Formally, for any E in $\mathscr{B}(R^l)$ one has

$$(4.2) \qquad \theta(E; p) = \mu(\{\omega \in \Omega : \zeta(\omega; p) \in E\}).$$

$\theta(E; p)$ is the probability (note that it does not depend on t) that the excess demand vector corresponding to p will be in the set E.

The first assumption is the following continuity property of h:

ASSUMPTION 1: *The adjustment function $h: R^l \times S \to S$ is continuous.*

For any set A in $\mathscr{B}(S)$ define

$$(4.3) \qquad h^{-1}(A)_p = \{z \in R^l ; h(z, p) \in A\}.$$

Note that $h^{-1}(A)_p$ is in $\mathscr{B}(R^l)$. Hence, define for any A in $\mathscr{B}(S)$ and p in S, the *kernel*

$$(4.4) \qquad \lambda(A; p) = \theta(h^{-1}(A)_p; p).$$

$\lambda(A; p)$ is the probability that the price vector will be in the set A in the next period, given that it is p in this period. Alternatively, for a given p in S, $\lambda(\ ; p)$ is *the distribution of the random vector* $h[\zeta(\ ; p); p]$ (defined on Ω), and is an element of $M(S)$.

For each ω, $\zeta(\omega, p)$ is a continuous function on S (recall the properties of demand functions listed in Section 3). Hence if a sequence p_n of price vectors in S converges to p in S, then $\zeta(\omega; p_n)$ converges to $\zeta(\omega; p)$ for every ω. In particular (see [**4**, p. 33]), *$\zeta(\ ; p_n)$ converges in distribution to $\zeta(\ ; p)$*; in other words, *the sequence*

$\theta(\ ;p_n)$ of probability measures in $M(S)$ converges to $\theta(\ ;p)$ in $M(S)$ in the weak topology as p_n in S converges to p in S. Using the continuity of h, it is immediate that convergence of p_n (in S) to p (in S) also implies that the sequence $h[\zeta(\ ;p_n);p_n]$ of random vectors converges to $h[\zeta(\ ;p);p]$; this in turn guarantees that $\lambda(\ ;p_n)$ (the distribution of $h[\zeta(\ ;p_n);p_n]$) *converges to* $\lambda(\ ;p)$ (the distribution of $h[\zeta(\ ;p);p]$) *in the weak topology as p_n in S converges to p in S.*

It is now necessary to verify that for any E in $\mathscr{B}(S)$, $\lambda(E;\ )$ is $\mathscr{B}(S)$ measurable. The proof of the following lemma uses a result of Varadarajan [**13**].

LEMMA 4.1: *For any p in S, $\lambda(\ ;p)$ is a probability measure on $\mathscr{B}(S)$; for any E in $\mathscr{B}(S)$, $\lambda(E;\ )$ is $\mathscr{B}(S)$ measurable.*

PROOF: The first statement being obvious, note that for any fixed E in $\mathscr{B}(S)$, $\lambda(E;\ )$ is a function from S into $[0, 1]$. It is representable as a composition of two functions. Define a function $\phi: S \to M(S)$ as

$$\phi(p) = \lambda(\ ;p). \tag{4.5}$$

Next, define $\Psi: M(S) \to [0, 1]$ by

$$\Psi(m) = m(E). \tag{4.6}$$

Clearly, $\lambda(E, p) = \Psi(\phi(p))$. It has already been pointed out that if $M(S)$ is endowed with the topology of weak convergence, then ϕ is a continuous function, hence it is surely $\mathscr{B}(S)$ measurable. Measurability of Ψ is precisely the result of Varadarajan [**13**]. Hence, being the composition of two measurable maps, $\lambda(E;\ )$ is measurable.

Q.E.D.

Lemma 4.1 ensures that the process $p_t(\ )$ with an initial distribution π_0 and the *kernel* $\lambda(A;p)$ is in fact a *Markov Process* (see, e.g. [**10**, p. 365]). The *distribution of the stochastic process* is entirely determined by the initial distribution and the kernel. Suppose the distribution of $p_t(\ )$ is π_t; the distribution of p_{t+1} can be determined easily. For any A in $\mathscr{B}(S)$,

$$\pi_{t+1}(A) = \int_S \lambda(A;p)\, d\pi_t(p). \tag{4.7}$$

It is necessary to remember that for a given A, $\lambda(A;)$ is measurable and bounded so that the integral in (4.7) is well-defined.

Starting with an initial distribution (which may of course be a measure assigning the mass 1 to a single point p^*), the distribution of prices at successive dates can be obtained by applying (4.7) repeatedly.

A *stochastic equilibrium* of the process is a measure π^* such that π^* is not identically zero on $\mathscr{B}(S)$ and

$$\pi^*(A) = \int_S \lambda(A;p)\, d\pi^*(p) \qquad \text{for } all\ A \text{ in } \mathscr{B}(S). \tag{4.8}$$

In other words, a stochastic equilibrium is a time-invariant distribution of prices.

If in some period the distribution of prices happens to satisfy (4.8), it will remain the same for all subsequent periods.

The first task is to prove the existence of such a stochastic equilibrium. One possibility is to impose direct restrictions on the kernel $\lambda(A;p)$. We shall, however, keep the discussion closer in spirit to that of the deterministic adjustment processes and proceed by imposing conditions on h. Ideally, one would like to derive these assumed conditions from some behavior rules of the agents, but it may be too premature to introduce so many complications at this stage. Besides for the existence question restrictions are in the nature of some mild continuity and boundedness properties.

ASSUMPTION 2: *For each commodity i, there exists* $\delta_i > 0$ *such that if p is in S and* $p_i \leqslant \delta_i$, *then* $\zeta_i(\omega;p) > 0$ *for all* ω *in* Ω.

This assumption is implied by global gross substitutability; for example, see Arrow and Hahn [**1**]. However, it is substantially weaker.[6]

ASSUMPTION 3: *For the* δ_i *defined in Assumption* 2,

$$\max_i |h_i(z,p) - p_i| < \min_i \delta_i.$$

Let ε be such that $\max_i |h_i(z,p) - p_i| < \varepsilon < \min_i \delta_i$. We will use the quantity ε later in the paper.

ASSUMPTION 4: $h_i(z,p) - p_i$ *is a sign preserving function of* z_i *for* $i = 1, \ldots, l$.

We can now prove the following lemma:

LEMMA 4.2: *Let* (δ_i) *be the numbers defined in Assumption* 2. *Then* $\Delta'' = \{p \in S: p_i \geqslant \delta_i \text{ for all } i\}$ *is nonempty.*

PROOF: For any fixed $\bar{\omega}$, there is a p^* such that $\zeta(\bar{\omega};p^*) = 0$ by the usual existence theorem (see, e.g., Debreu [**6**]). Clearly $p^* \in \Delta''$ since $\delta_i < p_i^*$. *Q.E.D.*

Define $\alpha_i = \delta_i - \varepsilon$ and $\Delta' = \{p \in S: p_i \geqslant \delta_i - \varepsilon\}$. Clearly Δ'' is a subset of Δ'. The next lemma provides one of the principal steps in the existence theorem which follows.

LEMMA 4.3: *For any p in* Δ', $\lambda(\Delta', p) = 1$.

PROOF: For p in Δ'', the uniform boundedness of the adjustment function (Assumption 3) insures that $h(\zeta(\omega;p), p)$ is also in Δ' for all ω in Ω, i.e., $\lambda(\Delta'', p) = 1$ for all p in Δ''.

[6] It is, however, much more restrictive than the usual desirability properties assumed in general equilibrium theory—for example (iii) of Section 3 above. Unfortunately, we have been unable to avoid its use. We are indebted to M. Morishima for helpful discussions of points related to this assumption.

Take p in $\Delta' - \Delta''$. If $p_i - \alpha_i < \varepsilon$ (i.e., $p_i < \delta_i$), then $\zeta_i(\omega, p) > 0$ for all ω in Ω (Assumption 2). Therefore, Assumption 4 implies $h_i(\zeta(\omega; p), p) - p_i > 0$ for all ω in Ω for such p. By definition of ε, $\{p' : p'_i > p_i$ for all i such that $p_i - \alpha_i < \varepsilon$, and $\max |p'_i - p_i| < \varepsilon\}$ is contained in Δ'. Thus, p in Δ' implies $h(\zeta(\omega, p); p)$ belongs to Δ' for all ω in Ω. Hence $\lambda(\Delta', p) = 1$ for all p in $\Delta' \setminus \Delta''$. *Q.E.D.*

The existence theorem will be proved by applying a theorem of Yosida [**14**]. Let $C(\Delta')$ be the space of all real-valued continuous functions on Δ'. For any f in $C(\Delta')$ one can define the *iterates*

$$f_1(p) = \int_{\Delta'} f \, d\lambda(\ ; p).$$

In general, taking $f_0 \equiv f$ one defines for any positive integer $t \geqslant 1$

$$f_t(p) = \int_{\Delta'} f_{t-1} \, d\lambda(\ ; p). \tag{4.9}$$

THEOREM 4.1: *Under Assumptions* (1) *through* (4), *there exists a stochastic equilibrium.*

PROOF: Note that a Markov Process can be defined with the state space $(\Delta', \mathscr{B}(\Delta'))$ and for A in $\mathscr{B}(\Delta')$ and p in Δ', the *kernel* $\lambda(A; p)$ can be defined in the same way as (4.4); for, by Lemma 4.3, $\lambda(\cdot, p)$ is a probability measure on $\mathscr{B}(\Delta')$ for any p in Δ', and—as before—$\lambda(A;\ )$ is a measurable function for any A in $\mathscr{B}(\Delta')$. Δ' is a compact (hence, separable) metric space whose bounded closed subsets are also compact. Furthermore, it is shown that

(4.10) *If f belongs to $C(\Delta')$, so does f_t for $t \geqslant 1$.*

Consider a sequence p_n in Δ' converging to p in Δ'; to show that for any f in $C(\Delta')$, $f_1(p_n)$ converges to $f_1(p)$, note that $\lambda(\ ; p_n)$ converges to $\lambda(\ ; p)$ in the weak topology. Since f being continuous is also bounded by compactness of Δ', one has, by Billingsley [**4**, p. 11], $f_1(p_n) = \int_{\Delta'} f \, d\lambda(\ , p_n)$ converges to $\int_{\Delta'} f \, d\lambda(\ ; p) = f_1(p)$. Thus f_1 belongs to $C(\Delta')$. The same argument is used to prove inductively that f_t belongs to $C(\Delta')$ for all positive integers $t \geqslant 1$, proving (4.10).

Since all the conditions of Theorem XIII, 4.1 of Yosida[7] [**14**, p. 395] are satisfied, *a necessary and sufficient condition for the non-existence of a non-trivial invariant distribution is that* $\lim_{n \to \infty} 1/n \sum_{t=0}^{n} f_t(p) = 0$ *for any f in $C(\Delta')$ and any p in Δ'*. To complete the proof, therefore, it is enough to take the function g in $C(\Delta')$ such that $g(p) \equiv 1$ for all p in Δ'. Since $\lambda(\Delta', p) = 1$ for all p in Δ', $g_t \equiv 1$ on Δ' for all positive integers $t \geqslant 1$. Hence $\lim_{n \to \infty} 1/n \sum_{t=0}^{n} g_t(p) = 1$ for *all* p in Δ'. *Q.E.D.*

[7] A technical point for the cautious reader: We have all along been considering Borel σ-fields. But Yosida's analysis is in terms of what he defines as the Baire σ-field. However, Δ' is a compact subset of R^l, and its Borel σ-field in our sense is the same as Yosida's Baire σ-field; see Yosida [**14**, p. 18] and the very useful synthesis of Billingsley [**4**, p. 11, problem 6].

5. UNIQUENESS AND GLOBAL STABILITY

We now take up the questions of uniqueness and global stability of stochastic equilibrium of the proposed adjustment process. One can say that the mere existence of a probability distribution having the property that if it happens to be the initial distribution of prices, then it will also be the distribution of prices at all subsequent periods, is by itself, not quite enough. Prices are not chosen randomly, and even if the existence of such a probability distribution is known, it is not clear how or whether the system will attain stationarity. Indeed, all that one can observe is the sequence of specific price vectors that emerge along any sample path according to the evolution of the environment—the stationary distribution is a statistical phenomenon.

Recall, however, that in the standard case of the completely deterministic Walrasian system, economists' interest in equilibrium is often justified by arguing that there may be forces at work to drive the economy back to an equilibrium, if it is not in equilibrium already. Characterization of the cases in which there is such an inherent tendency towards an equilibrium has been the subject of a large number of papers.[8] In the same spirit we can say that our equilibrium concept will take on added significance if we can delineate some cases in which the sequence π_t of the probability distributions of prices at date t (see (4.7)) converges to a stochastic equilibrium π^* (satisfying (4.8)) irrespective of the initial conditions. Better still, if we can show that for almost every realization of the environment, the sequence of sample distributions of observed prices tends to an invariant probability distribution π^* (satisfying (4.8)) independent of the starting point, we can certainly say that the limiting distribution π^* gives interesting insights into the asymptotic behavior of prices. It describes how the prices will be distributed in the distant future, it gives us an idea of how the prices vary over time, and by observing the actual distribution of prices for a long time we should be able to approximate the stochastic equilibrium arbitrarily closely.

Now, the technical difficulties in ensuring the global stability of a deterministic discrete-time tâtonnement process, in which prices are normalized and required to be non-negative, have already been spelled out in the literature. In view of the fact that our process (4.1) includes, as a special case, such a deterministic adjustment process, it would be somewhat näive to expect that global stability (implying uniqueness) can be obtained without making appropriately "strong" assumptions. Instead of aiming at the most general result, we shall present a typical stability theorem, and then indicate alternative formulations that can possibly lead to stronger local and global results.

The adjustment mechanism (4.1) can be described in an alternative way. For each ω in Ω, there is a function $g(\omega)$ from S into S such that for any price vector p in S at any date t, the price vector at the next date $t + 1$ is given by $g(\omega)(p)$ if ω occurs at date t. Note that the function $g(\omega)$ depends only on ω and not on t. Thus, (4.1) can also be represented as

(5.1) $$p_{t+1}(\ ) = g_{(\ )}(p_t(\ )).$$

[8] See Arrow-Hahn [1] or Morishima [11].

Uniqueness and global stability will be shown to follow from the following admittedly restrictive assumption:

ASSUMPTION 5: *There is a constant* $\delta > 0$ *such that for all* ω *in* Ω, $d(g_{(\omega)}(p), g_{(\omega)}(p')) \leqslant \delta d(p, p')$ *for all* p, p' *in* S *and* $\delta < 1$.

Because of the Banach fixed point theorem, this means that each deterministic economy as defined by the sample ω has a unique globally stable equilibrium. The strength of the assumption in the general equilibrium setting hardly needs emphasis. It is an assumption on $h(.,.)$ and $\zeta(.,.)$ jointly. However, without such an assumption, it is easy to see that our main result could not possibly be true. (Consider the trivial case in which Ω is a single point; then this assumption rules out expanding or stationary cobweb phenomenon.)

We now introduce a metric β on $M(S)$, the set of all probability measures on S, *convergence in which implies convergence in the weak topology on* $M(S)$. For a detailed discussion of this property and of the metric (and other properties) the reader is referred to Dudley [7]. Let $\mathscr{L}(S)$ be the space of all bounded continuous real valued functions on S satisfying

$$|f(x) - f(y)| \leqslant d(x, y)$$

for all x, y in S.

For any two elements Q, Q' in $M(S)$, define

$$\beta(Q, Q') = \sup_{f \in \mathscr{L}(S)} \left| \int f\, dQ - \int f\, dQ' \right|.$$

Consider the case in which the process starts at $t = 0$ from a single price vector $\bar{p}$ in S; i.e., $\pi_0(\bar{p}) = 1$ and $\pi_0(p) = 0$ for $p \neq \bar{p}$. Denote by $p_T(\ |\bar{p})$ the random price vector at date T, *conditional on the initial price vector being* $\bar{p}$. Let $\Pi_T(\ ,\bar{p})$ be the distribution of $p_T(\ |\bar{p})$. Note that $p_T(\ |\bar{p})$ can also be represented as a function from Ω^T into S, since every realization $\langle\omega_t\rangle$ of the environment (i.e., every element $\langle\omega_t\rangle$ of Ω^T) leads to a uniquely defined value $p_T(\langle\omega_t\rangle|\bar{p})$ of the price vector in period T starting from $\bar{p}$. We shall note a direct implication of the above Assumption. If two distinct price vectors $\bar{p}$ and $\bar{\bar{p}}$ are considered as starting points of the process, then for *any arbitrary element* $\langle\omega_t\rangle$ of Ω^T (i.e., an arbitrary realization of the environment for the first T periods) one has

$$(5.3) \qquad d(p_T(\langle\omega_t\rangle|\bar{p}), p_T(\langle\omega_t\rangle|\bar{\bar{p}})) \leqslant \delta^T d(\bar{p}, \bar{\bar{p}}).$$

The point to be emphasized is that the right-hand side of (5.3) is independent of $\langle\omega_t\rangle$. Since $\langle\omega_t\rangle$ was chosen arbitrarily we can thus argue that (5.3) holds for *all* elements of Ω^T. Taking any f in $\mathscr{L}(S)$ one computes the following: for an arbitrary $\langle\omega_t\rangle$ in Ω^T,

$$(5.4) \qquad \begin{aligned} |f(p_T(\langle\omega_t\rangle|\bar{p})) - f(p_T(\langle\omega_t\rangle|\bar{\bar{p}}))| &\leqslant d(p_T(\langle\omega_T\rangle|\bar{p}), p_T(\langle\omega_T\rangle|\bar{\bar{p}})) \\ &\leqslant \delta^T d(\bar{p}, \bar{\bar{p}}). \end{aligned}$$

Since the last bound (5.4) depends *neither on* $\langle\omega_t\rangle$ *nor on the function* f *chosen from* (S), *it follows that*

$$\sup_{f\in\mathscr{L}(S)} [Ef(p_T(\ |\bar{p})] - Ef(p_T(\ |\bar{\bar{p}})) \leqslant \delta^T d(\bar{p},\bar{\bar{p}}). \tag{5.5}$$

But by a well-known formula (see [**10**, p. 166]) one also has

$$Ef(p_T(\ |\bar{p})) = \int_S f\, d\Pi_T(\ ,\bar{p}).$$

Hence, $\beta(\Pi_T(\ ,\bar{p});\Pi_T(\ ,\bar{\bar{p}})) \leqslant \delta^T d(\bar{p},\bar{\bar{p}})$. Since $0<\delta<1$, one immediately has

$$\lim_{T\to\infty} \beta(\Pi_T(\ ,\bar{p});\Pi_T(\ ,\bar{\bar{p}})) = 0. \tag{5.6}$$

Suppose now that the initial price vector is chosen according to an arbitrary probability distribution Π_0 on $\mathscr{B}(S)$. As before, Π_T is the distribution of $p_T(\ )$. By a standard result on conditional expectation, for any f in $\mathscr{L}(S)$ one has

$$\int_S f\, d\Pi_T = \int_S \left[\int_S f\, d\Pi_T(\ ,p)\right] d\Pi_0(p).$$

But if $\bar{p}$ is a *fixed* element of S,

$$\int_S f\, d\Pi_T(\ ,\bar{p})$$

is a constant which, upon integration with respect to a *probability measure* Π_0, remains the same, i.e.,

$$\int_S \left[\int_S f\, d\Pi_T(\ ,\bar{p})\right] d\Pi_0(p) = \int_S f\, d\Pi_T(\ ,\bar{p}).$$

Hence,

$$\left|\int f\, d\Pi_T - \int f\, d\Pi_T(\ ,\bar{p})\right| = \left|\int_S \left[\int f\, d\Pi_T(\ ,p) - \int f\, d\Pi_T(\ ,\bar{p})\, d\Pi_0(p)\right]\right|. \tag{5.7}$$

From (5.5) and (5.7) it follows immediately that

$$\lim_{T\to\infty} \beta(\Pi_T,\Pi_T(\ ,\bar{p})) = 0 \quad \text{for any given } \bar{p} \text{ in } S. \tag{5.8}$$

Now, if there happens to be an invariant distribution Π^*, then by choosing the initial distribution $\Pi_0 = \Pi^*$, one also has $\Pi_T = \Pi^*$ for all T, so that one necessarily has, for any given p in S

$$\lim_{T\to\infty} \beta(\Pi^*,\Pi_T(\ ,p)) = 0. \tag{5.9}$$

Thus, starting from any given p in S, the price vector $p_T(\ |p)$ converges in distribution to Π^*. But if the initial distribution of the process is an arbitrary Π_0 on $\mathscr{B}(S)$, then taking a given $\bar{p}$ in S,

$$\beta(\Pi^*, \Pi_T) \leqslant \beta(\Pi^*, \Pi_T(\ ,\bar{p})) + \beta(\Pi_T(\ ,\bar{p}), \Pi_T).$$

By repeated application of (5.8) one has

$$\lim_{T\to\infty} \beta(\Pi^*, \Pi_T) = 0.$$

Thus, irrespective of the initial distribution Π_0, the distribution Π_T of the random vector $p_T(\ )$ converges (in weak topology) to the invariant distribution Π^*, under Assumption 5. Clearly, one cannot have two distinct invariant distributions, since in that case for any given p in S, $\Pi_T(\ p)$ must converge in the weak topology to both these (by (5.9)), and this is impossible (see [**4**, p. 11]). We summarize the previous discussion by stating formally the following theorem:

THEOREM 5.1: *Under Assumptions* 1–5 *there exists a unique stochastic equilibrium of the process*, Π^*, *and for any initial distribution* Π_0 *of the process*, Π_T, *the distribution of* $p_T(\ )$, *satisfies*

$$\lim_{T\to\infty} \beta(\Pi_T, \Pi^*) = 0.$$

In particular, Π_T *converges to* Π^* *in the weak topology irrespective of the initial distribution* Π_0.

Define $\lambda_n(\ |\Pi_0)$ to be the sample distribution of prices after n periods if the initial price is chosen according to Π_0.

THEOREM 5.2: *For any* Π_0, *if the conditions of Theorem* 5.1 *hold, then* $\lambda_n(\ |\Pi_0) \xrightarrow{\text{a.s.}} \Pi^*$, *the unique, stable, stochastic equilibrium of the process.*

PROOF: This follows directly from Parthasarathy [**12**, Theorem 9.1], and our results above.

We have already remarked that it is possible to break away from the tradition completely and to define our adjustment process solely in terms of the kernel and the initial distribution. We have not pursued this direction, although it is likely that interesting results on convergence of the process to a stochastic equilibrium may be obtained by restricting the initial distribution to a suitable class of probability measures and imposing suitable conditions on the kernel. To consider a typical example briefly, note that for $T \geqslant 2$, one can define for any A in $\mathscr{B}(S)$ and p in S (see [**10**, pp. 366–9]):

$$\lambda^{(T)}(A; p) = \int \lambda(dp,\ ; p) \int \lambda(dp_2; p_1) \ldots \int \lambda(A; p_{T-1})\lambda(p_{T-2}; dp_{T-2}).$$

$\lambda^{(T)}(A; p)$ is to be interpreted as the probability of going from the price vector p into the set A in T periods, i.e., the probability that $p_T(\ )$ will be in A if the starting point is the price vector p.

For measuring the dependence of π_T, the distribution of p_T, upon the initial condition, we can use the following indicator Δ_T defined solely in terms of $\lambda^{(T)}(A;p)$:

$$\Delta_T = \sup_{A\in\beta(S)} \sup_{p,q\in S} |\lambda^{(T)}(A;p) - \lambda^{(T)}(A;q)|. \tag{5.10}$$

If $\Delta_T < 1$ for some T it is known that there exists a probability distribution π^* which is invariant such that for *any* A in $\mathscr{B}(S)$

$$|\pi_T(A) - \pi^*(A)| \leqslant \delta\, e^{-\beta T} \quad \text{for positive constants } \delta, \beta.$$

Of course, such exponential convergence is quite a strong and special property, not obtainable under weaker conditions (than $\Delta_T < 1$ for some positive T). It is possible, however, that weaker conditions will suffice to prove "local stability" in the sense of convergence of π_T to an invariant distribution if π_0 belongs to a well-chosen subset of $M(S)$.

6. FURTHER CHARACTERIZATION OF STOCHASTIC EQUILIBRIUM: SOME OPEN QUESTIONS

Suppose that the process is in stochastic equilibrium. What can we say about the time-paths of prices and excess demand vectors along alternative evolutions $\langle\omega_t\rangle$ of the economic environment? A natural question is to ask whether the excess demand vectors $\zeta(\ ,p_t(\ ))$ converge to zero for almost every realization of the process. We do not know reasonable sufficient conditions for this to happen. It is possible, however, to say something about the behavior of the time-averages.

It is known that a Markov process such as the one described above with an initial distribution that is invariant is itself a stationary stochastic process. Hence if τ is the invariant σ-field (see, e.g., Breiman [**4**, p. 108]) and $E(p_0|\tau)$ is the random vector which is the conditional expectation of p_0 given τ, the well-known ergodic theorem tells us that

$$\lim_{T\to\infty} 1/T \sum_{t=0}^{T} p_t(\ ) = E(p_0|\tau) \quad \text{a.s.} \tag{6.1}$$

One can also study the stationary process $\zeta(\ ;p_t(\ ))$ when the price process $p_t(\ )$ is stationary. Suppose Π^* is such that $E\|\zeta(\ ,p_0(\ ))\|$ is finite. Then

$$1/T \sum_{t=0}^{T} \zeta(\ ;p_t(\ ))$$

will converge almost surely to $E\zeta(\ ;p_0(\ )|\tau)$. If $E\zeta(\ ;p_0(\ )|\tau)$ is not zero, say it is negative in some component, then for a sufficiently large T, the sum of excess demands per period

$$\sum_{t=0}^{T} \zeta(\ ;p_t(\ ))$$

will almost surely be large and negative in that component, i.e., certain stocks will pile up. It is natural to ask how the agents are likely to respond, and what mechanism can possibly ensure that the process will continue in the same way. This remains a serious difficulty involving the concept of equilibrium as a stationary distribution of prices.

It seems that the difficulty will remain so long as we are unable to specify how the agents react to situations in which their plans are not realized and expectations not fulfilled. Work on this and related questions in a Bayesian setting is now in progress. Models of this type may be useful for studies of job-search equilibria and persistent unemployment, in partial equilibrium interpretations in which expected excess demand is negative in equilibrium. In contrast with the deterministic models, the steady state is consistent with observation of changing prices (and is possibly more appealing from a descriptive point of view). Clearly more knowledge about the characteristics of these steady states will lead to a better understanding of the working of exchange mechanisms in the presence of stochastic factors.

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