Working Paper 19-051

Estimating Models of Supply and Demand: Instruments and Covariance Restrictions

Alexander MacKay
Nathan H. Miller

Estimating Models of Supply and Demand: Instruments and Covariance Restrictions

Alexander MacKay
Harvard Business School

Nathan H. Miller
Georgetown University

Working Paper 19-051

Funding for this research was provided in part by Harvard Business School.

Estimating Models of Supply and Demand: Instruments and Covariance Restrictions*

Alexander MacKay
Harvard University†

Nathan H. Miller
Georgetown University‡

February 1, 2023

Abstract

We consider the identification of empirical models of supply and demand with imperfect competition. We show that a covariance restriction on unobserved demand and cost shocks resolves endogeneity and identifies the price parameter. We demonstrate how to employ this approach in estimation, and we provide a comparison to instrumental variables approaches. Our formal results also indicate how weaker assumptions about the covariance term can be used to construct bounds on the price parameter. We illustrate the covariance restriction approach with applications to ready-to-eat cereal, cement, and airlines.

JEL Codes: C13, C36, D12, D22, D40, L10

Keywords: Identification, Demand Estimation, Covariance Restrictions, Instrumental Variables

---

*We thank Chris Adams, Steven Berry, Charles Murry, Chuck Romeo, Gloria Sheu, Karl Schurter, Jesse Shapiro, Jeff Thurk, Andrew Sweeting, Matthew Weinberg, and Nathan Wilson for helpful comments. We also thank seminar and conference participants at Harvard University, MIT, the University of Maryland, the Barcelona GSE Summer Forum, and the NBER Summer Institute. Previous versions of this paper were circulated with the title "Instrument-Free Demand Estimation."

†Harvard University, Harvard Business School. Email: amackay@hbs.edu.

‡Georgetown University, McDonough School of Business. Email: nathan.miller@georgetown.edu.

1 Introduction

A fundamental challenge in identifying models of supply and demand is that firms can adjust markups in response to demand shocks. Even if marginal costs are constant, this source of price endogeneity generates upward-sloping supply in settings with imperfect competition. Thus, the empirical relationship between prices and quantities does not represent a demand curve but rather a mixture of demand and supply. Researchers typically address this challenge by using supply-side instruments to estimate demand, and then using the supply model to recover marginal costs and simulate counterfactuals (e.g., Berry et al., 1995; Nevo, 2001).

In this paper, we develop an alternative identification strategy that exploits covariance restrictions between demand-side and supply-side structural error terms. We first establish a function linking the endogenous price coefficient and the covariance of unobservable cost and demand shocks for a broad class of oligopoly models. We then show how this relationship can be used in estimation to identify the price parameter and pin down the slope of demand. A key distinction between our approach and the use of instrumental variables is that we interpret the endogenous variation in prices and quantities through the lens of the model, rather than relying on an additional (observed) variable to isolate exogenous variation in price. The core intuition is that the supply side of the model dictates how prices respond to demand shocks, shaping the relative variation of quantities and prices in the data.

In Section 2, we outline the data generating process for our baseline model and provide formal identification results. The baseline model features differentiated-products Bertrand competition between multi-product oligopolists with constant marginal costs. The demand side of the model either nests or trivially extends to logit, nested logit, and random coefficients logit, among other demand systems. The supply side similarly extends to a generalized model of oligopolistic competition that nests Cournot competition. In this setting, prices are endogenous because that they respond to a demand shock (the demand-side "structural error term") that is unobserved to the econometrician.

We prove that the price parameter solves a quadratic equation in which the coefficients are functions of observables and the covariance of demand and cost shocks. With a restriction on the covariance term, the price parameter is identified up to (at most) two points. Under reasonable conditions, the price parameter is the more negative solution, and point identification is obtained. The price parameter can be computed directly from our analytical solution, or the covariance restriction can be recast as an orthogonality condition and estimation can proceed with the method of moments. In a method-of-moments framework, our results show that the covariance restriction does indeed point identify the price parameter. The price parameter can be inferred from the relative empirical variation in (transformed) quantities and prices. The greater the relative variation in quantities, the more elastic demand must be, all else equal.[1]

[1]This is true even if the empirical relationship between prices and quantities is upward sloping. Our method will still recover the correct downward-sloping demand curve.

We then compare covariance restrictions to instrumental variables and discuss the credibility of the underlying identifying assumptions (Section 3). A shared feature is that the econometrician makes assumptions about the relationship between costs and unobserved demand shocks. In both approaches, fixed effects may be employed to absorb obviously confounding relationships, such as higher-quality products being more costly to produce. To the extent that residual variation in costs are orthogonal to the residual demand shocks, both instrumental variables and covariance restrictions may provide valid approaches to resolving price endogeneity.

As typically defined, valid instruments are observed variables that satisfy an exclusion condition and a relevance condition (e.g., Wooldridge, 2010). Estimation relies on the relationship between quantity and the exogenous portion of price that is isolated by the instrument. In small samples, bias can exist if the instrument and price are weakly correlated. By contrast, pairing a covariance restriction with a supply-side model exploits all of the price variation. After formalizing these distinctions, we compare our identification strategy to an approach that uses "residual instruments" recovered under a covariance restriction (Hausman and Taylor, 1983) and to the oft-used instruments of Hausman (1996). Finally, we use Monte Carlo simulations to show that a covariance restriction outperforms an instrument variables approach in finite samples, especially when instruments become weak.

In Section 4, we consider extensions to the baseline model. First, we consider a more general class of covariance restrictions that might be employed in practice. For example, for the Hausman instruments—prices in related markets—to be valid, three economic assumptions about the correlation structure between unobserved shocks must be satisfied. We discuss how one could employ these assumptions directly in a method-of-moments estimator, rather than relying on an instrumental variables implementation using observed prices.

We then provide two ways to make progress when costs are not constant in quantity. In such cases, the response of prices to demand shocks will be mediated by the slope of the cost curve. One approach to resolve this issue is to explicitly model the (non-constant) marginal cost function. A covariance restriction may then be credible. However, identification requires additional moments for any parameters that enter the non-constant portion of marginal costs. The other path involves shifting to a bounds analysis. If the econometrician can sign the correlation between unobserved demand and cost shocks—for example, a positive correlation could exist due to capacity constraints—then one-sided bounds can be placed on the price parameter. Furthermore, interpreting the data through the lens of the model can allow the econometrician to rule out some values of the price parameter without any assumption on this correlation.

We apply these methods in a series of empirical applications (Section 5). Each of the three settings that we have selected—ready-to-eat (RTE) cereals, cement, and airlines—differ in a variety of ways that influence our implementation. With RTE Cereals, marginal costs can plausibly be modeled as constant, so we proceed with estimation under a covariance restriction, using fixed effects to absorb potentially confounding variation. With cement, capacity constraints imply that marginal costs may increase with quantities. We follow an approach developed in the literature and model this effect explicitly, after which we view a covariance restriction as credible. Finally, with airlines, we apply a bounds approach. In each case, we show how a covariance restriction supports inferences about the price parameter.

Together, our results help address a significant obstacle for empirical research—that of finding valid instruments for price. Empirical models of imperfect competition typically have other key parameters that characterize heterogeneity of consumer preferences. To identify these parameters, researchers have used micro-moments constructed from the observed behavior of individual consumers (e.g., Backus et al., 2021; Döpper et al., 2022) or "second-choice" data on what consumers view as their next-best option (e.g., Grieco et al., 2021). These strategies identify the consumer heterogeneity parameters but do not resolve price endogeneity (Berry and Haile, 2020). Thus, the covariance restriction we examine is a useful complement to the use of detailed consumer data.[2] For example, Döpper et al. (2022) combine covariance restrictions with micro-moments to estimate demand for consumer products in over 1,800 distinct estimation samples.

To put our results in context, covariance restrictions were analyzed in early research at the Cowles Foundation on the identification of linear systems of equations, including supply and demand models of perfect competition (e.g., Koopmans, 1949; Koopmans et al., 1950).[3] With perfect competition, the supply curve is upward-sloping due to increasing costs of production. With upward-sloping supply and downward-sloping demand, two separate restrictions are required for identification (Hausman and Taylor, 1983). If instead price endogeneity arises due to the markup adjustments that occur in models of imperfect competition, then (as we show) only a single restriction is sufficient for identification.

The strategy of using supply-side restrictions to reduce identification requirements has parallels in a handful of other articles. Leamer (1981) examines a linear model of perfect competition, and provides conditions under which the price parameters can be bounded using only the endogenous variation in prices and quantities. Feenstra (1994) considers the case of monopolistic competition with constant markups, and a number of application in the trade literature extend this constant-markup approach (e.g., Broda and Weinstein, 2006, 2010; Soderbery, 2015).[4] Zoutman et al. (2018) return to perfect competition and show that, under a standard assumption in models of taxation, both supply and demand can be estimated with exogenous variation in a single tax rate. At a high level, our approach to estimation with covariance restric-

[2]Alternatively, if instruments constructed from the characteristics of competing products (e.g., Berry et al., 1995; Gandhi and Houde, 2020), then the covariance restriction could be incorporated using the generalized method of moments (GMM) as an additional identifying restriction.

[3]Many articles advanced this research agenda in subsequent decades (e.g., Fisher, 1963, 1965; Wegge, 1965; Rothenberg, 1971; Hausman and Taylor, 1983; Hausman et al., 1987). More recently, Matzkin (2016) examines covariance restrictions in semi-parametric models.

[4]There are interesting historical antecedents to this trade literature. Leamer attributes an early version of his results to Schultz (1928). The identification argument of Feenstra (1994) is also proposed in Leontief (1929). Frisch (1933) provides an important econometric critique.

tions relates to Petterson et al. (2022), which shows how to bound structural parameters based on beliefs about the magnitudes of unobserved shocks. Our research builds on these articles by developing results for imperfect competition with adjustable markups.[5]

2 Identification with Covariance Restrictions

2.1 Data Generating Process

The model examines supply and demand across markets that can be conceptualized to represent (for example) separate geographies, time periods, or both. In each market t, there is a set $\mathcal{J}_t = \{0, 1, \ldots, J_t\}$ products available for purchase. The market $t = 1, \ldots, T$ is defined by $(\mathcal{J}_t, \chi_t)$, where

$$\chi_t = \{\boldsymbol{X}_t, \boldsymbol{\xi}_t, \boldsymbol{\eta}_t\}$$

is a set that contains non-price characteristics of the product and market. Among these, $\boldsymbol{X}_t = (\boldsymbol{x}_{1t}, \ldots, \boldsymbol{x}_{J_t t})$ is a matrix of characteristics that is observable to the econometrician, and $\boldsymbol{\xi}_t = (\xi_{1t}, \ldots, \xi_{J_t t})$ and $\boldsymbol{\eta}_t = (\eta_{1t}, \ldots, \eta_{J_t t})$ are mean-zero vectors of unobservable product-level or market-level characteristics. We sometimes refer to the unobservable characteristics as "structural error terms." Let each $\xi_{jt}, \eta_{jt} \in \mathbb{R}$, each $\boldsymbol{x}_{jt} \in \mathbb{R}^K$, and the support of χ_t be χ. Without loss of generality, let $\mathcal{J}_t = \mathcal{J} = \{0, 1, \ldots, J\}$ going forward.

Prices and quantities satisfy equilibrium conditions. Let $\boldsymbol{p}_t = (p_{1t}, \ldots, p_{Jt})$ be a vector of prices and $\boldsymbol{q}_t = (q_{1t}, \ldots, q_{Jt})$ be a vector of quantities, with $p_{jt}, q_{jt} \in \mathbb{R}$. On the demand side, we assume that the quantity sold of each product is determined by $q_{jt} = \sigma_{jt}(\boldsymbol{p}_t; \chi_t, \boldsymbol{\theta})$, where each $\sigma_{jt}(\cdot)$ is a differentiable, invertible demand function and $\boldsymbol{\theta}$ is a vector of parameters. We place the following restriction on demand:

$$h_{jt}(\boldsymbol{q}_t; \chi_t, \boldsymbol{\theta}) \equiv \sigma_{jt}^{-1}(\boldsymbol{q}_t; \chi_t, \boldsymbol{\theta}) = \alpha p_{jt} + \boldsymbol{x}'_{jt}\boldsymbol{\beta} + \xi_{jt} \tag{1}$$

We assume downward-sloping demand (i.e., $\alpha < 0$). Models that satisfy this restriction are used regularly in the empirical literature of industrial organization.[6]

On the supply side, we decompose prices into additive markups and marginal costs:

$$p_{jt} = \mu_{jt}(\chi_t, \boldsymbol{\theta}) + mc_{jt}(\chi_t, \boldsymbol{\theta}) \tag{2}$$

[5] Some applications in industrial organization identify demand-side parameters with the assistance of supply-side assumptions (e.g., Thomadsen, 2005; Cho et al., 2018; Li et al., 2022). Among these, Thomadsen assumes there are not unobserved demand shocks and Cho et al. assume that there are no unobserved cost shocks; both are special cases of the covariance restriction approach.

[6] For example, with logit demand, we have $\sigma_{jt}^{-1}(\cdot) \equiv \ln(s_{jt}) - \ln(s_{0t})$, where we follow convention and use markets shares, s_{jt}, in place of quantities. Nested logit, random coefficients logit, linear demand, and constant elasticity demand are nested or accommodated with trivial generalizations (Appendix A).

We initially maintain that marginal costs are a linear function of characteristics:

$$mc_{jt}(\chi_t, \boldsymbol{\theta}) = \boldsymbol{x}'_{jt}\boldsymbol{\gamma} + \eta_{jt} \tag{3}$$

Marginal costs can vary with quantities through a non-zero correlation of the the demand-side and supply-side structural error terms (e.g., as in Berry et al., 1995; Ciliberto et al., 2021).

Consistent with the first order conditions that arise in a broad class of oligopoly models, we assume that profit-maximizing markups take the form:

$$\mu_{jt}(\chi_t, \boldsymbol{\theta}) = -\frac{1}{\alpha}\lambda_{jt}(\boldsymbol{q}_t, D(\boldsymbol{q}_t, \chi_t); \boldsymbol{\theta}), \tag{4}$$

where λ_{jt} is a function of quantities and $D(\boldsymbol{q}_t, \chi_t)$, the $J \times J$ matrix of partial derivatives $\left[\frac{\partial \sigma_{kt}(\boldsymbol{p}_t, \chi_t)}{\partial p_{lt}}\right]_{k,l}$. We show how to construct λ_{jt} in a variety of specific contexts in our applications and in Appendix A.[7]

Combining equations (2)-(4), the supply-side of the model satisfies the following relationship for each product j and market t:

$$\lambda_{jt}(\boldsymbol{q}_t, D(\boldsymbol{q}_t, \chi_t); \boldsymbol{\theta}) = -\alpha p_{jt} + \alpha \boldsymbol{x}'_{jt}\boldsymbol{\gamma} + \alpha \eta_{jt} \tag{5}$$

This *supply relationship* characterizes how equilibrium prices and quantities respond to shifts in demand (holding fixed α and marginal costs). Unlike the supply curve for perfectly competitive firms, this relationship accounts for the market power of individual firms and lies above the marginal cost curve.[8]

Together, equations (1) and (5) provide the equilibrium conditions that determine prices and quantities.

2.2 Identification

Stacking objects across markets, the econometrician observes vectors of prices and quantities ($\boldsymbol{p}$ and $\boldsymbol{q}$) and a matrix of non-price characteristics ($\boldsymbol{X}$). The vectors of structural error terms are $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. From the quantities and the model, the econometrician also can obtain the quantity transformations that appear in demand and supply ($\boldsymbol{h}$ and λ). The parameters to be estimated are $(\alpha, \boldsymbol{\beta}, \boldsymbol{\gamma})$. As the latter are trivially identified given the first, we focus on the identification

[7]With single-product Bertrand pricing, we have $\mu_{jt} = -\frac{1}{dq_{jt}/dp_{jt}} q_{jt} = -\frac{1}{\alpha}\frac{dh_{jt}}{dq_{jt}} q_{jt}$. If, in addition, demand is logit, then $\lambda_{jt} \equiv \frac{dh_{jt}}{dq_{jt}} q_{jt} = \frac{1}{1-s_{jt}}$. Appendix A covers nested logit, random coefficients logit, constant elasticity demand, and linear demand. On the supply-side, it covers the construction of λ_{jt} with multi-product firms and under a generalized model of oligopoly that nests both Bertrand competition in prices and Cournot competition in quantities.

[8]The difference between marginal costs and the supply relationship is the (perceived) inframarginal loss in revenue for selling an additional unit of quantity. Bresnahan (1982) refers to the inverse of this equation, with price on the left-hand side, as the "supply relation" and notes that it generalizes to different models of firm conduct. See Appendix A.6 for a figure and additional discussion.

of the price parameter, α. In some models there may be additional demand parameters in $\boldsymbol{\theta}$, in which case the quantity transformations may obtain conditional on those other parameters.[9]

We assume that the structural error terms can be decomposed as

$$\xi_{jt} = \xi_j + \xi_t + \Delta\xi_{jt} \tag{6}$$

$$\eta_{jt} = \eta_j + \eta_t + \Delta\eta_{jt} \tag{7}$$

where demand and cost shocks have product-specific persistent components (e.g., higher quality or higher cost), market-specific components (greater demand in a year and/or region), and an orthogonal mean-zero residual term.[10] Let $\tilde{\boldsymbol{X}}$ denote the $JT \times (K + J + T - 2)$ matrix that includes, in addition to the K observed covariates, a full set of dummy variables for products $(J - 1)$ and markets $(T - 1)$. We assume these characteristics are exogenous in the sense that $\mathbb{E}[\Delta\xi_{jt}|\tilde{\boldsymbol{X}}_t] = \mathbb{E}[\Delta\eta_{jt}|\tilde{\boldsymbol{X}}_t] = 0$ for all $j = 1, \ldots, J_t$.

We assume that $\mathbb{E}[\boldsymbol{W}'\boldsymbol{W}]$ has full rank, where $\boldsymbol{W} = [\boldsymbol{p}\ ;\ \tilde{\boldsymbol{X}}]$ is a $JT \times (K + J + T - 1)$ matrix. This is a standard (and mild) condition. An implication is that $\mathbb{E}[\tilde{\boldsymbol{X}}'\tilde{\boldsymbol{X}}]$ must also have full rank, allowing us to construct, for example, the residuals of a regression of $\boldsymbol{p}$ on $\tilde{\boldsymbol{X}}$ as

$$\boldsymbol{p}^* = \boldsymbol{p} - \tilde{\boldsymbol{X}}[\tilde{\boldsymbol{X}}'\tilde{\boldsymbol{X}}]^{-1}[\tilde{\boldsymbol{X}}'\boldsymbol{p}]. \tag{8}$$

Henceforth, we will use the superscript * to denote the residuals obtained from a regression of some variable on $\tilde{\boldsymbol{X}}$.

The rank condition on $\mathbb{E}[\boldsymbol{W}'\boldsymbol{W}]$ requires that the observable non-price characteristics and the dummy variables are not collinear, and also that they do not fully explain prices—at least one of the residual structural error terms must matter. Letting the absence of subscripts denote scalar random variables, we can express this implication as $Var(p^*) > 0$.

We now formalize our first identification result, which links the OLS estimate to the price parameter. The probability limit ($T \to \infty$) of the OLS estimate of α obtained from a regression of $\boldsymbol{h}$ on $\boldsymbol{W}$ (i.e., on $\boldsymbol{p}$ and $\tilde{\boldsymbol{X}}$) is

$$\alpha^{OLS} \equiv \frac{Cov(p^*, h)}{Var(p^*)} = \alpha + \frac{Cov(p^*, \Delta\xi)}{Var(p^*)}. \tag{9}$$

The corresponding OLS residuals are given by $\boldsymbol{\Delta\xi}^{OLS} = \boldsymbol{h} - \boldsymbol{W}[\boldsymbol{W}'\boldsymbol{W}]^{-1}[\boldsymbol{W}'\boldsymbol{h}]$.

We now construct a function that maps the price coefficient to a specific value for the co-

[9]An example is the random coefficients logit model of Berry et al. (1995). The model incorporates "linear parameters" (analogs to $\alpha, \boldsymbol{\beta}, \boldsymbol{\gamma}$) and also "nonlinear parameters" that influence consumer substitution parameters. The identification of these nonlinear parameters through micro-moments and other data has been a focus of other research, e.g., Berry and Haile (2020). Though they are not our focus, in Section 5.1, we show that additional covariance restrictions can identify these parameters in one of our applications.

[10]This decomposition is not strictly necessary, and our results go through in the cases for which ξ_j and η_j equal zero or ξ_t and η_t equal zero (or both). If so, the included dummy variables need to be adjusted accordingly. We present this as the baseline as we expect it to be the most common empirical implementation of our approach.

variance of the residual structural error terms:

Proposition 1. *The probability limit of the OLS estimate can be written as a function of α, the residuals from an OLS regression, prices and quantities, and a covariance term:*

$$\alpha^{OLS} = \alpha - \frac{1}{\alpha + \frac{Cov(p^*,\lambda)}{Var(p^*)}} \frac{Cov\left(\Delta\xi^{OLS},\lambda\right)}{Var(p^*)} + \alpha \frac{1}{\alpha + \frac{Cov(p^*,\lambda)}{Var(p^*)}} \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} \tag{10}$$

Therefore, α solves the following quadratic equation:

$$\begin{aligned} 0 = {} & \alpha^2 \\ & + \left(\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)} - \alpha^{OLS} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)\alpha \\ & + \left(-\alpha^{OLS}\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)} - \frac{Cov\left(\Delta\xi^{OLS},\lambda\right)}{Var(p^*)}\right) \end{aligned} \tag{11}$$

All proofs are in Appendix C. The terms in equation (11) are well defined under our rank condition and, aside from α and $Cov(\Delta\xi,\Delta\eta)$, they have straightforward empirical analogs.

Note that are at most two solutions for α for any given value of $Cov(\Delta\xi,\Delta\eta)$. Further, in most empirical models, α is likely to be the lower root. The following result provides formal conditions under which this is guaranteed:

Proposition 2. *The parameter α is the lower root of equation* (11) *if and only if*

$$-\frac{1}{\alpha}Cov(\Delta\xi,\Delta\eta) \leq Cov\left(p^*, \Delta\eta - \frac{1}{\alpha}\Delta\xi\right) \tag{12}$$

and, furthermore, α is the lower root of equation (11) *if*

$$0 \leq \alpha^{OLS} Cov\left(p^*,\lambda\right) + Cov\left(\Delta\xi^{OLS},\lambda\right) \tag{13}$$

In the first condition, it is helpful to think of $-\frac{1}{\alpha}\Delta\xi$ as the residual demand-side structural error term, scaled so that units are equivalent to those of marginal costs (and price). If $Cov(\Delta\xi,\Delta\eta) = 0$, the condition holds as long as prices tend to increase with demand and marginal costs, as occurs in most empirical models. For example, the condition holds when demand is linear. Thus, α is likely the lower root of equation (11) in most applications.

The second condition is derived using properties of the quadratic formula. Because the terms in equation (13) are constructed from data, the sufficient condition can be estimated and used to test (and possibly reject) the null hypothesis that multiple negative roots exist. Henceforth, we assume that α is the lower root of equation (11).

The implication of this result—a one-to-one function mapping α to $Cov(\Delta\xi, \Delta\eta)$—is that the price coefficient can be recovered with information about the correlation between residual demand and cost shocks in models with imperfect competition. Conversely, moments that pin down the price parameter also pin down the value of $Cov(\Delta\xi, \Delta\eta)$.

2.3 Estimation

We now illustrate how the above results can be used to construct estimates of the price parameter. Estimation can proceed with the method of moments (a general approach) by recasting the information about the covariance term as an orthogonality condition. One possibility is that that demand-side and supply-side structural error terms are uncorrelated: $Cov(\Delta\xi, \Delta\eta) = 0$. Equivalently, this can be expressed as $\mathbb{E}[\Delta\xi_{jt}\Delta\eta_{jt}] = 0$.

Under this assumption, the method-of-moments estimator uses the empirical analog of this condition and attempts to minimize its contribution to the GMM objective function. To illustrate this, for a case with one moment and one parameter, the method-of-moments estimate of α is given by

$$\hat{\alpha}^{CR} = \arg\min_{\tilde{\alpha}<0} \left[\frac{1}{T}\frac{1}{J}\sum_{t}\sum_{j\in\mathcal{J}} \Delta\xi_{jt}(\tilde{\alpha})\Delta\eta_{jt}(\tilde{\alpha}) \right]^2, \tag{14}$$

where $\Delta\xi_{jt}(\tilde{\alpha})$ and $\Delta\eta_{jt}(\tilde{\alpha})$ may be recovered from residualized (transformed) quantities and prices given the candidate parameter under consideration. GMM may also be used with additional moments and to estimate multiple parameters jointly, in which case the sample moment may be weighted against other components of the GMM objective function using the standard approach. Some care must be taken to ensure convergence to the lower root.

An implication of Proposition 1 and a key contribution of our paper is that this estimate is consistent for the price parameter, i.e., $\hat{\alpha}^{CR} \to \alpha$. This is useful because, in general, the inclusion of a moment in a method-of-moments approach does not imply the consistent identification of an additional parameter.[11] By contrast, we demonstrate that this particular moment provides identification of the price parameter *a priori*. In fact, one could use the analytical expression in equation (11) to directly compute the coefficient estimate.[12]

This method-of-moments approach is employed in our first application and also by Döpper et al. (2022) to estimate models with random-coefficients logit demand and Bertrand pricing. In these models, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ can be estimated via OLS regression once $\hat{\alpha}^{CR}$ is obtained. Döpper

[11] To highlight this, consider that Berry and Haile (2020) identify a class of moment conditions (micro-moments) that can pin down consumer heterogeneity but provide *no identifying information* about the price parameter.

[12] The probability limit of the coefficient estimate is given by:

$$\alpha^{CR} = \frac{1}{2}\left(\alpha^{OLS} - \frac{Cov\,(p^*, \lambda)}{Var(p^*)} - \sqrt{\left(\alpha^{OLS} + \frac{Cov\,(p^*, \lambda)}{Var(p^*)}\right)^2 + 4\frac{Cov\,(\Delta\xi^{OLS}, \lambda)}{Var(p^*)}}\right), \tag{15}$$

which obtains from an application of the quadratic formula.

et al. (2022) also illustrate how additional “nonlinear” parameters can be incorporated in GMM with the nested fixed point approach of Berry et al. (1995). Denote these additional parameters $\boldsymbol{\theta}_2$ (as in Nevo, 2001). For each candidate $\tilde{\boldsymbol{\theta}}_2$, the covariance restrictions estimator is applied to obtain $\hat{\alpha}^{CR}(\tilde{\boldsymbol{\theta}}_2)$. In the outer loop, an estimate of $\boldsymbol{\theta}_2$ is pinned down by the inclusion of micro-moments in the GMM objective function. More generally, covariance restriction can be applied in conjunction with instruments, and additional moments allow for efficiency improvements and specification tests (e.g., Hausman, 1978; Hansen, 1982).

The empirical variation that identifies α is the relative variance of (transformed) quantities and prices. When $Cov(\Delta\xi, \Delta\eta) = 0$, we obtain the following formal result:

Proposition 3. *If $Cov(\Delta\xi, \Delta\eta) = 0$, then a first-order approximation to probability limit of the method-of-moments estimator is*

$$\alpha^{CR} \approx -\sqrt{\frac{Var(h^*)}{Var(p^*)}}. \tag{16}$$

Intuition can be gleaned from the simultaneous equations representation of the model using equations (1) and (5). Rearranging these to obtain inverse demand and inverse supply relationships, we have:

$$p_{jt}^D = \frac{1}{\alpha} h_{jt}(\boldsymbol{q}_t; \chi_t, \boldsymbol{\theta}) - \frac{1}{\alpha}\boldsymbol{x}_{jt}'\boldsymbol{\beta} - \frac{1}{\alpha}\xi_{jt} \tag{17a}$$

$$p_{jt}^S = -\frac{1}{\alpha}\lambda_{jt}(\boldsymbol{q}_t, D(\boldsymbol{q}_t, \chi_t); \chi_t, \boldsymbol{\theta}) + \boldsymbol{x}_{jt}'\boldsymbol{\gamma} + \eta_{jt} \tag{17b}$$

By inspection, α determines the slope of both. A large α corresponds to a flatter inverse demand schedule (i.e., price sensitive consumers) and a flatter inverse supply relationship (i.e., less market power).[13] Uncorrelated shifts in such schedules tend to generate more variation in quantity than price. By contrast, a small α corresponds to steeper inverse demand and inverse supply, such that uncorrelated shifts generate more variation in price than quantity. Connecting these observations formally generates an approximation of the lower root based on the ratio of variances. Estimation with a covariance restriction converts the endogenous variation in prices and quantity into consistent estimates.

3 Relationship to Instrumental Variables

The covariance restriction approach to estimation that we have outlined above exploits all of the endogenous variation in quantities and prices, which is interpreted through the lens of the economic model. The approach differs from the instrumental variables approach, which seeks to isolate exogenous variation in prices. It also employs different assumptions than those used

[13] See Figure A.1 in the appendix for an illustration.

to obtain “residual instruments” or Hausman instruments. We provide formal distinctions in this section and include Monte Carlo simulations to illustrate.

3.1 Motivating Example

We begin with an economic model for which a traditional instrumental variable approaches and the covariance restriction approach can yield consistent estimates. Consider a simplified setting in which a profit-maximizing monopolist faces linear demand in each of $t = 1, \ldots, T$ markets. The demand schedule takes the form:

$$q_{jt}^D = \alpha p_{jt} + \xi_j + \xi_t + \Delta\xi_{jt} \tag{18}$$

and marginal costs take the form

$$mc_{jt} = \eta_j + \eta_t + \Delta\eta_{jt}. \tag{19}$$

where quantity demanded (q_{jt}) and price (p_{jt}) are observed (we retain the subscript j for notational consistency). Thus, the model includes product- and market-specific shocks, but no observable characteristics other than price.

The key identification challenge for estimating α is that prices are often chosen by firms to reflect unobserved demand shocks. In our example, equilibrium prices are given by $p_{jt} = \frac{1}{2}mc_{jt} + \frac{1}{2|\alpha|}(\xi_j + \xi_t + \Delta\xi_{jt})$. Thus, prices are higher for higher quality products ($Cov(p_{jt}, \xi_j) > 0$) and in high-demand markets ($Cov(p_{jt}, \xi_t) > 0$). In settings with sufficient observations, these correlations can be accounted for with fixed effects. However, the primary concern about price endogeneity remains as long as $Cov(p_{jt}, \Delta\xi_{jt}) \neq 0$.

The prior literature primarily resolves this issue of price endogeneity with instruments. One common approach is to obtain auxiliary data that measures a component of costs that is orthogonal to $\Delta\xi_{jt}$. For illustrative purposes, suppose that one could measure $\Delta\eta_{jt}$ directly. Then, $\Delta\eta_{jt}$ would be a valid instrument for p_{jt} in the demand equation as long as $Cov(\Delta\xi_{jt}, \Delta\eta_{jt}) = 0$. It would typically not be sufficient to use a measure of η_j or η_t as an instrument because these components of costs would be absorbed by the fixed effects used to control for product quality and market-level demand differences.

Instead, Proposition 1 indicates that one could use the restriction $Cov(\Delta\xi_{jt}, \Delta\eta_{jt}) = 0$ directly in estimation. To explore the conditions under which this is valid, we build on the economic model using the following stylized example.

Consider an online retailer that sells coffee tables made from two different materials, e.g., wicker and solid wood. Consumers may prefer one product to another (ξ_j) and overall demand for the retailer's products may vary across markets (ξ_t). The online retailer sells the products for different prices in each market, which reflect the above features and also idiosyncratic variation in tastes for products across markets, given by $\Delta\xi_{jt}$. On the supply side, products vary in the procurement and distribution costs (η_j), and marginal costs vary across markets due to differences in distribution networks and fuel costs (η_t). For the online retailer, residual product-market variation in costs is due to the interaction of product characteristics with features of the local distribution networks. Specifically, similarly-sized coffee tables can differ significantly in weight, depending on the material. Thus, the product-market cost variation ($\Delta\eta_{jt}$) is approximately given by weight$_j$ $\times$ (fuel cost)$_t$.

The demand for these products does not have any obvious link to differential fluctuations in distribution costs for the two products. Based on this, one could estimate demand by first obtaining data on product-level characteristics (weight) and market-level features (fuel costs), and then using the interaction of the two to generate cost-shifter instruments in a standard instrumental variables approach. When controlling for product quality and market-level demand, it would be necessary to construct a measure with idiosyncratic across-market variation by product; otherwise, the instrument would be fully absorbed by the fixed effects.

The covariance restrictions approach would leverage the same logic—that idiosyncratic product-market differences in costs are orthogonal to the idiosyncratic product-market differences in preferences—to obtain identification. A key distinction is that the covariance restrictions approach applies the identifying assumption $Cov(\Delta\xi_{jt}, \Delta\eta_{jt}) = 0$ directly, without the need to collect additional data and construct an instrument. Moreover, the specific identifying assumptions are distinct across the approaches, which we formalize in the following sections.

In practice, the econometrician often observes only a portion of marginal costs, in which case the instrument $\Delta\eta^{(1)}$ can be expressed as a component of the full structural error, $\Delta\eta_{jt} = \omega\Delta\eta^{(1)}_{jt} + (1-\omega)\Delta\eta^{(2)}_{jt}$. The unobserved component, $\Delta\eta^{(2)}_{jt}$, may be interpreted as measurement error—for example, if the interaction of fuel costs with weight is only a first-order approximation to actual shipping costs. The fact that the instrument does not use all of the variation can be an advantage when $Cov(\Delta\xi_{jt}, \Delta\eta^{(1)}_{jt}) = 0$ but $Cov(\Delta\xi_{jt}, \Delta\eta^{(2)}_{jt}) \neq 0$. In this case, the instrument isolates the exogenous variation and yields a consistent estimate, but the covariance restriction approach may be biased. This bias becomes small when the orthogonal component explains a greater share of idiosyncratic cost shocks ($\omega \to 1$). Thus, in assessing the credibility of the covariance restrictions approach, it is helpful to understand the components that contribute the most to the marginal cost residual, even if they are unobserved in data.

The above example mirrors the approach taken by Döpper et al. (2022) when estimating demand for consumer products. Döpper et al. (2022) employ a rich set of fixed effects to account for obvious linkages between demand and costs. The residual supply-side structural error features two sources of variation that have been exploited as instruments in recent research: product-specific changes in input costs (Backus et al., 2021) and product-specific changes in distribution costs (Miller and Weinberg, 2017). The demand elasticities that obtain are comparable those from the literature (e.g., for beer, cereals, and yogurt), which indicates that both approaches are consistent with the underlying economic model. Thus, these examples—along with the applications in Section 5—demonstrate how a similar justification for the validity of instrumental variables may be used to motivate the covariance restrictions approach.

3.2 Excluded Instruments

An instrument is an observable variable that satisfies an exclusion condition and a relevance condition (e.g., Wooldridge, 2010). In our model, the exclusion and relevance conditions can be expressed as:

$$\mathbb{E}[\Delta\xi_{jt} z_{jt}] = 0 \tag{20a}$$

$$\mathbb{E}[p^*_{jt} z_{jt}] \neq 0. \tag{20b}$$

where we focus on the case of a single instrument, z, for expositional purposes. There are two perhaps obvious differences between the instrumental variables approach to identification and the covariance restrictions approach. The first is that implementation of the instrumental variables approach (e.g., with 2SLS) requires that an excluded instrument be observed in the data. The second is that there may be settings in which the orthogonality condition $\mathbb{E}[\Delta\xi_{jt} z_{jt}] = 0$ holds but the covariance restriction $\mathbb{E}[\Delta\xi_{jt}\Delta\eta_{jt}] = 0$ does not. For example, in the motivating example, a valid instrument may be possible to construct using a component of $\Delta\eta_{jt}$.

In this section, we develop a third distinction that is more related to the relevance condition of equation (20b). The condition states that some residual variation in prices is explained by the instrument. This a stronger condition than what is required by the covariance restriction approach, which works even in the case when $\mathbb{E}[p^*_{jt}\Delta\eta_{jt}] = 0$.

To show this, we start with the 2SLS estimate of α given instrument z:

$$\alpha^{IV} = \frac{Cov(h^*, z)}{Cov(p^*, z)} = \frac{Cov(h^*, \hat{p}^*)}{Var(\hat{p}^*)} \tag{21}$$

where $\hat{\boldsymbol{p}} = \boldsymbol{Z}[\boldsymbol{Z}'\boldsymbol{Z}]^{-1}\boldsymbol{Z}'\boldsymbol{p}$ is the predicted value of price for $\boldsymbol{Z} = [\boldsymbol{z}\ ;\ \tilde{\boldsymbol{X}}]$ and $\hat{\boldsymbol{p}}^*$ is the residual from the projection of $\hat{\boldsymbol{p}}$ on $\tilde{\boldsymbol{X}}$. The first equality shows that the 2SLS estimate equals the ratio of coefficient obtained in the reduced-form regression of h^* on z to the coefficient obtained in a first-stage regression of p^* on z. For α^{IV} to be well defined, the relevance condition must hold, so that $Cov(p^*, z) \neq 0$. The second equation provides a familiar reformulation that is useful for our purposes: the 2SLS estimate equals the coefficient from a regression of h^* on $\hat{p}^*$, the (residualized) first-stage predicted values. The relevance condition similarly is re-expressed as $Var(\hat{p}^*) > 0$. This is a stronger condition than $Var(p^*) > 0$, the assumption that is necessary for the covariance restrictions approach.[14]

[14] As we discuss in Section 2.2, $Var(p^*) > 0$ is an implication of $\mathbb{E}[\boldsymbol{W}'\boldsymbol{W}]$ having full rank.

Moreover, even if condition (20b) is satisfied in the limit, the instrumental variables estimator can exhibit asymptotic bias in finite samples (e.g., Keane and Neal, 2022). This is an important condition in practice, and many papers have been devoted to address the "weak instrument" problem when this condition is tenuously satisfied (e.g., Bound et al., 1995; Staiger and Stock, 1997; Stock and Yogo, 2005). The covariance restriction approach can side-step this issue because all of the variation in p is used to construct the estimate. From the Frisch-Waugh-Lovell Theorem, we have $Var(p^*) \geq Var(\hat{p}^*)$, and indeed in applications $Var(\hat{p}^*)$ may be much smaller than $Var(p^*)$ if z is constructed from one of many components of costs.

Thus far, we have framed a valid instrument as satisfying $\mathbb{E}[\Delta\xi_{jt}z_{jt}] = 0$. However, in the context of the model, an alternative is use an instrument that satisfies $\mathbb{E}[\Delta\eta_{jt}z_{jt}] = 0$. Such an instrument could be taken from the demand-side of the model or constructed based on markup shifters, as in Berry et al. (1995). This approach uses the supply-side of the model, specifically the first-order conditions of equation (5), to estimate the price parameter. We highlight this possibility because it shows how either demand-side variation or supply-side variation can be used to pin down the price parameter, so long as the appropriate exclusion restriction can be applied. The covariance restriction approach employs both cost-side variation and demand-side exogenous variation implicitly in a single restriction.

3.3 Residual Instruments

We now compare our approach to existing results on the identification of simultaneous equations with covariance restrictions. Wooldridge (2010, p. 258) focuses on the case of two linear equations.[15] For comparability, we build on our motivating example in this section while incorporating observable characteristics ($\boldsymbol{x}$) into demand and marginal costs, yielding the following linear demand and supply relationships:

$$q_{jt}^D = \alpha_1 p_{jt} + \boldsymbol{x}_{jt}'\boldsymbol{\beta} + \xi_j + \xi_t + \Delta\xi_{jt} \tag{22a}$$

$$q_{jt}^S = -\alpha_2 p_{jt} + \alpha_2\left(\boldsymbol{x}_{jt}'\boldsymbol{\gamma} + \eta_j + \eta_t + \Delta\eta_{jt}\right) \tag{22b}$$

where, again, the supply relationship comes directly from the monopolist's first-order condition for profit maximization. Because we first consider the general simultaneous equations approach, the price coefficients are allowed to vary across the equations.

The "residual instruments" approach to identification relies on the following identifying

[15]The discussion in Wooldridge (2010) builds on a substantial literature on covariance restrictions in linear simultaneous equation models (e.g., Hausman and Taylor, 1983; Hausman et al., 1987). More recent research generalizes these results (Matzkin, 2016).

moments for some observable x^k:

$$\mathbb{E}[x^k_{jt}\Delta\eta_{jt}] = 0 \tag{23a}$$
$$\mathbb{E}[x^k_{jt}p^{*k}_{jt}] \neq 0 \tag{23b}$$
$$\mathbb{E}[\Delta\xi_{jt}\Delta\eta_{jt}] = 0. \tag{23c}$$

where p^{*k} is the residual from a regression of p on $\boldsymbol{x}$ excluding the k^{th} characteristic, x^k. The first two moments correspond to the assumption that x^k is a valid instrument for the supply equation (22b). This requires that $\boldsymbol{x}$ is exogenous and there exists some x^k for which $\gamma_k = 0$ and $\beta_k \neq 0$. These moments can be used to obtain a consistent estimate of α_2.

The third moment corresponds to the additional assumption that $Cov(\Delta\xi, \Delta\eta) = 0$. Under this covariance restriction, the estimated residuals $\widehat{\Delta\eta}$—which can be obtained after identifying α_2 in the first step—can be used as instruments in the demand equation (22a) to obtain α_1. Thus, in this framework, covariance restrictions have been interpreted as providing excluded instruments.[16]

We note that, in this approach, the covariance restriction can only yield the residual instruments when a valid instrument x^k also exists. Thus, two identifying assumptions are required. Furthermore, both x^k and $\widehat{\Delta\eta}$ must explain p to a substantial degree, otherwise each step could suffer from the weak instruments problem.

By contrast, our approach to estimation with covariance restrictions recognizes a theoretical connection between the slopes of demand and supply that is implied by the economic model: $\alpha_1 = \alpha_2 = \alpha$. In this case, α is point identified with only the covariance restriction $Cov(\Delta\xi, \Delta\eta) = 0$. Thus, our approach eliminates the need for the excluded instrument (x^k), providing a path for identification under a weaker set of identifying assumptions, while also avoiding the finite-sample challenges of weak instruments.

3.4 Hausman Instruments

A number of articles in industrial organization have relied on prices in related markets as instruments in demand estimation (Gandhi and Nevo, 2021). Typically, the "related markets" refer to distinct geographic areas. In that setting, the price of a product in some market s can be a valid instrument for the price of the same product in market t if marginal costs are correlated across markets (e.g., due to shared production facilities) but demand is not. Such instruments often are referred to as "Hausman instruments" due to their use in Hausman (1996).

[16]This interpretation has been influential. For example, McFadden states in lecture notes (dated 1999) that "Even covariance matrix restrictions can be used in constructing instruments. For example, if you know that the disturbance in an equation you are trying to estimate is uncorrelated with the disturbance in another equation, then you can use a consistently estimated residual from the second equation as an instrument." See `https://eml.berkeley.edu/~mcfadden/e240b_f01/ch6.pdf`.

Here, we assume that the implementation employs product and market fixed effects.[17] Formally, the conditions needed for validity are that there exist pairs of markets t, s such that:

$$\mathbb{E}[\Delta\xi_{jt}\Delta\xi_{js}] = 0 \tag{24a}$$

$$\mathbb{E}[\Delta\xi_{jt}\Delta\eta_{st}] = 0 \tag{24b}$$

$$\mathbb{E}[\Delta\eta_{jt}\Delta\eta_{js}] \neq 0. \tag{24c}$$

Condition (24a) states that the demand-side error terms are uncorrelated across markets, condition (24b) states that the demand-side error of one market is uncorrelated with the supply-side error term in another market, and condition (24c) states that supply-side error terms are correlated across markets.

If these conditions are satisfied, then p_{js} is a valid (excluded) instrument for p_{jt} in the demand equation. Thus, analogous to the residual instruments approach in the previous section, this approach leverages assumptions about the correlation structure of unobservables to generate excluded instruments. This approach can suffer from the weak instruments problem, as we discuss in Section 3.2.

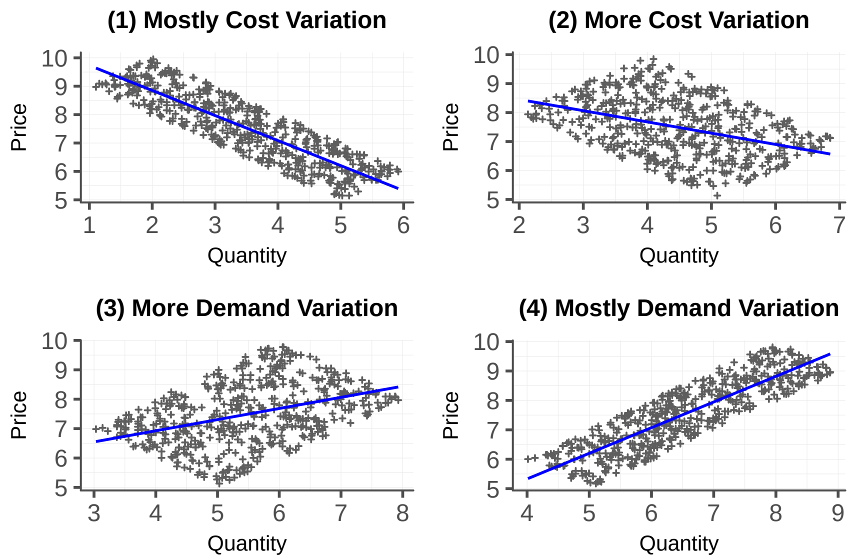
Similar to our approach, the Hausman instruments are justified by assumptions about the correlation structure of demand and cost shocks. The set of assumptions are distinct: the three above conditions could be met when $\mathbb{E}[\Delta\xi_{jt}\Delta\eta_{jt}] \neq 0$, or conditions (24a)–(24c) may not be satisfied while $\mathbb{E}[\Delta\xi_{jt}\Delta\eta_{jt}] = 0$. One advantage of our approach is that only a single restriction is required. A second advantage is that our approach exploits all of the variation implied by the identifying moment and avoids the finite-sample issues with excluded instruments. That is, our approach employs conditions (24a)–(24c) directly, rather than using them to employ an observed variable (p_{js}) as an instrument. We explore generalizations of our approach with these and other covariance restrictions in Section 4.1.

3.5 Finite Sample Comparison

We use Monte Carlo simulations to illustrate the finite sample performance of covariance restrictions relative to excluded instruments. For excluded instruments, we consider both traditional supply-side instruments and also demand-side instruments discussed above in Section 3.2. In both cases, we assume that the available instrument is fully efficient in that it captures all of the relevant exogenous variation—i.e., $z = \Delta\eta$ for a supply-side instrument and $z = \Delta\xi$ for the demand-side instrument. That is, we allow the econometrician to fully observe cost shocks when estimating demand (but not demand shocks or the supply model) or demand shocks when estimating the supply relationship (but not cost shocks or the demand model). By contrast, the covariance restrictions approach does not treat either as observed, but it uses both demand and

[17] In practice, researchers sometimes use Hausman instruments that reflect variation across all markets, in which case it is necessary to assume that $\xi_j = 0$.

Figure 1: Prices and Quantities in the Monopoly Model



Notes: This figure displays equilibrium prices and quantities under four different specifications for the distribution of unobserved shocks to demand and marginal costs. The line in each figure indicates the slope obtained by OLS regression.

supply in conjunction with the observed variation in prices and quantities. We consider four specifications where we vary the relative contributions of demand-side and cost-side variation.

For our simulations, we consider the demand relation given by equation (18) and the corresponding supply relation:

$$q_{jt}^S = -\alpha p_{jt} + \alpha \left(\eta_j + \eta_t + \Delta \eta_{jt} \right), \tag{25}$$

For this single-product case, we normalize ξ_t and η_t to zero. We specify $\alpha = -1$ and that $\xi_{jt} = \xi_j + \Delta\xi_{jt}$ and $\eta_{jt} = \eta_j + \Delta\eta_{jt}$ have independent uniform distributions. We consider four specifications: (i) $\xi \sim 10 + U(0,2)$ and $\eta \sim U(0,8)$, (ii) $\xi \sim 10 + U(0,4)$ and $\eta \sim U(0,6)$, (iii) $\xi \sim 10 + U(0,6)$ and $\eta \sim U(0,4)$, and (iv) $\xi \sim 10 + U(0,8)$ and $\eta \sim U(0,2)$. Moving from (i) to (iv), demand-side variation increases and supply-side variation decreases.

As is well known, if both cost and demand variation is present, then equilibrium outcomes provide a "cloud" of data points that do not necessarily correspond to the demand curve. To illustrate, we present one simulation of 500 observations from each specification in Figure 1, along with the fit of an OLS regression of quantity on price. The expected values of the OLS estimator in each scenario are -0.882, -0.385, 0.385, and 0.882. With greater demand-side variation, the endogeneity bias is larger.[18] In addition, with greater demand-side variation, the

[18]Inspection of Figure 1 further suggests that there may be connection between OLS bias and goodness-of-fit. Indeed, starting with equation (16), a few lines of additional algebra obtain $\alpha \approx -\left|\alpha^{OLS}\right| / \sqrt{R^2}$ where R^2 is from the residual OLS regression of h^* on p^*. The approximation is exact with linear demand. This reformulation fails if $R^2 = 0$, but numerical results indicate robustness for values of R^2 that are approximately zero. We thank Peter Hull

Table 1: Small-Sample Properties: Relative Variation in Demand and Supply Shocks

(a) Covariance Restrictions

Observations	(i) $Var(\eta) \gg Var(\xi)$		(ii) $Var(\eta) > Var(\xi)$		(iii) $Var(\eta) < Var(\xi)$		(iv) $Var(\eta) \ll Var(\xi)$	
25	-1.004	(0.098)	-1.017	(0.201)	-1.018	(0.206)	-1.005	(0.099)
50	-1.001	(0.068)	-1.008	(0.136)	-1.007	(0.135)	-1.001	(0.068)
100	-1.001	(0.047)	-1.003	(0.094)	-1.004	(0.093)	-1.001	(0.047)
500	-1.000	(0.021)	-1.001	(0.041)	-1.001	(0.042)	-1.000	(0.021)

(b) Supply Shifters (IV-1)

Observations	(i) $Var(\eta) \gg Var(\xi)$		(ii) $Var(\eta) > Var(\xi)$		(iii) $Var(\eta) < Var(\xi)$		(iv) $Var(\eta) \ll Var(\xi)$	
25	-1.004	(0.105)	-1.039	(0.303)	-1.310	(2.629)	-0.899	(13.921)
50	-1.001	(0.072)	-1.018	(0.201)	-1.113	(1.135)	-1.392	(10.890)
100	-1.001	(0.050)	-1.008	(0.138)	-1.048	(0.332)	-1.432	(5.570)
500	-1.000	(0.022)	-1.001	(0.060)	-1.009	(0.138)	-1.061	(0.411)

(c) Demand Shifters (IV-2)

Observations	(i) $Var(\eta) \gg Var(\xi)$		(ii) $Var(\eta) > Var(\xi)$		(iii) $Var(\eta) < Var(\xi)$		(iv) $Var(\eta) \ll Var(\xi)$	
25	-0.881	(12.794)	-1.295	(3.087)	-1.040	(0.312)	-1.006	(0.106)
50	-1.448	(10.980)	-1.112	(0.596)	-1.016	(0.198)	-1.001	(0.073)
100	-1.597	(5.837)	-1.045	(0.333)	-1.009	(0.136)	-1.001	(0.050)
500	-1.070	(0.414)	-1.008	(0.137)	-1.002	(0.060)	-1.000	(0.022)

Notes: Results are based on 10,000 simulations of data for each specification and number of observations. The demand curve is $q_{jt} = \alpha p_{jt} + \xi_{jt}$ with $\alpha = -1$ and $\xi_{jt} = \xi_j + \Delta\xi_{jt}$. Marginal costs are $mc_{jt} = \eta_{jt}$ where $\eta_{jt} = \eta_j + \Delta\eta_{jt}$. We consider a single product ($j = 1$) and vary the number of markets/observations from 25 to 500. IV-1 estimates are calculated using 2SLS with cost shocks ($\Delta\eta$) as an instrument in the demand equation. Analogously, IV-2 estimates are calculated using 2SLS with demand shocks ($\Delta\xi$) as an instrument in the supply relationship. We consider four specifications: (i) $\xi \sim 10 + U(0, 2)$ and $\eta \sim U(0, 8)$, (ii) $\xi \sim 10 + U(0, 4)$ and $\eta \sim U(0, 6)$, (iii) $\xi \sim 10 + U(0, 6)$ and $\eta \sim U(0, 4)$, and (iv) $\xi \sim 10 + U(0, 8)$ and $\eta \sim U(0, 2)$. Moving from (i) to (iv), demand-side variation increases and supply-side variation decreases.

supply-side instrument is less predictive of price.

We consider sample sizes of 25, 50, 100, and 500 observations. For each specification and sample size, we randomly draw 10,000 datasets, and with each we estimate the model with a covariance restriction, with a supply-side instrument, and with a demand-side instrument. For the covariance restriction, we assume $Cov(\Delta\xi, \Delta\eta) = 0$. The estimate of α can be constructed as $-\sqrt{Var(q)/Var(p)}$ because the approximation of equation (16) is exact in this context. For a supply-side instrument, we estimate demand with 2SLS using the cost shock $\Delta\eta$ as the instrument. That is, we use all of the supply-side variation to estimate demand. For the demand-side instrument, we estimate the supply relationship with 2SLS using $\Delta\xi$ as the instrument. Note that all three approaches rely on an identical orthogonality condition: $\mathbb{E}[\Delta\xi_{jt}\Delta\eta_{jt}] = 0$.

Table 1 provides the mean and (empirical) standard error of the point estimates for each

for suggesting this connection.

specification and approach.[19] Panel (a) shows that the covariance restriction approach to estimation yields estimates that are consistently close to the true value. Panel (b) shows that, with supply-side instruments, small sample bias becomes substantial with smaller datasets and less variance in the cost shock. This is due to a weak instrument—for example, the mean first-stage F-statistics in specification (iv) are 2.6, 4.2, 7.3, and 32.6 for markets with 25, 50, 100, and 500 observations, respectively. Panel (c) shows that, with demand-side instruments, small sample bias becomes substantial with smaller datasets and less variance in the demand shock, which also is due to a weak instruments problem.

Thus, in settings where instruments perform poorly, a covariance restriction may still provide a precise estimate. The approach exploits both demand and cost variation without treating (a component of) either as separately observed. Instead, all of the price and quantity variation is interpreted through the lens of the model.

One potential risk of the covariance restrictions approach is that (for example) the supply side could be misspecified. This could generate bias, while an instruments-based approach that uses a cost-shifter to estimate demand only requires an informal understanding of supply.[20] To asssess the potential for bias, we perform additional Monte Carlo exercises with a misspecified supply model in Appendix B. In our simulations, the bias from supply-side misspecification is small; it is mitigated by the fact that covariance restrictions approach also uses demand.

4 Extensions

In this section, we explore three extensions of our main results. First, we discuss a broader class of potential covariance restrictions. Second, we consider the case of non-constant marginal costs. Third, we show how our results can be used to bound the coefficients of interests even with weaker information about the covariance of unobserved shocks. We demonstrate the use of each these extensions in the applications of Section 5.

4.1 Generalized Covariance Restrictions

Thus far, our analysis has focused on covariance restrictions between own demand and cost shocks. Our results demonstrate that this moment is expected to generate a consistent estimate of the price parameter. We now consider different covariance restrictions that generalize the approach. Though other covariance restrictions do not provide a similar guarantee of point identification, they may work well in certain settings.

[19] To avoid outliers arising from the weak instrument problem, we bound the estimates of α on the range $[-100, 100]$. For specifications that suffer from weak instruments, this biases the standard errors toward zero. This affects specifications where the estimated standard error is greater than one, i.e., in 9 of 48 specifications.

[20] Note that, in practice, empirical research often employs supply models to calculate markups or conduct counterfactuals. In these cases, supply-side misspecification cannot be fully addressed by estimating demand separately.

Consider the assumptions (24a)–(24c) that are required for the Hausman instruments. Rather than using these assumptions to motivate the use of an instrument, the assumptions could be employed directly in a method-of-moments estimator, where, as in equation (14), the estimated residuals are generated from econometric model for a candidate parameter. This approach has the advantage over the Hausman instruments approach in that the estimator would utilize all of the variation implied by the identifying moments.

Alternatively, it may be reasonable to assume that the variance of the demand shock does not depend on the level of the cost shock, and vice versa, which generates the moments $\mathbb{E}_{jt}[\Delta\xi_{jt}^2\Delta\eta_{jt}]$ and $\mathbb{E}_{jt}[\Delta\xi_{jt}\Delta\eta_{jt}^2]$. Or it may be reasonable to assume that average shocks are uncorrelated across groups of products, i.e., $\mathbb{E}_{gt}[\overline{\Delta\xi}_{gt}\overline{\Delta\eta}_{gt}] = 0$, where $\overline{\Delta\xi}_{gt}$ and $\overline{\Delta\eta}_{gt}$ are the mean demand and cost shocks for products in group g.

Finally, it may be useful to consider cross-product covariance restrictions, i.e.,

$$\mathbb{E}_t[\Delta\xi_{jt}\Delta\eta_{kt}] = 0 \quad \forall j \neq k. \tag{26}$$

These restrictions state that the demand shock for product j is uncorrelated with the cost shock for product k. The expectation in equation (26) can be taken over t and k to obtain J restrictions or over markets (as written above) to obtain $J \times (J-1)$ restrictions.[21]

These alternative covariance restrictions may be used to identify the price parameter when one is concerned that own-product demand and cost shocks are correlated. However, as noted above, our results do not provide a direct link between these moments and the price parameter. We therefore suspect that, even when true, the assumptions do not always suffice to identify the price parameter.

Additionally, alternative restrictions may be employed to pin down other parameters of interest (e.g., $\boldsymbol{\theta}$), in addition to the price parameter. We explore this possibility in the first and third applications in Section 5.

We also note that our analytical results do not require that the value of the covariance terms are equal to zero. For example, Proposition 1 can be used to construct consistent estimates for any ς for which $\mathbb{E}[\Delta\xi_{jt}\Delta\eta_{jt}] = \varsigma$. In certain situations, it may be possible to employ an estimate of the correlation in demand and cost shocks to identify the price parameter. For example, (Berry et al., 1995) report that this correlation is 0.17. For a similar empirical setting, it may be reasonable to invoke Proposition 1 to obtain an estimate of α conditional on this value.

4.2 Other Marginal Cost Functions

In Section 3, we discussed the credibility of the covariance restrictions approach under the maintained assumption that marginal costs are constant in quantities. We now relax that assumption. Here, we show how our approach to estimation extends if the econometrician

[21] Similarly, the own-product restrictions may assumed to hold separately by product, providing J restrictions.

has knowledge of the marginal cost function. Then, in the next subsection, we provide a bounds approach that can be applied if the marginal cost function is unknown. These three cases—constant marginal costs, known marginal cost functions, and unknown marginal cost functions—are mirrored in the three empirical applications that we present in the final section of the paper.

If marginal costs depend on quantities then estimation based on the covariance restriction $Cov(\Delta\xi, \Delta\eta) = 0$ may not be credible unless the econometrician can model the relationship between quantity and marginal cost explicitly. Consider the case in which marginal costs can be expressed as the following function:

$$mc_{jt}(q_{jt}; \chi_t, \boldsymbol{\theta}) = \boldsymbol{x}'_{jt}\boldsymbol{\gamma} + g(q_{jt}; \boldsymbol{\tau}) + \eta_{jt} \tag{27}$$

where $g(q_{jt}; \boldsymbol{\tau})$ is a known function that depends on quantity and the vector of parameter, $\boldsymbol{\tau}$. The supply relationship becomes:

$$\lambda_{jt}(\boldsymbol{q}_t, D(\boldsymbol{q}_t, \chi_t); \boldsymbol{\theta}) = -\alpha p_{jt} + \alpha \boldsymbol{x}'_{jt}\boldsymbol{\gamma} + \alpha g(q_{jt}; \boldsymbol{\tau}) + \alpha \eta_{jt} \tag{28}$$

In this augmented model, both markup adjustments and non-constant marginal costs can contribute to price endogeneity. If the econometrician omits $g(\cdot)$ from the model, then the residual cost shock is $\widetilde{\Delta}\eta_{jt} \equiv g(q_{jt}; \boldsymbol{\tau}) + \Delta\eta_{jt}$. Then, $Cov(\Delta\xi, \Delta\eta) = 0$ does not imply that $Cov(\Delta\xi, \widetilde{\Delta}\eta) = 0$, and the approach developed in Section 2 may not produce consistent estimates.

We now trace through the steps developed in Section 2.2 and show that α is identified by the covariance restriction $Cov(\Delta\xi, \Delta\eta) = 0$ for any value of $\boldsymbol{\tau}$. The OLS regression of $\boldsymbol{h}$ on $\boldsymbol{p}$ and $\boldsymbol{x}$ yields a price coefficient with the following probability limit:

$$\alpha^{OLS} = \alpha - \frac{1}{\alpha}\frac{Cov\left(\Delta\xi, \lambda\right)}{Var(p^*)} + \frac{Cov(\Delta\xi, g(q; \boldsymbol{\tau}))}{Var(p^*)} + \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)} \tag{29}$$

This equation can be reformulated such that the demand-side error term, $\Delta\xi$, is replaced with the probability limit of OLS residuals, $\Delta\xi^{OLS}$, creating an analog to equation (10). Rearranging terms and assuming $Cov(\Delta\xi, \Delta\eta) = 0$ then yields an analog to equation (11):

Corollary 1. *If marginal costs take the semi-linear form of equation* (27) *and* $Cov(\Delta\xi, \Delta\eta) = 0$*,* *then α solves the following quadratic equation:*

$$
\begin{aligned}
0 = \quad & \left(1 - \frac{Cov(p^*, g(q;\boldsymbol{\tau}))}{Var(p^*)}\right)\alpha^2 \\
& + \left(\frac{Cov\,(p^*,\lambda)}{Var(p^*)} - \alpha^{OLS} + \alpha^{OLS}\frac{Cov(p^*, g(q;\boldsymbol{\tau}))}{Var(p^*)} + \frac{Cov(\Delta\xi^{OLS}, g(q;\boldsymbol{\tau}))}{Var(p^*)}\right)\alpha \\
& + \left(-\alpha^{OLS}\frac{Cov\,(p^*,\lambda)}{Var(p^*)} - \frac{Cov\left(\Delta\xi^{OLS},\lambda\right)}{Var(p^*)}\right)
\end{aligned}
$$

There are at most two solutions for α for any $\boldsymbol{\tau}$ and the lower root can fairly be targeted in most applications. In estimation, the method-of-moments can be used to jointly estimate α and $\boldsymbol{\tau}$, using the covariance restriction to identify α and auxiliary moments to identify $\boldsymbol{\tau}$. The auxiliary moments can consist of excluded instruments or the generalized covariance restrictions discussed above. Thus, it is possible to control for a direct relationship between quantity and marginal costs with additional structure. We explore a cost function approach to estimation in an application to cement (Section 5.2).

4.3 Analysis of Bounds

We now present the bounds analysis, which may be useful for inference when a covariance restriction along the lines of $Cov(\Delta\xi, \Delta\eta) = 0$ is not credible. We first consider bounds that utilize prior knowledge of the sign of the correlation between $\Delta\xi$ and $\Delta\eta$. Next, we show how the model and the data together may bound the price coefficient without any additional information.

For the first case, we assume that the econometrician can sign the correlation between $\Delta\xi$ and $\Delta\eta$. This situation might arise, for example, if factor prices are influenced by macroeconomic conditions, such that there is a link between the unobserved demand-side and supply-side error terms that is difficult to model explicitly. With a prior of the sign of $Cov(\Delta\xi, \Delta\eta)$, bounds can be placed on α. The reason is that there is a one-to-one mapping between the value of $Cov(\Delta\xi, \Delta\eta)$ and the lower root of equation (11):

Lemma 1. ***(Monotonicity)*** *Under assumptions 1 and 2, a valid lower root of equation* (11) ***(i.e., one that is negative)*** *is decreasing in $Cov(\Delta\xi, \Delta\eta)$. The range of the function is $(0, -\infty)$.*

Thus, if idiosyncratic demand and costs are correlated, such as through capacity constraints ($Cov(\Delta\xi, \Delta\eta) \geq 0$), then one-sided bounds can be placed on α. More generally, let $r(m)$ be the lower root of the quadratic in equation (11), evaluated at $Cov(\Delta\xi, \Delta\eta) = m$. Then $Cov(\Delta\xi, \Delta\eta) \geq m$ produces $\alpha \in (-\infty, r(m)]$, and $Cov(\Delta\xi, \Delta\eta) \leq m$ produces $\alpha \in [r(m), 0)$. The lower root, $r(m)$, can be estimated with the method-of-moments.[22]

[22] Nevo and Rosen (2012) develop conceptually similar bounds for estimation with imperfect instruments, defined as instruments that are less correlated with the structural error term than the endogenous regressor.

For the second case, it can be that some values of the price parameter are unable to rationalize the data for *any* amount of correlation between $\Delta\xi$ and $\Delta\eta$. These values can be ruled out. Thus, the demand and supply assumptions alone may be informative about the plausible range of α. Formally, this occurs when the quadratic from equation (11) does not have a lower root, and thus no valid solution for α. To see how this can occur, represent the quadratic from equation (11) as $az^2 + bz + c$. By assumption, one root is $\alpha < 0$. As $a = 1$, the quadratic forms a $\cup$-shaped parabola. If $c < 0$ then the existence of a negative root is guaranteed. However, if $c > 0$ then b must be positive and sufficiently large for a negative root to exist. This places restrictions on $Cov(\Delta\xi, \Delta\eta)$, which is a component of b. From the monotonicity result (Lemma 1), we can use the excluded values of $Cov(\Delta\xi, \Delta\eta)$ from this result to rule out values of α.

We now state the result formally:

Proposition 4. ***(Model-Based Bound)*** *The model and data alone may bound $Cov(\Delta\xi, \Delta\eta)$ from below. The bound is given by:*

$$Cov(\Delta\xi, \Delta\eta) > Var(p^*)\alpha^{OLS} - Cov(p^*, \lambda) + 2Var(p^*)\sqrt{\left(-\alpha^{OLS}\frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)}\right)}$$

The bound exists if and only if the term inside the radical is non-negative. Further, through equation (11)*, this lower bound on $Cov(\Delta\xi, \Delta\eta)$ provides an upper bound on α.*

From the monotonicity result (Lemma 1), the excluded values of $Cov(\Delta\xi, \Delta\eta)$ rule out values of α. A model-based upper bound for α is obtained by evaluating the lower root of equation (11) at the model-based bound of $Cov(\Delta\xi, \Delta\eta)$. In practice, priors over the the covariance of unobserved shocks may be combined with model-based bounds to further restrict the identified set. We explore bounds analyses in an application to airlines (Section 5.3).

5 Empirical Applications

We provide three empirical applications to demonstrate how covariance restrictions can inform inference. The three settings—ready-to-eat (RTE) cereals, cement, and airlines—differ in a variety of ways that influence our implementation. With RTE Cereals we proceed with estimation under $Cov(\Delta\xi, \Delta\eta) = 0$, assuming constant marginal costs and using fixed effects to absorb potentially confounding variation, as discussed in Section 3. With cement, capacity constraints imply that marginal costs can increase with quantities. We follow an approach developed in the literature and model this effect explicitly, after which $Cov(\Delta\xi, \Delta\eta) = 0$ becomes credible (as in Section 4.2). Finally, with airlines, the relationship between demand shocks and prices can be complicated; instead of modeling it directly we apply a bounds approach (as in Section 4.3).

5.1 Ready-to-Eat (RTE) Cereals

We choose RTE cereals for our first application because, with panel data and appropriate fixed effects, a covariance assumption appears credible, for reasons that we explain below. Furthermore, it allows us to develop the covariance restrictions approach to estimation in the context of the random coefficients logit demand model (Berry et al., 1995). We use the pseudo-real cereals data of Nevo (2000) and compare estimates obtained with a covariance restriction to those obtained with the provided instruments. There are 24 products, 47 cities, and 2 quarters.[23]

Let the indirect utility that consumer i receives from product j in market t (a combination of a quarter and a city) be given by

$$u_{ijt} \quad = \quad \delta_{jt}(p_{jt};\beta) + \mu_{ijt}(p_{jt}, \boldsymbol{D}_i, \boldsymbol{v}_i; \boldsymbol{\Pi}, \boldsymbol{\Sigma}) + \epsilon_{ijt} \tag{30}$$

where the mean utility of each product, $\delta_{jt}(\cdot)$, and contribution of demographics to consumer-specific deviations, $\mu_{ijt}(\cdot)$, respectively are given by

$$\begin{aligned} \delta_{jt}(p_{jt};\alpha,\boldsymbol{\beta}) &= \alpha p_{jt} + \boldsymbol{x}'_{jt}\boldsymbol{\beta} + \xi_j + \Delta\xi_{jt} \\ \mu_{ijt}(p_{jt}, \boldsymbol{D}_i, \boldsymbol{v}_i; \boldsymbol{\Pi}, \boldsymbol{\Sigma}) &= [p_{jt}, \boldsymbol{x}_{jt}]' * (\boldsymbol{\Pi}\boldsymbol{D}_i + \boldsymbol{\Sigma}\boldsymbol{v}_i) \end{aligned}$$

with $\boldsymbol{D}_i$ and $\boldsymbol{v}_i$ being vectors of consumer-specific demographic characteristics.

The probability with which consumer i selects product j is

$$s_{ijt}(\boldsymbol{\delta}_t, p_{jt}, \boldsymbol{D}_i, \boldsymbol{v}_i; \boldsymbol{\Pi}, \boldsymbol{\Sigma}) = \frac{\exp(\delta_{jt}(p_{jt};\alpha,\boldsymbol{\beta}) + \mu_{ijt}(p_{jt}, \boldsymbol{D}_i, \boldsymbol{v}_i; \boldsymbol{\Pi}, \boldsymbol{\Sigma}))}{1 + \sum_{k=1}^{J_{kt}} \exp(\delta_{kt}(p_{kt};\alpha,\boldsymbol{\beta}) + \mu_{ikt}(p_{kt}, \boldsymbol{D}_i, \boldsymbol{v}_i; \boldsymbol{\Pi}, \boldsymbol{\Sigma}))} \tag{31}$$

where $\boldsymbol{\delta}_t = (\delta_{1t}, \delta_{2t}, \dots)$ is the vector of mean utilities. The market share of product j is obtained by integrating over the joint distribution of consumer demographics:

$$s_{jt}(\boldsymbol{\delta}_t, p_{jt}; \boldsymbol{\Pi}, \boldsymbol{\Sigma}) = \frac{1}{I}\sum_i s_{ijt}(\boldsymbol{\delta}_t, p_{jt}, \boldsymbol{D}_i, \boldsymbol{v}_i; \boldsymbol{\Pi}, \boldsymbol{\Sigma}) \tag{32}$$

We now connect to the modeling framework of Section 2. Let the set of products sold by the same firm as product j be given by $\mathcal{J}_{f(j)}$. Then, under the assumption of Bertrand price competition, we have:

$$\lambda_{jt}(\boldsymbol{s}_t, \boldsymbol{p}_t; \boldsymbol{\theta}) = \frac{s_{jt}}{\frac{1}{I}\sum_i s_{ijt}(1 - s_{ijt})} - \sum_{k \in \mathcal{J}_{f(j)} \setminus j} \frac{s_{kt}}{\frac{1}{I}\sum_i s_{ijt} s_{ikt}} \tag{33}$$

where the denominators integrate over the (product of) consumer-specific choice probabilities.

[23] See also Dubé et al. (2012), Knittel and Metaxoglou (2014), and Conlon and Gortmaker (2020). We focus on the "restricted" specification of Conlon and Gortmaker (2020), which addresses a multicollinearity problem by imposing that the parameter on $Price \times Income^2$ takes a value of zero. We refer readers to Nevo (2000) for details on the data.

From an econometric standpoint, λ_{jt} is free from the price parameter α because it depends only on market shares and consumer-specific choice probabilities. The market shares are data. From equation (31), the consumer-specific choice probabilities depend on $\mu_{ijt}(\cdot)$, which obtains immediately from data and $(\mathbf{\Pi}, \mathbf{\Sigma})$, and on $\boldsymbol{\delta}_t(\cdot)$, which obtains from the contraction mapping of Berry et al. (1995), again given data and $(\mathbf{\Pi}, \mathbf{\Sigma})$.

Marginal costs in the Nevo (2000) model are given by

$$mc_{jmt} \quad = \quad \eta_j + \Delta\eta_{jt} \tag{34}$$

where η_j is a product fixed effect, and $\Delta\eta_{jt}$ is the supply-side structural error term.

We use the covariance restriction $Cov(\Delta\xi_{jt}, \Delta\eta_{jt}) = 0$ in estimation. The supply-side structural error term incorporates some of the cost-shifter instruments that have been used in the recent literature, including time-varying, product-specific shipping costs (Miller and Weinberg, 2017) and the time-varying prices of product-specific ingredients (Backus et al., 2021). Given the fixed effects, these cost-shifters can be conceptualized as providing the variation that is exploited in estimation. Furthermore, it may be reasonable to think that marginal costs are roughly constant with consumer products, as is sometimes maintained in the literature (Villas-Boas, 2007; Chevalier et al., 2003; Hendel and Nevo, 2013; Miller and Weinberg, 2017; Backus et al., 2021).

The parameters for estimation include $(\alpha, \boldsymbol{\beta})$, as in our baseline model from Section 2, and also $(\mathbf{\Pi}, \mathbf{\Sigma})$. Therefore, additional identifying assumptions are needed. Some recent applications use micro-moments constructed from the observed behavior of individual consumers (e.g., Backus et al., 2021; Döpper et al., 2022) or "second-choice" data on what consumers view as their next-best option (e.g., Grieco et al., 2021). Both of these strategies identify $(\mathbf{\Pi}, \mathbf{\Sigma})$ but not the price parameter (Berry and Haile, 2020). This separability allows for a two-step approach to estimation in which the price parameter is estimated after the other parameters. An alternative strategy is to use instruments constructed from competitor characteristics (e.g., Berry et al., 1995; Gandhi and Houde, 2020) to identify the additional parameters. As none of these options are available to us given the data and specification, we pursue an alternative approach based on a generalization of the covariance restriction assumption.

Specifically, we extend the assumption that residual demand and cost shocks are uncorrelated to all cross-product pairs, such that $Cov(\Delta\xi_{jt}, \Delta\eta_{kt}) = 0$ for all j, k. The joint restrictions are valid if the demand shock of each product is orthogonal to its own marginal cost shock and those of all other products. As there are 24 products in each market, the full covariance matrix of demand and cost shocks provides sufficient moments to estimate the 12 nonlinear parameters in the specification.

Table 2 summarizes the results of estimation based on the instruments (panel (a)) and covariance restrictions (panel (b)). Both identification strategies yield similar mean own-price demand elasticities: -3.70 with instruments and -3.61 with covariance restrictions. Overall,

Table 2: Point Estimates for Ready-to-Eat Cereal

(a) Available Instruments

		Standard	Interactions with Demographics		
Variable	Means	Deviations	Income	Age	Child
Price	-32.019	1.803	4.187	–	11.755
	(2.304)	(0.920)	(4.638)		(5.198)
Constant	–	0.120	3.101	1.198	–
		(0.163)	(1.105)	(1.048)	
Sugar	–	0.004	-0.190	0.028	–
		(0.012)	(0.035)	(0.032)	
Mushy	–	0.086	1.495	-1.539	–
		(0.193)	(0.648)	(1.107)	

(b) Covariance Restrictions

		Standard	Interactions with Demographics		
Variable	Means	Deviations	Income	Age	Child
Price	-36.230	1.098	14.345	–	26.906
	(1.122)	(1.067)	(1.677)		(1.384)
Constant	–	0.051	-0.156	1.072	–
		(0.230)	(0.286)	(0.240)	
Sugar	–	0.003	-0.084	-0.004	–
		(0.014)	(0.018)	(0.010)	
Mushy	–	0.130	0.301	-0.845	–
		(0.162)	(0.196)	(0.103)	

Notes: This table reports point estimates for the random-coefficients logit demand system estimated using the Nevo (2000) dataset. Panel (a) employs the available instruments. Panel (b) employs covariance restrictions.

the different approaches produce similar patterns for the coefficients. Most of the point estimates under covariance restrictions fall in the 95 percent confidence intervals implied by the specification with instruments, including that of the mean price parameter. The standard errors are noticeably smaller with covariance restrictions, which likely reflects that the covariance restrictions approach to estimation more fully exploits the variation that is present in the data. We conclude that in this setting—where a covariance restriction appears credible—estimation with covariance restrictions and with instruments indeed produce similar results.

5.2 The Portland Cement Industry

Our second empirical application considers a setting in which marginal costs increase with output. We build on the marginal cost specification from Section 4.2, in which the upward-sloping part of the cost function can be modeled explicitly. To illustrate, we consider the setting and data of Fowlie et al. (2016) ["FRR"], which examines market power in the cement industry.

The data are a balanced panel of 520 region-year observations for 20 regions over 1984-

2009, with the regions corresponding to selected urban areas in the United States. There are an average of 4.65 cement firms located in each region-year.[24]

The model features Cournot competition among cement plants facing capacity constraints. Let the market demand curve in region r and year t have a logit form:

$$h_{rt}(Q_{rt}) \equiv \ln(Q_{rt}) - \ln(M_r - Q_{rt}) = \boldsymbol{x}_{rt}'\boldsymbol{\beta} + \alpha p_{rt} + \xi_r + \Delta\xi_{rt} \tag{35}$$

where $Q_{rt} = \sum_{j \in \mathcal{J}} q_{jrt}$ is total quantity and M_r is the "market size" of the region.[25] Further, we allow marginal costs to vary with quantity according to

$$mc_{jrt} = \boldsymbol{x}_{jrt}'\boldsymbol{\gamma} + g_{jrt}(q_{jrt}; \tau) + \Delta\eta_{jrt} \tag{36}$$

In particular, we follow FRR and assume that that g is a "hockey stick" function, $g_{jrt}(q_{jrt}; \tau) = 2\tau 1\{q_{jrt}/k_{jr} > 0.9\}(q_{jrt}/k_{jr} - 0.9)$, where k_{jr} and q_{jrt}/k_{jr} are capacity and utilization, respectively. Marginal costs are constant if utilization is less than 90%. Above this threshold, marginal costs increase linearly in quantities at a rate determined by $\tau \geq 0$.

As in our baseline model, correlation between price and the demand-side structural error term can arise both due to markup adjustments and the effect of demand on marginal costs. However, due to the presence of $g_{jrt}(\cdot)$ in the cost function, the latter channel exists even under the covariance restriction $Cov(\Delta\xi_{rt}, \overline{\Delta\eta}_{rt}) = 0$, where $\overline{\Delta\eta}_{rt} = \frac{1}{J}\Delta\eta_{jrt}$. If $g_{jrt}(\cdot)$ is known or can identified with additional moments, then the covariance restriction is sufficient to resolve price endogeneity, as the model informs the markup adjustments. In estimation, we maintain the covariance restriction at the market level.

Our demand and supply framework of equations (1) and (5) readily admits Cournot competition. As only market-level price and costs measures are observed, one must use the mean firm-level quantity $\overline{q}_{rt} = \frac{1}{J}Q_{rt}$ to obtain an expression for mean market-level markups and λ. In particular, when firms compete in quantities, we obtain $\lambda_{rt} = \frac{1}{J}\frac{dh}{dq}Q_{rt}$.

Section 4.2 establishes the necessary results to incorporate increasing marginal costs in our framework. In our implementation, we assume that $\psi = 800$, such that our $g_{jrt}(\cdot)$ function is close to what is used in Fowlie et al. (2016). We solve the quadratic equation expressed in Proposition 1 to recover $\hat{\alpha}$, under the assumption that residual demand and cost shocks are uncorrelated. Here, whether the covariance restriction is reasonable depends primarily on the relationship between construction activity (unobserved demand) and the prices of coal and electricity (unobserved supply costs). In this context, there is a theoretical basis for orthogonality: for example, if coal suppliers have limited market power and roughly constant marginal costs, then coal prices should not respond much to demand for cement. Indeed, this is the

[24]See FRR for details on the data, which are available for downloaded as part of the replication package.

[25]We use logit demand rather than the constant elasticity demand of FRR to allow for adjustable markups. The 2SLS results are unaffected by the choice. In our implementation, we assume $M_r = 2 \times \max_t\{Q_{rt}\}$.

identification argument of FRR, as coal and electricity prices are included in the set of excluded instruments. Consistent with this, the time-series of coal prices over 1980-2010 is not obviously correlated with macroeconomic conditions (e.g., Miller et al., 2017).

We find that the covariance restrictions approach yields a demand elasticity of -1.15, with a standard error of 0.18.[26] This is nearly identical to the 2SLS estimate of -1.07 (standard error 0.19) that we obtain using the FRR instruments: coal prices, natural gas prices, electricity prices, and wage rates. This similarity reflects, we believe, that the identifying assumptions are actually quite similar, with the main difference being whether the cost shifters are treated as observed (2SLS) or unobserved (covariance restrictions). By contrast, we obtain a demand elasticity of -0.47 (standard error of 0.15) using OLS. And, if we use the covariance restriction without accounting for the presence of $g_{jrt}(\cdot)$, we obtain a demand elasticity of -0.90 (standard error of 0.13), which is in between the OLS and 2SLS estimates.

5.3 The Airline Industry

In our third empirical application, we examine demand for airline travel using the setting and data of Aguirregabiria and Ho (2012) ["AH"]. The economics of the industry suggest that the covariance restriction $Cov(\Delta\xi, \Delta\eta) = 0$ would not be credible. The reason is that airlines bear an opportunity cost when they sell a seat because it can no longer be sold at a higher price to another passenger (Williams, 2022). Thus, all else equal, greater demand generates higher marginal costs, inclusive of the opportunity cost. Absent a model of these opportunity costs, would be difficult to achieve point identification using the covariance restriction. Instead, we illustrate the bounds approach to identification, using multiple sets of bounds.

The data feature 2,950 city-pairs in the United States observed over the four quarters 0f 2004. A market is a city-pair-quarter combination. Products are classified into the following groups: nonstop flights, one-stop flights, and the outside good. On average, there are 7.98 products per market (not including the outside good) including 2.04 nonstop products.[27]

The nested logit demand system can be expressed as

$$\ln s_{jmt} - \ln s_{0mt} - \sigma \ln \bar{s}_{jmt|g} = \alpha p_{jmt} + \boldsymbol{x}'_{jmt}\boldsymbol{\beta} + \xi_{a(j)} + \xi_{mt} + \Delta\xi_{jmt} \tag{37}$$

where s_{jmt} is the market share of product j in market m in period t. The conditional market share, $\bar{s}_{j|g} = s_j / \sum_{k \in g} s_k$, is the the choice probability of product j given that its "group" of products, g, is selected. The outside good is indexed as $j = 0$. Consumer preferences vary by airline ($\xi_{a(j)}$) and by route-quarter (ξ_{mt}). Higher values of σ increase within-group consumer

[26] We obtain bootstrapped standard errors based on 200 random samples constructed by drawing from the data with replacement.

[27] We thank Victor Aguirregabiria for providing the data, which derive from the DB1B database maintained by the Department of Transportation. Replication is not exact because the sample differs somewhat from what is used in the AH publication and because we employ a different set of fixed effects in estimation.

substitution relative to across-group substitution.[28]

We impose three sets of bounds. First, we assume that product-level demand and cost shocks are weakly positive, i.e., $Cov(\Delta\xi_{jmt}, \Delta\eta_{jmt}) \geq 0$, based on the role of opportunity costs in the industry. Second, if the correlation in product-level shocks is weakly positive, it is reasonable to also assume that the correlation in group-level shocks is also weakly positive. That is, overall demand for non-stop flights in a market may drive up the opportunity costs for non-stop flights. Thus, building on Section 4.1, we apply the group-level inequality

$$E_{gmt}[\overline{\Delta\xi}_{gmt}\overline{\Delta\eta}_{gmt}] \geq 0, \tag{38}$$

where $\overline{\Delta\xi}_{gmt} = \frac{1}{|g|}\sum_{j\in g}\Delta\xi_{jmt}$ and $\overline{\Delta\eta}_{gmt} = \frac{1}{|g|}\sum_{j\in g}\Delta\eta_{jmt}$ are the mean demand and cost shocks within a group-market-period. Finally, we combine these bounds with the model-based bounds developed in Section 4.3. We then construct an identified set by rejecting values of the parameters (α, σ) that fail to generate the data or that deliver negative correlations between costs and demand.

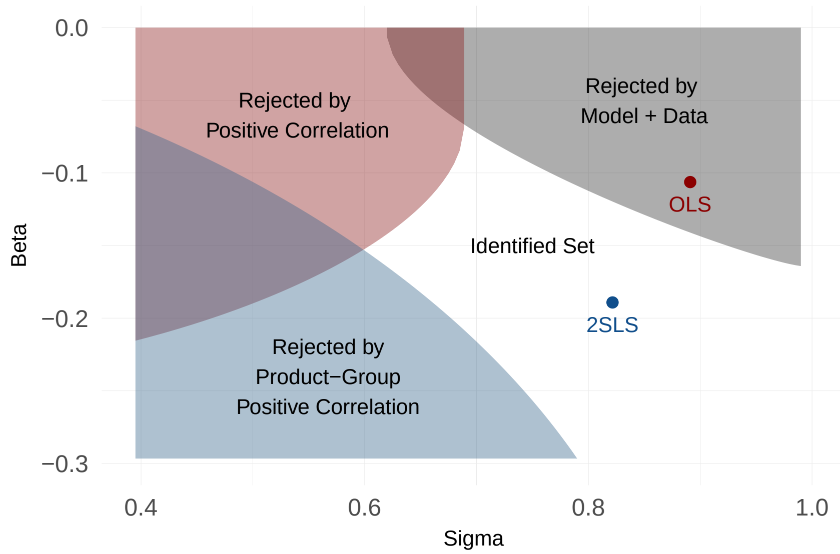
Figure 2 displays the rejected regions based on the model and our assumptions on unobserved shocks. The gray region corresponds to the parameter values rejected by the model-based bounds; the model itself rejects some values of α if $\sigma \geq 0.62$. As σ becomes larger, a more negative α is required to rationalize the data. The dark red region corresponds to parameter values that generate negative correlation between demand and supply shocks. The region is rejected under the prior that $Cov(\Delta\xi_{jmt}, \Delta\eta_{jmt}) \geq 0$. The dark blue region provides the corresponding set for the prior $Cov(\overline{\Delta\xi}_{gmt}, \overline{\Delta\eta}_{gmt}) \geq 0$ and is similarly rejected.

The three regions overlap, but no region is a subset of another. The non-rejected values provide the identified set. We rule out values of σ less than 0.599 for any value of α, as these lower values cannot generate positive correlation in both product-level and product-group-level shocks. Thus, the bounds serve to reject the logit model ($\sigma = 0$) in favor of nested logit, even with relatively limited information about the covariance of shocks.

Similarly, we obtain an upper bound on α of -0.067 across all values of σ. Combined, these bounds indicate that the mean own-price elasticity is less than -0.537. For context, we plot the OLS and the 2SLS estimates in Figure 2. The OLS estimate falls in a rejected region and can be ruled out by the model alone. The 2SLS estimate falls within the identified set. This result is not mechanical, as these point estimates are generated with non-nested assumptions.

[28] The covariates include an indicator for nonstop itineraries, the distance between the origin and destination cities, and a measure of the airline's "hub sizes" at the origin and destination cities. In estimation, we include airline fixed effects and route$\times$quarter fixed effects. Market size, which determines the market share of the outside good, is equal to the total population in the origin and destination cities.

Figure 2: Analysis of Bounds in the Airlines Industry



Notes: This figure displays candidate parameter values for (σ, α). The gray region indicates the set of parameters that cannot generate the observed data from the assumptions of the model. The red region indicates the set of parameters that generate $Cov(\Delta\xi, \Delta\eta) < 0$, and the blue region indicates parameters that generate $Cov(\overline{\Delta\xi}, \overline{\Delta\eta}) < 0$. The identified set is obtained by rejecting values in the above regions under the assumption of (weakly) positive correlation. For context, the OLS and the 2SLS estimates are plotted. The parameter σ can only take values on $[0, 1)$.

6 Conclusion

We have shown that covariance restrictions between unobserved demand and cost shocks can resolve price endogeneity and allow for consistent estimation in models of imperfect competition. The covariance restrictions approach is notable in part because, unlike approaches with instrumental variables, an exogenous portion of price is not isolated and then used in estimation; instead the endogenous variation in quantity and price is interpreted through the lens of the model to recover the structural parameters. As this is somewhat novel, we provide three empirical applications to demonstrate how covariance restrictions can be applied and evaluated.

More broadly, our analysis shows how imposing a supply-side model provides feasible paths to identification. In addition to covariance restrictions, our formal results illustrate how demand-side instruments can be sufficient to resolve price endogeneity. We also establish model-free bounds, in which the model and the data jointly can reject certain values of the price parameter, without the need for additional identifying assumptions. Conditional on meeting these bounds, there is typically a unique mapping between the price coefficient and the covariance of demand and costs shocks. In such settings, the covariance term can act as a free parameter to rationalize different values of the price coefficient.

The appeal of the covariance restrictions approach relative to alternatives will depend on data availability and the institutional details of the industry under study. In cases where cost-shifters are observed, instrumental variables has the advantage that only an informal understanding of supply is required to estimate demand. By contrast, the covariance restrictions approach leverages all of the observed variation in prices and quantities, but it requires a formal supply-side model. Our results provide paths to identification that may facilitate research in areas for which strong supply-side instruments are unavailable. As is true with most empirical work, the reliability of research that employs a particular approach depends on the appropriateness of the identifying assumptions.

References

AGUIRREGABIRIA, V. AND C.-Y. HO (2012): "A Dynamic Oligopoly Game of the US Airline Industry: Estimation and Policy Experiments," *Journal of Econometrics*, 168, 156–173.

BACKUS, M., C. CONLON, AND M. SINKINSON (2021): "Common Ownership and Competition in the Ready-to-Eat Cereal Industry," Working paper.

BERRY, S., J. LEVINSOHN, AND A. PAKES (1995): "Automobile Prices in Market Equilibrium," *Econometrica*, 63, 841.

BERRY, S. T. (1994): "Estimating Discrete-Choice Models of Product Differentiation," *RAND Journal of Economics*, 25, 242–262.

BERRY, S. T. AND P. A. HAILE (2020): "Nonparametric Identification of Differentiated Products Demand Using Micro Data," Working paper.

BOUND, J., D. A. JAEGER, AND R. A. BAKER (1995): "Problems with Instrumental Variables Estimation When the Correlation Between the Instruments and the Endogenous Explanatory Variable Is Weak," *Journal of the American Statistical Association*.

BRESNAHAN, T. F. (1982): "The Oligopoly Solution Concept is Identified," *Economics Letters*, 10, 87–92.

BRODA, C. AND D. E. WEINSTEIN (2006): "Globalization and the Gains from Variety," *The Quarterly journal of economics*, 121, 541–585.

——— (2010): "Product Creation and Destruction: Evidence and Price Implications," *American Economic Review*, 100, 691–723.

CARDELL, S. N. (1997): "Variance Components Structures for the Extreme-Value and Logistic Distributions with Applications to Models of Heterogeneity," *Econometric Theory*, 13, 185–213.

CHEVALIER, J. A., A. K. KASHYAP, AND P. E. ROSSI (2003): "Why Don't Prices Rise during Periods of Peak Demand? Evidence from Scanner Data," *American Economic Review*, 93, 15–37.

CHO, S., G. LEE, J. RUST, AND M. YU (2018): "Optimal Dynamic Hotel Pricing," Working paper.

CILIBERTO, F., C. MURRY, AND E. TAMER (2021): "Market Structure and Competition in Airline Markets," *Journal of Political Economy*, 129, 2995–3038.

CONLON, C. AND J. GORTMAKER (2020): "Best Practices for Differentiated Products Demand Estimation with PyBLP," *RAND Journal of Economics*, 51, 1108–1161.

De Loecker, J. (2011): "Product Differentiation, Multiproduct Firms, and Estimating the Impact of Trade Liberalization on Productivity," *Econometrica*, 79, 1407–1451.

Deaton, A. and J. Muellbauer (1980): "An Almost Ideal Demand System," *The American Economic Review*, 70, pp. 312–326.

Dixit, A. K. and J. E. Stiglitz (1977): "Monopolistic Competition and Optimum Product Diversity," *American Economic Review*, 67, 297–308.

Döpper, H., A. MacKay, N. Miller, and J. Stiebale (2022): "Rising Markups and Consumer Preferences," Working paper.

Doraszelski, U. and J. Jaumandreeu (2013): "R&D and Productivity: Estimating Endogenous Productivity," *Review of Economic Studies*, 80, 1338–1383.

Dubé, J.-P., , J. T. Fox, and C.-L. Su (2012): "Improving the Numerical Performance of Static and Dynamic Aggregate Discrete Choice Random Coefficients Demand Estimation," *Econometrica*, 80, 2231–2267.

Feenstra, R. C. (1994): "New Product Varieties and the Measurement of International Prices," *The American Economic Review*, 157–177.

Fisher, F. (1963): "Uncorrelated Disturbances and Identifiability Criteria," *International Economic Review*, 4, 134–152.

——— (1965): "Near-Identifiability and the Variances of the Disturbance Terms," *Econometrica*, 33, 409–419.

Fowlie, M., M. Reguant, and S. P. Ryan (2016): "Market-Based Emissions Regulation and Industry Dynamics," *Journal of Political Economy*, 124, 249–302.

Frisch, R. (1933): *Pitfalls in the Statistical Construction of Demand and Supply Curves.* Leipzig: Hans Buske, Verlag.

Gandhi, A. and J.-F. Houde (2020): "Measuring Substitution Pattern in Differentiated Products Industries," Working paper.

Gandhi, A. and A. Nevo (2021): "Empirical Models of Demand and Supply in Differentiated Products Industries," In Ho, K., Hortacsu, A., and Lizzeri, A. (Eds.), *Handbook of Industrial Organization, 4th Edition*. Elsevier.

Grieco, P. L. E., C. Murry, and A. Yurukoglu (2021): "The Evolution of Market Power in the US Automobile Industry," Working Paper.

HANSEN, L. P. (1982): "Large Sample Properties of Generalized Method of Moments Estimators," *Econometrica*, 50, 1029–1054.

HAUSMAN, J. (1996): "Valuation of New Goods Under Perfect and Imperfect Competition," In *The Economics of New Goods, Studies in Income and Wealth* Vol 58, ed. by T. Bresnahan and R. Gordon. Chicago: National Bureau of Economic Research.

HAUSMAN, J. A. (1978): "Specification Tests in Econometrics," *Econometrica*, 46, 1251–1271.

HAUSMAN, J. A., W. K. NEWEY, AND W. E. TAYLOR (1987): "Efficient Estimation and Identification of Simultaneous Equation Models with Covariance Restrictions," *Econometrica*, 55, 1377–1398.

HAUSMAN, J. A. AND W. E. TAYLOR (1983): "Identification in Linear Simultaneous Equations Models with Covariance Restrictions: An Instrumental Variables Interpretation," *Econometrica*, 51, 1527–1549.

HENDEL, I. AND A. NEVO (2013): "Intertemporal Price Discrimination in Storable Goods Markets," *American Economic Review*, 103, 2722–2751.

KEANE, M. AND T. NEAL (2022): "A Practical Guide to Weak Instruments," Working Paper.

KNITTEL, C. R. AND K. METAXOGLOU (2014): "Estimation of Random-Coefficients Demand Models: Two Empiricists' Perspective," *The Review of Economics and Statistics*, 96, 34–59.

KOOPMANS, T. C. (1949): "Identification Problems in Economic Model Construction," *Econometrica*, 17, 125–144.

KOOPMANS, T. C., H. RUBIN, AND R. B. LEIPNIK (1950): *Statistical Inference in Dynamic Economic Models (Cowles Commission Monograph 10)*, New York: John Wiley & Sons, chap. Measuring the Equation Systems of Dynamic Economics.

LEAMER, E. E. (1981): "Is it a Demand Curve, Or is it a Supply Curve? Partial Identification through Inequality Constraints," *The Review of Economics and Statistics*, 63, 319–327.

LEONTIEF, W. (1929): "Ein Versuch Zur Statistichen Analyse von Angebot und Nachfrage," *Weltvirtschaftliches Archiv*, 30, 1–53.

LI, S., J. MAZUR, Y. PARK, J. ROBERTS, A. SWEETING, AND J. ZHANG (2022): "Repositioning and Market Power after Airline Mergers," *RAND Journal of Economics*, 53, 166–199.

MATZKIN, R. L. (2016): "On Independence Conditions in Nonseparable Models: Observable and Unobservable Instruments," *Journal of Econometrics*, 191, 302–311.

MILLER, N., M. OSBORNE, AND G. SHEU (2017): “Pass-Through in a Concentrated Industry: Empirical Evidence and Regulatory Implications,” *RAND Journal of Economics*, 48, 69–93.

MILLER, N. H. AND M. WEINBERG (2017): “Understanding the Prices Effects of the MillerCoors Joint Venture,” *Econometrica*, 85, 1763–1791.

NEVO, A. (1998): “Identification of the Oligopoly Solution Concept in a Differentiated-Products Industry,” *Economics Letters*, 59, 391–395.

——— (2000): “A Practitioner’s Guide to Estimation of Random-Coefficients Logit Models of Demand,” *Journal of Economics & Management Strategy*, 9, 513–548.

——— (2001): “Measuring Market Power in the Ready-to-Eat Cereal Industry,” *Econometrica*, 69, 307–342.

NEVO, A. AND A. ROSEN (2012): “Identification with Imperfect Instruments,” *Review of Economics and Statistics*, 94, 659–671.

PETTERSON, M. S., D. G. SEIM, AND J. M. SHAPIRO (2022): “Bounds on a Slope from Size Restrictions on Economic Shocks,” *American Economic Journal: Microeconomics*, forthcoming.

ROTHENBERG, T. J. (1971): “Identification in Parametric Models,” *Econometrica*, 39, 577–592.

SCHULTZ, H. (1928): *Statistical Laws of Demand and Supply*, The University of Chicago Press.

SODERBERY, A. (2015): “Estimating Import Supply and Demand Elasticities: Analysis and Implications,” *Journal of International Economics*, 96, 1–17.

STAIGER, D. AND J. H. STOCK (1997): “Instrumental Variables with Weak Instruments,” *Econometrica*, 65, 557–586.

STOCK, J. H. AND M. YOGO (2005): “Testing for Weak Instruments in Linear IV Regression,” *Identification and Inference for Econometric Models: Essays in Honor of Thomas Rothenberg*.

THOMADSEN, R. (2005): “The Effect of Ownership Structure on Prices in Geographically Differentiated Industries,” *RAND Journal of Economics*, 36, 908–929.

VILLAS-BOAS, S. B. (2007): “Vertical Relationships Between Manufacturers and Retailers: Inference with Limited Data,” *Review of Economic Studies*, 74, 625–652.

WEGGE, L. (1965): “Identifiability Criteria for a System of Equations as a Whole,” *The Australian Journal of Statistics*, 7, 67–77.

WILLIAMS, K. R. (2022): “The Welfare Effects of Dynamic Pricing: Evidence from Airline Markets,” *Econometrica*, 92, 831–858, forthcoming.

WOOLDRIDGE, J. M. (2010): "Econometric Analsysis of Cross Section and Panel Data," .

ZOUTMAN, F. T., E. GAVRILOVA, AND A. O. HOPLAND (2018): "Estimating Both Suppy and Demand Elasticities Using Variation in a Single Tax Rate," *Econometrica*, 86, 763–771.

Appendix

For Online Publication

A Demand System Applications

The demand system of equation (1) is sufficiently flexible to nest monopolistic competition with linear demands (e.g., as in the motivating example) and the discrete choice demand models that support much of the empirical research in industrial organization. The demand assumption can also be modified to allow for semi-linearity in a transformation of prices, $f(p_{jt})$:

$$h_{jt}(\boldsymbol{q}_t, \boldsymbol{p}_t, \boldsymbol{x}_t, \boldsymbol{\xi}_t; \boldsymbol{\theta}) = \alpha f(p_{jt}) + \boldsymbol{x}'_{jt}\boldsymbol{\beta} + \xi_j + \xi_t + \Delta\xi_{jt} \tag{A.1}$$

Under this modified assumption, it is possible to employ a method-of-moments approach to estimate the structural parameters. When $f(p_{jt}) = \ln p_{jt}$, it is straightforward to extend our analytical identification results, under the modified assumptions that $\Delta\xi$ is orthogonal to $\ln X$ and that $\ln \eta$ and $\Delta\xi$ are uncorrelated. To see this, note that the optimal price for these demand systems takes the form $p_{jt} = \mu_{jt} c_{jt}$, where μ_{jt} is a markup that reflects demand parameters and (in general) demand shocks. It follows that the probability limit of an OLS regression of h^* on $\ln p$ is given by:

$$\alpha^{OLS} = \alpha - \frac{1}{\alpha}\frac{Cov(\ln\mu, \Delta\xi)}{Var(\ln p^*)} + \frac{Cov(\ln\Delta\eta, \Delta\xi)}{Var(\ln p^*)}. \tag{A.2}$$

Therefore, the results developed in this paper extend in a straightforward manner.

We provide some typical examples below for single-product firms with Bertrand competition. We then show how multi-product firms and other models of competition fit within the framework of Section 2.

A.1 Nested Logit Demand

Following the exposition of Cardell (1997), let the firms be grouped into $g = 0, 1, \ldots, G$ mutually exclusive and exhaustive sets, and denote the set of firms in group g as $\mathscr{J}_g$. An outside good, indexed by $j = 0$, is the only member of group 0. Then the left-hand-side of equation (1) takes the form

$$h_{jt}(\boldsymbol{q}_t, \boldsymbol{p}_t, \boldsymbol{x}_t, \boldsymbol{\xi}_t; \boldsymbol{\theta}) \equiv \ln(s_j) - \ln(s_0) - \sigma \ln(\bar{s}_{j|g})$$

where $\bar{s}_{j|g,t} = \sum_{j\in\mathscr{J}_g} \frac{s_{jt}}{\sum_{j\in\mathscr{J}_g} s_{jt}}$ is the market share of firm j within its group. The parameter $\sigma \in [0, 1)$ determines the extent to which consumers substitute disproportionately among firms within the same group. If $\sigma = 0$ then the logit model obtains. We can construct the markup by calculating the total derivative of h with respect to s. For single-product firms at the Bertrand-Nash equilibrium,

$$\lambda_{jt} = \frac{dh_{jt}}{ds_{jt}} s_{jt} = \frac{1}{\frac{1}{1-\sigma} - s_{jt} - \frac{\sigma}{1-\sigma}\bar{s}_{j|g,t}}.$$

In our third application, we use the nested logit model to estimate bounds on the structural parameters (Section 5.3).

A.2 Random Coefficients Logit Demand

Building on the notation of Berry (1994) and Nevo (2000), let the indirect utility that consumer $i = 1, \ldots, I$ receives from product j be

$$\begin{aligned} u_{ij} =& \alpha p_j + \boldsymbol{x}_j' \boldsymbol{\beta} + \Delta\xi_j + p_j(\sigma_0 \zeta_i^0 + \pi_{01} D_{i1} + \ldots + \pi_{0d} D_{id}) \\ &+ \sum_k x_j^k (\sigma_k \zeta_i^k + \pi_{k1} D_{i1} + \ldots + \pi_{kd} D_{id}) + \epsilon_{ij} \end{aligned}$$

where x_k^k is the kth element of x_j, ζ_i^k captures mean-zero consumer-specific tastes for characteristic k (including price), D_i captures consumer-specific mean-zero demographic characteristics, and ϵ_{ij} is a logit error. We have suppressed market subscripts and fixed effect terms for notational simplicity. Decomposing the right-hand side of the indirect utility equation into $\delta_j = \alpha p_j + \boldsymbol{x}_j' \boldsymbol{\beta} + \Delta\xi_j$ and $\phi_{ij} = p_j(\sigma_0 \zeta_i^0 + \pi_{01} D_{i1} + \ldots + \pi_{0d} D_{id}) + \sum_k x_j^k (\sigma_k \zeta_i^k + \pi_{k1} D_{i1} + \ldots + \pi_{kd} D_{id})$, the probability that consumer i selects product j is given by the standard logit formula

$$s_{ij} = \frac{\exp(\delta_j + \phi_{ij})}{\sum_k \exp(\delta_k + \phi_{ik})}.$$

Integrating yields the market shares: $s_j = \frac{1}{I} \sum_i s_{ij}$. Berry et al. (1995) prove that a contraction mapping recovers, for any candidate parameter vector $\tilde{\sigma}$, the vector $\delta(s, \tilde{\sigma})$ that equates these market shares to those observed in the data. This "mean valuation" is $h(\cdot)$ in our notation. An estimator can be applied to recover the price coefficient, again taking some $\tilde{\sigma}$ as given. For single-product firms at the Bertrand-Nash equilibrium, λ_j takes the form

$$\lambda_j = \frac{dh_j}{ds_j} s_j = \frac{s_j}{\frac{1}{I} \sum_i s_{ij}(1 - s_{ij})}.$$

Thus, with the uncorrelatedness assumption the linear parameters can be recovered given the candidate parameter vector $\tilde{\sigma}$. We demonstrate how to estimate these parameters using additional covariance restrictions in our first application (Section 5.1), which also incorporates multi-product firms.

A.3 Constant Elasticity Demand

A special case that is often estimated in empirical work is when h and $f(p)$ are logarithms. With the modified demand assumption of equation (A.1), the constant elasticity of substitution (CES) demand model of Dixit and Stiglitz (1977) can be incorporated:

$$\ln(q_{jt}/q_t) = \beta + \alpha \ln\left(\frac{p_{jt}}{\Pi_t}\right) + \xi_{jt}$$

where q_t is an observed demand shifter, Π_t is a price index, and α provides the constant elasticity of demand. This model is often used in empirical research on international trade and firm productivity (e.g., De Loecker, 2011; Doraszelski and Jaumandreeu, 2013). Due to the constant elasticity, profit-maximization and uncorrelatedness imply $Cov(p, \xi) = 0$, and OLS produces unbiased estimates of the demand parameters.[29] Indeed, this is an excellent illustration of our basic argument: so long as the data-generating process is sufficiently well understood, it is possible to characterize the bias of OLS estimates.

A.4 Other Demand Systems

The demand assumption in equation (1) accommodates many rich demand systems. Consider the linear demand system, $q_{jt} = \beta_j + \sum_k \alpha_{jk} p_k + \xi_{jt}$, which sometimes appears in identification proofs (e.g., Nevo, 1998) but is seldom applied empirically due to the large number of price coefficients. In principle, the system could be formulated such that $h(q_{jt}, w_{jt}; \sigma) \equiv q_{jt} - \sum_{k \neq j} \alpha_{jk} p_k$. In addition to the own-product uncorrelatedness restrictions that could identify α_{jj}, one could impose cross-product covariance restrictions to identify α_{jk} $(j \neq k)$. We discuss these cross-product covariance restrictions in our first application (Section 5.1). A similar approach could be used with the almost ideal demand system of Deaton and Muellbauer (1980).

A.5 Multi-Product Firms

We illustrate how our framework incorporates multi-product firms with the case of Bertrand pricing. Let K^m denote the set of products owned by multi-product firm m. When the firm sets prices on each of its products to maximize joint profits, there are $|K^m|$ first-order conditions, which can be expressed as

$$\sum_{k \in K^m} (p_k - mc_k) \frac{\partial q_k}{\partial p_j} = -q_j \;\; \forall j \in K^m.$$

The market subscript, t, is omitted to simplify notation. For demand systems satisfying equation (1), $\frac{\partial q_k}{\partial p_j} = \alpha \frac{1}{\frac{dh_j}{dq_k}}$, where the derivative $\frac{dh_j}{dq_k}$ is calculated holding the prices of other products fixed. Therefore, the set of first-order conditions can be written as

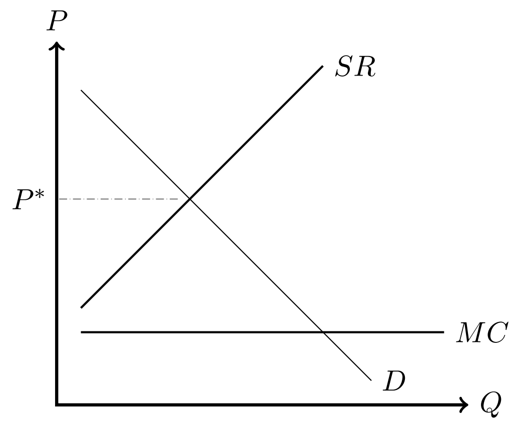
$$\sum_{k \in K^m} (p_k - mc_k) \frac{1}{dh_j / dq_k} = -\frac{1}{\alpha} q_j \;\; \forall j \in K^m.$$

For each firm, stack the first-order conditions, writing the left-hand side as the product of a matrix A^m of loading components and a vector of markups, $(p_j - mc_j)$, for products owned by the firm. The loading components are given by $A^m_{i(j),i(k)} = \frac{1}{dh_j / dq_k}$, where $i(\cdot)$ indexes products within a firm. Next, invert the loading matrix to solve for markups as function of the loading components and $-\frac{1}{\alpha}\boldsymbol{q}^m$, where $\boldsymbol{q}^m$ is a vector of the multi-product firm's quantities. Equilibrium prices equal marginal costs plus a markup, where the markup is determined by the inverse of A^m $((A^m)^{-1} \equiv \Lambda^m)$, quantities, and the price parameter:

$$p_j = mc_j - \frac{1}{\alpha} \left(\Lambda^m \boldsymbol{q}^m\right)_{i(j)}. \tag{A.3}$$

[29] The international trade literature following Feenstra (1994) consider non-constant marginal costs, which requires an additional restriction. See section 5.2 for an extension of our methodology to non-constant marginal costs.

Figure A.1: Supply Relationship



Notes: Figure plots an illustrative example of demand (D), marginal costs (MC), and the supply relationship described in the paper (SR). The supply relationship can be interpreted as the opportunity cost to the firm of selling an additional unit. The opportunity cost is the sum of the marginal cost and the inframarginal losses of lowering price. The equilibrium price (P^*) is determined by the intersection of D and SR.

Here, $(\Lambda^m \boldsymbol{q}^m)_{i(j)}$ provides the entry corresponding to product j in the vector $\Lambda^m \boldsymbol{q}^m$. As the matrix Λ^m is not a function of the price parameter after conditioning on observables, this form of the first-order condition allows us to solve for α. Letting $\lambda \equiv (\Lambda^m \boldsymbol{q}^m)_{i(j)}$, we see that multi-product Bertrand fits in the class of models specified by equation (4).

A.6 Alternative Models of Competition

Our restriction on additive markups from equation (4) applies to a broad set of competitive assumptions. Consider, for example, Nash competition among profit-maximizing firms that have a single choice variable, a, and constant marginal costs. The individual firm's objective function is:

$$\max_{a_j | a_i, i \neq j} (p_j(a) - c_j) q_j(a).$$

This generalized model of Nash competition nests Bertrand ($a = p$) and Cournot ($a = q$). The first-order condition, holding fixed the actions of the other firms, is given by:

$$p_j(a) = c_j - \frac{{p_j}'(a)}{{q_j}'(a)} q_j(a).$$

In equilibrium, we obtain the structural decomposition $p = c + \mu$, where μ incorporates the structure of demand and its parameters. This decomposition provides a restriction on how prices move with demand shocks, aiding identification. Using restrictions about demand, such as those imposed by equation (1), one can construct the appropriate values of λ and solve for the price coefficient. Related first-order conditions can be obtained in other contexts, such as consistent conjectures.

Bresnahan (1982) refers to the above equation as the "supply relation," and notes that it generalizes to many different forms of conduct. Figure A.1 plots the supply relationship along with the demand curve for an illustrative setting. The supply relationship lies above the marginal cost curve, and the difference is given by the inframarginal loss in revenue for selling an additional unit (i.e., the gap between price and marginal revenue). As the inframarginal loss has an opportunity cost interpretation, the supply relation can be conceptualized as the sum of the marginal cost curve and the firm's opportunity cost curve, with the latter incorporating any market power that the firm has. Equilibrium price is determined by the intersection of demand and the supply relationship. This is equivalent to the equilibrium price that obtains if the firm sets price to equate marginal revenue and marginal cost.

B Supply-Side Misspecification

To illustrate how supply-side misspecification can affect the performance of the estimators, we simulate duopoly markets in which the standard assumption of Bertrand price competition may not match the data-generating process.[30] We assume the demand system is logit, providing consumers with a differentiated discrete choice, and we allow them to select an outside option in addition to a product from each firm. The quantity demanded of firm j in market t is

$$q_{jt} = \frac{\exp(2 - p_{jt} + \Delta\xi_{jt})}{1 + \sum_{k=j,i} \exp(2 - p_{kt} + \Delta\xi_{kt})}$$

On the supply side, marginal costs are $c_{kt} = \Delta\eta_{kt}$ $(k = j, i)$. Firm j sets price to maximize $\pi_j + \kappa\pi_i$, and likewise for firm i, where $\kappa \in [0, 1]$ is a conduct parameter (e.g., Miller and Weinberg, 2017). The first-order conditions take the form

$$\begin{bmatrix} p_j \\ p_i \end{bmatrix} = \begin{bmatrix} c_j \\ c_i \end{bmatrix} - \left[\begin{pmatrix} 1 & \kappa \\ \kappa & 1 \end{pmatrix} \circ \left(\frac{\partial q}{\partial p} \right)^T \right]^{-1} \begin{bmatrix} q_j \\ q_i \end{bmatrix}$$

where $\frac{\partial q}{\partial p}$ is a matrix of demand derivatives and $\circ$ denotes element-by-element multiplication. The model nests Bertrand competition ($\kappa = 0$) and joint price-setting behavior ($\kappa = 1$), as well as capturing (non-micro-founded) intermediate cases.

We generate data with different conduct parameters: $\kappa \in \{0, 0.2, 0.4, 0.6, 0.8, 1.0\}$. For each specification, we simulate datasets with 400 observations (200 markets $\times$ two firms), and estimate the model under the (erroneous) assumption of Bertrand price competition ($\kappa = 0$), thus generating supply-side misspecification. We then estimate the model using the covariance restrictions approach assuming $Cov(\Delta\xi, \Delta\eta) = 0$, using $\Delta\eta$ as an (observed) excluded instrument for demand, and using $\Delta\xi$ as an (observed) excluded instrument for supply. Across all specifications, $\Delta\xi \sim U(0, 0.5)$ and $\Delta\eta \sim U(0, 0.5)$.[31]

Table B.1 displays the results. As expected, supply-side misspecification can introduce bias into the covariance restrictions approach. The bias does not appear to be meaningful for modest values of κ (i.e., 0.6 or less). When the true nature of conduct is $\kappa = 1$ (joint price setting), but we assume Bertrand price competition, the bias is -3.8 percent. Likewise, the demand-side instruments (IV-2), which invoke the formal assumption about conduct in estimation, perform worse when the true κ is farther from the assumed value. The demand-side instruments perform poorly when the true conduct is $\kappa = 1$, with a mean bias of over 20 percent. By contrast, supply-

[30]Another form of misspecification could arise if prices or quantities are measured with error, in which case the demand and cost residuals might be correlated even if the underlying shocks are uncorrelated.

[31]We note that these are not mean zero, but it does not matter in this case. It is simply a normalization.

Table B.1: Small-Sample Properties: Supply-Side Misspecification

Estimation Method	(1) $\kappa = 0.0$	(2) $\kappa = 0.2$	(3) $\kappa = 0.4$	(4) $\kappa = 0.6$	(5) $\kappa = 0.8$	(6) $\kappa = 1.0$
Covariance Restrictions	-1.001	-1.002	-1.000	-1.003	-1.016	-1.038
	(0.050)	(0.052)	(0.053)	(0.054)	(0.053)	(0.051)
IV-1: Supply Shifters	-1.002	-1.000	-1.001	-1.001	-1.001	-1.002
	(0.076)	(0.077)	(0.077)	(0.076)	(0.073)	(0.071)
IV-2: Demand Shifters	-1.015	-1.017	-1.012	-1.025	-1.082	-1.220
	(0.153)	(0.155)	(0.159)	(0.178)	(0.213)	(0.298)
IV-1: First-stage *F*-statistic	1079.9	1335.3	1424.9	1277.8	1027.1	801.9
IV-2: First-stage *F*-statistic	99.0	108.4	111.4	100.8	77.8	50.8

Notes: Results are based on 10,000 simulations of 200 duopoly markets for each specification. The demand curve is $h_{jt} = 2 - p_{jt} + \Delta\xi_{jt}$, so that $\alpha = -1$, and marginal costs are $c_{jt} = \Delta\eta_{jt}$. Demand is logit: $h(q_{jt}) = \ln(q_{jt}) - \ln(q_{0t})$, where q_{0t} is consumption of the outside good. IV-1 estimates are calculated using two-stage least squares with marginal costs ($\Delta\eta$) as an instrument in the demand equation. Analogously, IV-2 estimates are calculated using two-stage least squares with demand shocks ($\Delta\xi$) as an instrument in the supply relationship. Across all specifications, $\Delta\xi \sim U(0, 0.5)$ and $\Delta\eta \sim U(0, 0.5)$. The data-generating process varies in the nature of competition across specifications, indexed by the conduct parameter κ. The coefficients are estimated under the (misspecified) assumption of Bertrand price competition ($\kappa = 0$).

side instruments do not use a formal assumption about conduct in estimation and provide consistent estimates across the specifications (IV-1). Consistent with the earlier simulations, the three-stage estimator outperforms IV-1 when conduct is correctly specified ($\kappa = 0$).

These results illustrate a key trade-off to the econometrician: if the supply-side assumptions are to be maintained, then covariance restrictions can offer better precision relative to instrument-based approaches. However, supply-side instruments are robust to misspecification of firm conduct, whereas covariance restrictions are not. We note that the covariance restriction approach, which uses both demand-side and supply-side variation, is not as susceptible to misspecification bias as demand-side instruments in our simulations. The estimator appears to place greater weight on the source of variation with more power. In specification (6), the mean coefficient of -1.038 is much closer to the supply-shifter mean of -1.002 than the demand-shifter mean of -1.220. Indeed, it is approximately equal to the IV-1 and IV-2 estimates weighted by the square root of the respective F-statistics. By placing greater weight on supply-side shocks as the demand-side instruments degrade, the covariance restriction approach receives some protection against bias from model misspecification.

C Proofs

C.1 A Consistent and Unbiased Estimator for $\Delta\xi$

Our proofs make use of the following lemma, which identifies a consistent and unbiased estimator for the unobserved term in a linear regression when one of the covariates is endogenous. Though demonstrated in the context of semi-linear demand, the proof also applies for any endogenous covariate, including when (transformed) quantity depends on a known transformation of price, as no supply-side assumptions are required. For example, we may replace p

with $\ln p$ everywhere and obtain the same results.

Lemma C.1. *A consistent and unbiased estimator of* $\Delta\xi$ *is given by* $\hat{\Delta\xi}_1 = \hat{\Delta}\xi^{OLS} + \left(\hat{\alpha}^{OLS} - \alpha\right) p^*$

For some intuition, note that we can construct both the true demand shock and the (probability limit of the) OLS residuals as:

$$\Delta\xi = h(q) - \alpha p - \boldsymbol{x}'\boldsymbol{\beta}$$
$$\Delta\xi^{OLS} = h(q) - \alpha^{OLS} p - \boldsymbol{x}'\boldsymbol{\beta}^{OLS}$$

where this holds even in small samples. Without loss of generality, we assume $E[\Delta\xi] = 0$. The true demand shock is given by $\Delta\xi_0 = \Delta\xi^{OLS} + (\alpha^{OLS} - \alpha)p + \boldsymbol{x}'(\boldsymbol{\beta}^{OLS} - \boldsymbol{\beta})$. We desire to show that this estimator of the demand shock, $\hat{\Delta\xi}_1 = \hat{\Delta}\xi^{OLS} + \left(\hat{\alpha}^{OLS} - \alpha\right) p^*$, is consistent and unbiased. (This eliminates the need to estimate the true β parameters). It suffices to show that $(\hat{\alpha}^{OLS} - \alpha)p^* = (\hat{\alpha}^{OLS} - \alpha)p + \boldsymbol{x}'(\hat{\boldsymbol{\beta}}^{OLS} - \boldsymbol{\beta}) + \Upsilon$, where Υ is such that $E[\Upsilon = 0]$ and $\Upsilon \to 0$ as T gets large. It is straightforward to show this using the projection matrices for p and $\boldsymbol{x}$.[32]

C.2 Proof of Proposition 1 (Set Identification)

From equation (9), we have $\hat{\alpha}^{OLS} \xrightarrow{p} \alpha + \frac{Cov(p^*,\Delta\xi)}{Var(p^*)}$. The general form for a firm's first-order condition is $p = mc + \mu$, where mc is the marginal cost and μ is the markup. We can write $p = p^* + \hat{p}$, where $\hat{p}$ is the projection of p onto the exogenous demand variables, X. By assumption, $c = \boldsymbol{x}'\boldsymbol{\gamma} + \eta$. If we substitute the first-order condition $p^* = \boldsymbol{x}'\boldsymbol{\gamma} + \eta + \mu - \hat{p}$ into the bias term from the OLS regression, we obtain

$$\begin{aligned}\alpha^{OLS} - \alpha = \frac{Cov(p^*, \Delta\xi)}{Var(p^*)} &= \frac{Cov(\Delta\xi, \boldsymbol{x}'\boldsymbol{\gamma} + \eta + \mu - \hat{p})}{Var(p^*)} \\ &= \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)} + \frac{Cov(\Delta\xi, \mu)}{Var(p^*)} \end{aligned} \tag{C.1}$$

where the second line follows from the exogeneity assumption ($E[\Delta\xi|X] = 0$).

From Lemma C.1, we can construct a consistent estimate of the unobserved demand shock as $\Delta\xi = \Delta\xi^{OLS} + \left(\alpha^{OLS} - \alpha\right) p^*$. We substitute this expression into $\frac{Cov(\Delta\xi,\mu)}{Var(p^*)}$, along with the above expression for $\left(\alpha^{OLS} - \alpha\right)$ to obtain

$$\begin{aligned}
\frac{Cov\left(\Delta\xi, \mu\right)}{Var(p^*)} &= \frac{Cov\left(\Delta\xi^{OLS}, \mu\right)}{Var(p^*)} + \left(\frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)} + \frac{Cov\left(\Delta\xi, \mu\right)}{Var(p^*)}\right)\frac{Cov(p^*, \mu)}{Var(p^*)} \\
\frac{Cov\left(\Delta\xi, \mu\right)}{Var(p^*)}\left(1 - \frac{Cov(p^*, \mu)}{Var(p^*)}\right) &= \frac{Cov\left(\Delta\xi^{OLS}, \mu\right)}{Var(p^*)} + \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\frac{Cov(p^*, \mu)}{Var(p^*)} \\
\frac{Cov\left(\Delta\xi, \mu\right)}{Var(p^*)} &= \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\Delta\xi^{OLS}, \mu\right)}{Var(p^*)} + \\
&\quad \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\frac{Cov(p^*, \mu)}{Var(p^*)}
\end{aligned}$$

[32]Please contact the authors if interested in the full proof.

Plugging this into equation (C.1) yields

$$\alpha^{OLS} = \alpha + \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)} + \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}} \frac{Cov\left(\Delta\xi^{OLS}, \mu\right)}{Var(p^*)} + \frac{\frac{Cov(p^*,\mu)}{Var(p^*)}}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}} \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}$$
$$\alpha^{OLS} = \alpha + \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}} \frac{Cov\left(\Delta\xi^{OLS}, \mu\right)}{Var(p^*)} + \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}} \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}$$

Thus, we obtain an expression for the plim of the OLS estimator in terms of the OLS residuals, the residualized prices, the markup, and the correlation between unobserved demand and cost shocks. If the markup can be parameterized in terms of observables and the correlation in unobserved shocks can be calibrated, we have a method to estimate α from the OLS regression. Under our supply and demand assumptions, $\mu = -\frac{1}{\alpha}\lambda$, and plugging in obtains the first equation of the proposition:

$$\alpha^{OLS} = \alpha - \frac{1}{\alpha + \frac{Cov(p^*,\lambda)}{Var(p^*)}} \frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)} + \alpha \frac{1}{\alpha + \frac{Cov(p^*,\lambda)}{Var(p^*)}} \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}.$$

The second equation in the proposition is obtained by rearranging terms. QED.

C.3 Proof of Proposition 2 (Point Identification)

Part (1). We first prove the sufficient condition, i.e., that under assumptions 1 and 2, α is the lower root of equation (11) if the following condition holds:

$$0 \leq \alpha^{OLS} \frac{Cov(p^*, \lambda)}{Var(p^*)} + \frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)} \tag{C.2}$$

Consider a generic quadratic, $ax^2 + bx + c$. The roots of the quadratic are $\frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$. Thus, if $4ac < 0$ and $a > 0$ then the upper root is positive and the lower root is negative. In equation (11), $a = 1$, and $4ac < 0$ if and only if equation (C.2) holds. Because the upper root is positive, $\alpha < 0$ must be the lower root, and point identification is achieved given knowledge of $Cov(\Delta\xi, \Delta\eta)$. QED.

Part (2). In order to prove the necessary and sufficient condition for point identification, we first state and prove a lemma:

Lemma C.2. *The roots of equation* (11) *are* α *and* $\frac{Cov(p^*,\Delta\xi)}{Var(p^*)} - \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}$.

Proof of Lemma C.2. We first provide equation (11) for reference:

$$\begin{aligned} 0 \quad = \quad & \alpha^2 \\ + \quad & \left(\frac{Cov(p^*, \lambda)}{Var(p^*)} + \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)} - \alpha^{OLS}\right)\alpha \\ + \quad & \left(-\alpha^{OLS}\frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)}\right) \end{aligned}$$

To find the roots, begin by applying the quadratic formula

$$\begin{aligned}
(r_1, r_2) &= \frac{1}{2}\left(-B \pm \sqrt{B^2 - 4AC}\right) \\
&= \frac{1}{2}\left(\alpha^{OLS} - \frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\right) \\
&\pm \frac{1}{2}\sqrt{\left(\alpha^{OLS} + \frac{Cov(p^*, \lambda)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)} + \left(\frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\left(\alpha^{OLS} - \frac{Cov(p^*, \lambda)}{Var(p^*)}\right)}
\end{aligned} \tag{C.3}$$

Looking inside the radical, consider the first part: $\left(\alpha^{OLS} + \frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\Delta\xi^{OLS},\lambda\right)}{Var(p^*)}$

$$\begin{aligned}
&\left(\alpha^{OLS} + \frac{Cov(p^*, \lambda)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)} \\
&= \left(\alpha^{OLS} + \frac{Cov(p^*, \lambda)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\Delta\xi - p^*(\alpha^{OLS} - \alpha), \lambda\right)}{Var(p^*)} \\
&= \left(\alpha^{OLS} + \frac{Cov(p^*, \lambda)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\Delta\xi, \lambda\right)}{Var(p^*)} - 4\frac{Cov(p^*, \Delta\xi)}{Var(p^*)}\frac{Cov\left(p^*, \lambda\right)}{Var(p^*)} \\
&= \left(\alpha^{OLS} + \frac{Cov(p^*, \lambda)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\Delta\xi, \lambda\right)}{Var(p^*)} - 4\left(\frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)} + \frac{Cov(\Delta\xi, -\frac{1}{\alpha}\lambda)}{Var(p^*)}\right)\frac{Cov\left(p^*, \lambda\right)}{Var(p^*)} \\
&= \left(\alpha^{OLS} + \frac{Cov(p^*, \lambda)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\Delta\xi, \lambda\right)}{Var(p^*)}\left(1 + \frac{1}{\alpha}\frac{Cov\left(p^*, \lambda\right)}{Var(p^*)}\right) - 4\frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\frac{Cov\left(p^*, \lambda\right)}{Var(p^*)}
\end{aligned} \tag{C.4}$$

To simplify this expression, it is helpful to use the general form for a firm's first-order condition, $p = c + \mu$, where c is the marginal cost and μ is the markup. We can write $p = p^* + \hat{p}$, where $\hat{p}$ is the projection of p onto the exogenous demand variables, X. By assumption, $c = \boldsymbol{x}'\boldsymbol{\gamma} + \eta$. It follows that

$$\begin{aligned}
p^* &= \boldsymbol{x}'\boldsymbol{\gamma} + \eta + \mu - \hat{p} \\
&= \boldsymbol{x}'\boldsymbol{\gamma} + \eta - \frac{1}{\alpha}\lambda - \hat{p}
\end{aligned}$$

Therefore

$$Cov(p^*, \Delta\xi) = Cov(\Delta\xi, \Delta\eta) - \frac{1}{\alpha}Cov(\Delta\xi, \lambda)$$

and

$$\begin{aligned}
Cov(\Delta\xi, \lambda) &= -\alpha\left(Cov(p^*, \Delta\xi) - Cov(\Delta\xi, \Delta\eta)\right) \\
\frac{Cov(\Delta\xi, \lambda)}{Var(p^*)} &= -\alpha\left(\frac{Cov(p^*, \Delta\xi)}{Var(p^*)} - \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\right)
\end{aligned} \tag{C.5}$$

Returning to equation (C.4), we can substitute using equation (C.5) and simplify:

$$\begin{aligned}
&\left(\alpha^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\,(\Delta\xi,\lambda)}{Var(p^*)}\left(1+\frac{1}{\alpha}\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\right)-4\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\\
&=\left(\alpha^{OLS}\right)^2+\left(\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\right)^2+2\alpha^{OLS}\frac{Cov\,(p^*,\lambda)}{Var(p^*)}-4\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\\
&\quad+4\frac{Cov\,(\Delta\xi,\lambda)}{Var(p^*)}+4\frac{1}{\alpha}\frac{Cov\,(\Delta\xi,\lambda)}{Var(p^*)}\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\\
&=\left(\alpha+\frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)}\right)^2+\left(\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\right)^2+2\left(\alpha+\frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)}\right)\frac{Cov\,(p^*,\lambda)}{Var(p^*)}-4\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\\
&\quad-4\alpha\left(\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}-\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)-4\left(\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}-\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\\
&=\left(\alpha+\frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)}\right)^2+\left(\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\right)^2+2\left(\alpha+\frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)}\right)\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\\
&\quad-4\alpha\left(\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}\right)-4\left(\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}\right)\frac{Cov\,(p^*,\lambda)}{Var(p^*)}+4\alpha\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\\
&=\alpha^2+\left(\frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)}\right)^2+\left(\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\right)^2+2\alpha\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\\
&\quad-2\alpha\frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)}-2\frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)}\frac{Cov\,(p^*,\lambda)}{Var(p^*)}+4\alpha\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\\
&=\left(\left(\alpha+\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\right)-\frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)}\right)^2+4\alpha\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}
\end{aligned}$$

Now, consider the second part inside of the radical in equation (C.3):

$$\begin{aligned}
&\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\left(\alpha^{OLS}-\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)\\
&=\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\left(\alpha+\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}-\frac{1}{\alpha}\frac{Cov(\Delta\xi,\lambda)}{Var(p^*)}-\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)\\
&=\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2-2\alpha\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}-2\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2+2\frac{1}{\alpha}\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(\Delta\xi,\lambda)}{Var(p^*)}+2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}\\
&=-\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2-2\alpha\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\left(\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}-\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)+2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}\\
&=\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\alpha-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}+2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}
\end{aligned}$$

Combining yields a simpler expression for the terms inside the radical of equation (C.3):

$$\begin{aligned}
&\left(\left(\alpha+\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\right)-\frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)}\right)^2+4\alpha\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\\
&+\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\alpha-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}+2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}\\
&=\left(\left(\alpha+\frac{Cov\,(p^*,\lambda)}{Var(p^*)}\right)-\frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)}\right)^2+\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2\\
&\quad+2\alpha\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}+2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}\\
&=\left(\alpha+\frac{Cov\,(p^*,\lambda)}{Var(p^*)}-\frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)}+\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2
\end{aligned}$$

Plugging this back into equation (C.3), we have:

$$\begin{aligned}(r_1, r_2) &= \frac{1}{2}\left(\alpha^{OLS} - \frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\right.\\ &\quad \left.\pm\sqrt{\left(\alpha + \frac{Cov\,(p^*, \lambda)}{Var(p^*)} - \frac{Cov\,(p^*, \Delta\xi)}{Var(p^*)} + \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\right)^2}\right)\\ &= \frac{1}{2}\left(\alpha + \frac{Cov\,(p^*, \Delta\xi)}{Var(p^*)} - \frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\right.\\ &\quad \left.\pm\sqrt{\left(\alpha + \frac{Cov\,(p^*, \lambda)}{Var(p^*)} - \frac{Cov\,(p^*, \Delta\xi)}{Var(p^*)} + \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\right)^2}\right)\end{aligned}$$

The roots are given by

$$\begin{aligned}&\frac{1}{2}\left(\alpha + \frac{Cov\,(p^*, \Delta\xi)}{Var(p^*)} - \frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\right) + \\ &\qquad \frac{1}{2}\left(\alpha + \frac{Cov\,(p^*, \lambda)}{Var(p^*)} - \frac{Cov\,(p^*, \Delta\xi)}{Var(p^*)} + \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\right) = \alpha\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\left(\alpha + \frac{Cov\,(p^*, \Delta\xi)}{Var(p^*)} - \frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\right) + \\ &\qquad \frac{1}{2}\left(-\alpha - \frac{Cov\,(p^*, \lambda)}{Var(p^*)} + \frac{Cov\,(p^*, \Delta\xi)}{Var(p^*)} - \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\right)\\ &\qquad\qquad = \frac{Cov\,(p^*, \Delta\xi)}{Var(p^*)} - \frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\end{aligned}$$

which completes the proof of the intermediate result. QED.

Part (3). Consider the roots of equation (11), α and $\frac{Cov(p^*,\Delta\xi)}{Var(p^*)} - \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}$. The price parameter α may or may not be the lower root.[33] However, α is the lower root iff

$$\begin{aligned}
\alpha &< \frac{Cov(p^*,\Delta\xi)}{Var(p^*)} - \frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} \\
\alpha &< -\alpha\frac{Cov(p^*,-\frac{1}{\alpha}\Delta\xi)}{Var(p^*)} + \alpha\frac{Cov(p^*,-\frac{1}{\alpha}\frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} \\
\alpha &< -\alpha\frac{Cov(p^*,-\frac{1}{\alpha}\Delta\xi)}{Var(p^*)} + \alpha\frac{Cov(p^*,p^*-c)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} \\
\alpha &< \alpha\frac{Var(p^*)}{Var(p^*)} - \alpha\frac{Cov(p^*,-\frac{1}{\alpha}\Delta\xi)}{Var(p^*)} - \alpha\frac{Cov(p^*,\Delta\eta)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} \\
0 &< -\alpha\frac{Cov(p^*,-\frac{1}{\alpha}\Delta\xi)}{Var(p^*)} - \alpha\frac{Cov(p^*,\Delta\eta)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} \\
0 &< \frac{Cov(p^*,-\frac{1}{\alpha}\Delta\xi)}{Var(p^*)} + \frac{Cov(p^*,\Delta\eta)}{Var(p^*)} + \frac{1}{\alpha}\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}
\end{aligned}$$

The third line relies on the expression for the markup, $p - c = -\frac{1}{\alpha}\frac{dh}{dq}q$. The final line holds because $\alpha < 0$ so $-\alpha > 0$. It follows that α is the lower root of equation (11) iff

$$-\frac{1}{\alpha}\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} \leq \frac{Cov\left(p^*,-\frac{1}{\alpha}\Delta\xi\right)}{Var(p^*)} + \frac{Cov\left(p^*,\Delta\eta\right)}{Var(p^*)}$$

in which case α is point identified given knowledge of $Cov(\Delta\xi,\Delta\eta)$. QED.

C.4 Proof of Proposition 3 (Approximation)

The demand and supply equations are given by:

$$\begin{aligned}
h =& \alpha p + \boldsymbol{x}'\boldsymbol{\beta} + \xi \\
p =& \boldsymbol{x}'\boldsymbol{\gamma} - \frac{1}{\alpha}\frac{dh}{dq}q + \eta
\end{aligned}$$

[33] Consider that the first root is the upper root if

$$\alpha + \frac{Cov\left(p^*,\lambda\right)}{Var(p^*)} - \frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} > 0$$

because, in that case,

$$\sqrt{\left(\alpha + \frac{Cov\left(p^*,\lambda\right)}{Var(p^*)} - \frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2} = \alpha + \frac{Cov\left(p^*,\lambda\right)}{Var(p^*)} - \frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}$$

When $\alpha + \frac{Cov\left(p^*,\lambda\right)}{Var(p^*)} - \frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} < 0$, then $\sqrt{\left(\alpha + \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(p^*,\Delta\xi)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2} = -\left(\alpha + \frac{Cov\left(p^*,\lambda\right)}{Var(p^*)} - \frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)$, and the first root is then the lower root (i.e., minus the negative value).

For ease of exposition, here we assume that ξ and η are exogenous (and x includes dummy variables to absorb fixed effects). Using an first-order expansion of h about q, $h \approx \overline{h} + \overline{\frac{dh}{dq}}\left(q - \overline{q}\right)$, we can solve for a reduced-form for p and h. It follows that

$$\begin{aligned}\overline{h} + \overline{\frac{dh}{dq}}\left(q - \overline{q}\right) \approx& \alpha p + \boldsymbol{x}'\boldsymbol{\beta} + \xi \\ \overline{\frac{dh}{dq}} q \approx& \alpha p + \boldsymbol{x}'\boldsymbol{\beta} + \xi - \overline{h} + \overline{\frac{dh}{dq}}\overline{q}\end{aligned}$$

Letting $\frac{dh}{dq}q = \frac{\tilde{dh}}{dq}q + \overline{\frac{dh}{dq}}q$, we have

$$\begin{aligned}p &\approx \boldsymbol{x}'\boldsymbol{\gamma} - \frac{1}{\alpha}\frac{\tilde{dh}}{dq}q - \frac{1}{\alpha}\left(\alpha p + \boldsymbol{x}'\boldsymbol{\beta} + \xi - \overline{h} + \overline{\frac{dh}{dq}}\overline{q}\right) + \eta \\ 2p &\approx \boldsymbol{x}'\boldsymbol{\gamma} + \frac{1}{\alpha}\boldsymbol{x}'\boldsymbol{\beta} - \frac{1}{\alpha}\overline{h} + \frac{1}{\alpha}\overline{\frac{dh}{dq}}\overline{q} - \frac{1}{\alpha}\frac{\tilde{dh}}{dq}q + \eta + \frac{1}{\alpha}\xi \\ p &\approx \frac{1}{2}\left(\boldsymbol{x}'\boldsymbol{\gamma} + \frac{1}{\alpha}\boldsymbol{x}'\boldsymbol{\beta} - \frac{1}{\alpha}\overline{h} + \frac{1}{\alpha}\overline{\frac{dh}{dq}}\overline{q} - \frac{1}{\alpha}\frac{\tilde{dh}}{dq}q + \eta + \frac{1}{\alpha}\xi\right).\end{aligned}$$

Let H^* denote the residual from a regression of $\frac{\tilde{dh}}{dq}q$ on $\boldsymbol{x}$. Then p^*, the residual from a regression of p on x, is

$$p^* \approx \frac{1}{2}\left(\eta + \frac{1}{\alpha}\xi - \frac{1}{\alpha}H^*\right). \tag{C.6}$$

Likewise, as $h - \overline{h} + \overline{\frac{dh}{dq}}\overline{q} \approx \overline{\frac{dh}{dq}}q$,

$$p \approx \boldsymbol{x}'\boldsymbol{\gamma} - \frac{1}{\alpha}\frac{\tilde{dh}}{dq}q - \frac{1}{\alpha}\overline{\frac{dh}{dq}}q + \eta$$

$$\begin{aligned}h \approx& \alpha\left(\boldsymbol{x}'\boldsymbol{\gamma} - \frac{1}{\alpha}\frac{\tilde{dh}}{dq}q - \frac{1}{\alpha}\overline{\frac{dh}{dq}}q + \eta\right) + \boldsymbol{x}'\boldsymbol{\beta} + \xi \\ h \approx& \alpha\boldsymbol{x}'\boldsymbol{\gamma} + \boldsymbol{x}'\boldsymbol{\beta} - \frac{\tilde{dh}}{dq}q - \left(h - \overline{h} + \overline{\frac{dh}{dq}}\overline{q}\right) + \alpha\eta + \xi \\ 2h \approx& \alpha\boldsymbol{x}'\boldsymbol{\gamma} + \boldsymbol{x}'\boldsymbol{\beta} - \frac{\tilde{dh}}{dq}q + \overline{h} - \overline{\frac{dh}{dq}}\overline{q} + \alpha\eta + \xi.\end{aligned}$$

Similarly, the residual from a regression of h on x is:

$$h^* \approx \frac{1}{2}\left(\alpha\eta + \xi - H^*\right). \tag{C.7}$$

Equations (C.6) and (C.7) provide an approximation for α.

$$-\sqrt{\frac{Var(h^*)}{Var(p^*)}} \approx -\sqrt{\frac{\frac{1}{4}Var\left(\alpha\eta+\xi-H^*\right)}{\frac{1}{4}Var\left(\eta+\frac{1}{\alpha}\xi-\frac{1}{\alpha}H^*\right)}}$$
$$\approx -\sqrt{\frac{\alpha^2 Var\left(\eta+\frac{1}{\alpha}\xi-\frac{1}{\alpha}H^*\right)}{Var\left(\eta+\frac{1}{\alpha}\xi-\frac{1}{\alpha}H^*\right)}}$$
$$\approx \alpha$$

QED.

C.5 Proof of Corollary 1 (Marginal Cost Functions)

Under the semi-linear marginal cost schedule of equation (27) and the assumption that $Cov(\Delta\xi, \Delta\eta) = 0$, the plim of the OLS estimator is equal to

$$\text{plim } \hat{\alpha}^{OLS} = \alpha + \frac{Cov(\Delta\xi, g(q))}{Var(p^*)} - \frac{1}{\alpha}\frac{Cov\left(\Delta\xi,\lambda\right)}{Var(p^*)}.$$

This is obtain directly by plugging in the first–order condition for p: $Cov(p^*, \Delta\xi) = Cov(g(q) + \eta - \frac{1}{\alpha}\lambda - \hat{p}, \Delta\xi) = Cov(\Delta\xi, g(q)) - \frac{1}{\alpha}Cov(\Delta\xi,\lambda)$ under the assumptions. Next, we re-express the terms including the unobserved demand shocks in in terms of OLS residuals. As shown by Lemma C.1, the estimated residuals are given by $\Delta\xi^{OLS} = \Delta\xi + \left(\alpha - \alpha^{OLS}\right)p^*$. As $\alpha - \alpha^{OLS} = \frac{1}{\alpha}\frac{Cov(\Delta\xi,\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,g(q))}{Var(p^*)}$, we obtain $\Delta\xi^{OLS} = \Delta\xi + \left(\frac{1}{\alpha}\frac{Cov(\Delta\xi,\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,g(q))}{Var(p^*)}\right)p^*$. This implies

$$Cov\left(\Delta\xi^{OLS},\lambda\right) = \left(1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)Cov(\Delta\xi,\lambda) - \frac{Cov(p^*,\lambda)}{Var(p^*)}Cov(\Delta\xi,g(q))$$
$$Cov\left(\Delta\xi^{OLS},g(q)\right) = \frac{1}{\alpha}\frac{Cov(p^*,g(q))}{Var(p^*)}Cov\left(\Delta\xi,\lambda\right) + \left(1-\frac{Cov(p^*,g(q))}{Var(p^*)}\right)Cov(\Delta\xi,g(q))$$

We write the system of equations in matrix form and invert to solve for the covariance terms that include the unobserved demand shock:

$$\begin{bmatrix} Cov(\Delta\xi,\lambda) \\ Cov(\Delta\xi,g(q)) \end{bmatrix} = \begin{bmatrix} 1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)} & -\frac{Cov(p^*,\lambda)}{Var(p^*)} \\ \frac{1}{\alpha}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1-\frac{Cov(p^*,g(q))}{Var(p^*)} \end{bmatrix}^{-1} \begin{bmatrix} Cov(\Delta\xi^{OLS},\lambda) \\ Cov(\Delta\xi^{OLS},g(q)) \end{bmatrix}$$

where

$$\begin{bmatrix} 1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)} & -\frac{Cov(p^*,\lambda)}{Var(p^*)} \\ \frac{1}{\alpha}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1-\frac{Cov(p^*,g(q))}{Var(p^*)} \end{bmatrix}^{-1} =$$
$$\frac{1}{1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(p^*,g(q))}{Var(p^*)}} \begin{bmatrix} 1-\frac{Cov(p^*,g(q))}{Var(p^*)} & \frac{Cov(p^*,\lambda)}{Var(p^*)} \\ -\frac{1}{\alpha}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)} \end{bmatrix}.$$

Therefore, we obtain the relations

$$Cov(\Delta\xi,\lambda)=\frac{\left(1-\frac{Cov(p^*,g(q))}{Var(p^*)}\right)Cov(\Delta\xi^{OLS},\lambda)+\frac{Cov(p^*,\lambda)}{Var(p^*)}Cov(\Delta\xi^{OLS},g(q))}{1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)}-\frac{Cov(p^*,g(q))}{Var(p^*)}}$$

$$Cov(\Delta\xi,g(q))=\frac{-\frac{1}{\alpha}\frac{Cov(p^*,g(q))}{Var(p^*)}Cov(\Delta\xi^{OLS},\lambda)+\left(1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)Cov(\Delta\xi^{OLS},g(q))}{1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)}-\frac{Cov(p^*,g(q))}{Var(p^*)}}.$$

In terms of observables, we can substitute in for $Cov(\Delta\xi,g(q))-\frac{1}{\alpha}Cov\left(\Delta\xi,\lambda\right)$ in the plim of the OLS estimator and simplify:

$$\begin{aligned}&\left(1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)}-\frac{Cov(p^*,g(q))}{Var(p^*)}\right)\left(Cov(\Delta\xi,g(q))-\frac{1}{\alpha}Cov\left(\Delta\xi,\lambda\right)\right)\\ =&-\frac{1}{\alpha}\frac{Cov(p^*,g(q))}{Var(p^*)}Cov(\Delta\xi^{OLS},\lambda)+\left(1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)Cov(\Delta\xi^{OLS},g(q))\\ &-\frac{1}{\alpha}\left(1-\frac{Cov(p^*,g(q))}{Var(p^*)}\right)Cov(\Delta\xi^{OLS},\lambda)-\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)}Cov(\Delta\xi^{OLS},g(q))\\ =&Cov(\Delta\xi^{OLS},g(q))-\frac{1}{\alpha}Cov(\Delta\xi^{OLS},\lambda).\end{aligned}$$

Thus, we obtain an expression for the probability limit of the OLS estimator,

$$\text{plim}\hat{\alpha}^{OLS}=\alpha-\frac{\frac{Cov(\Delta\xi^{OLS},\lambda)}{Var(p^*)}-\alpha\frac{Cov(\Delta\xi^{OLS},g(q))}{Var(p^*)}}{\alpha+\frac{Cov(p^*,\lambda)}{Var(p^*)}-\alpha\frac{Cov(p^*,g(q))}{Var(p^*)}},$$

and the following quadratic α.

$$\begin{aligned}0=&\left(1-\frac{Cov(p^*,g(q))}{Var(p^*)}\right)\alpha^2\\ &+\left(\frac{Cov(p^*,\lambda)}{Var(p^*)}-\hat{\alpha}^{OLS}+\frac{Cov(p^*,g(q))}{Var(p^*)}\hat{\alpha}^{OLS}+\frac{Cov(\Delta\xi^{OLS},g(q))}{Var(p^*)}\right)\alpha\\ &+\left(-\frac{Cov(p^*,\lambda)}{Var(p^*)}\hat{\alpha}^{OLS}-\frac{Cov(\Delta\xi^{OLS},\lambda)}{Var(p^*)}\right).\end{aligned}$$

QED.

C.6 Proof of Lemma 1 (Monotonicity in $Cov(\Delta\xi,\Delta\eta)$)

We return to the quadratic formula for the proof. The lower root of a quadratic ax^2+bx+c is $L\equiv\frac{1}{2}\left(-b-\sqrt{b^2-4ac}\right)$. In our case, $a=1$.

We wish to show that $\frac{\partial L}{\partial\gamma}<0$, where $\gamma=Cov(\Delta\xi,\Delta\eta)$. We evaluate the derivative to obtain

$$\frac{\partial L}{\partial\gamma}=-\frac{1}{2}\left(1+\frac{b}{(b^2-4c)^{\frac{1}{2}}}\right)\frac{\partial b}{\partial\gamma}.$$

We observe that, in our setting, $\frac{\partial b}{\partial \gamma} = \frac{1}{Var(p^*)}$ is always positive. Therefore, it suffices to show that

$$1 + \frac{b}{(b^2 - 4c)^{\frac{1}{2}}} > 0. \tag{C.8}$$

We have two cases. First, when $c < 0$, we know that $\left| \frac{b}{(b^2-4c)^{\frac{1}{2}}} \right| < 1$, which satisfies equation (C.8). Second, when $c > 0$, it must be the case that $b > 0$ also. Otherwise, both roots are positive, invalidating the model. When $b > 0$, it is evident that the left-hand side of equation (C.8) is positive. This demonstrates monotonicity.

Finally, we obtain the range of values for L by examining the limits as $\gamma \to \infty$ and $\gamma \to -\infty$. From the expression for L and the result that $\frac{\partial b}{\partial \gamma}$ is a constant, we obtain

$$\lim_{\gamma \to -\infty} L = 0$$
$$\lim_{\gamma \to \infty} L = -\infty$$

When $c < 0$, the domain of the quadratic function is $(-\infty, \infty)$, which, along with monotonicity, implies the range for L of $(0, -\infty)$. When $c > 0$, the domain is not defined on the interval $(-2\sqrt{c}, 2\sqrt{c})$, but L is equal in value at the boundaries of the domain. QED.

Additionally, we note that the upper root, $U \equiv \frac{1}{2}\left(-b + \sqrt{b^2 - 4ac}\right)$ is increasing in γ. When the upper root is a valid solution (i.e., negative), it must be the case that $c > 0$ and $b > 0$, and it is straightforward to follow the above arguments to show that $\frac{\partial U}{\partial \gamma} > 0$ and that the range of the upper root is $[-\frac{1}{2}b, 0)$.

C.7 Proof of Proposition 4 (Covariance Bound)

The proof involves an application of the quadratic formula. Any generic quadratic, $ax^2 + bx + c$, with roots $\frac{1}{2}\left(-b \pm \sqrt{b^2 - 4ac}\right)$, admits a real solution if and only if $b^2 \geq 4ac$. Given the formulation of equation (11), real solutions satisfy the condition:

$$\left(\frac{Cov(p^*, \lambda)}{Var(p^*)} + \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)} - \alpha^{OLS}\right)^2 \geq 4\left(-\alpha^{OLS}\frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)}\right).$$

As $a = 1$, a solution is always possible if $c < 0$. This is the sufficient condition for point identification from the text. If $c \geq 0$, it must be the case that $b \geq 0$; otherwise, both roots are positive. Therefore, a real solution is obtained if and only if $b \geq 2\sqrt{c}$, that is

$$\left(\frac{Cov(p^*, \lambda)}{Var(p^*)} + \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)} - \alpha^{OLS}\right) \geq 2\sqrt{-\alpha^{OLS}\frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)}}.$$

Solving for $Cov(\Delta\xi, \Delta\eta)$, we obtain the model-based bound,

$$Cov(\Delta\xi, \Delta\eta) \geq Var(p^*)\alpha^{OLS} - Cov(p^*, \lambda) + 2Var(p^*)\sqrt{-\alpha^{OLS}\frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)}}.$$

This bound exists if the expression inside the radical is positive, which is the case if and only if the sufficient condition for point identification from Proposition 2 fails. QED.