



The impact of horizontal mergers and acquisitions in price competition models

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Abstract

The question of what impact mergers and acquisitions have on key equilibrium performance measures is fundamental to our understanding of competitive dynamics in an oligopolistic industry. We address these questions in the context of price competition models with differentiated goods and asymmetric firms allowing for general non-linear demand and cost functions merely assuming that both the pre- and post-merger competition games are supermodular along with two minor technical conditions. We show that, in the absence of cost synergies, post-merger equilibrium prices exceed their pre-merger levels. Moreover, the post-merger equilibrium profit of the merged firms exceeds the aggregate of the pre-merger equilibrium profits of the merging firms. The equilibrium profit of the non-merging firms increases as well. We establish our results, at first, for settings where each firm in the industry offers a single product; we then generalize them to industries with multi-product firms. We also derive conditions under which cost synergies, by themselves, result in *lower* equilibrium prices than otherwise observe post-merger, and discuss how the combined effect of increased market concentration and cost synergies can be assessed efficiently.

1 Introduction and Summary

The question of what impact mergers and acquisitions have on key equilibrium performance measures is fundamental to our understanding of competitive dynamics in an oligopolistic industry. These include the aggregate profits of the merging firms, those of the other firms in the industry (hereafter referred to as the “remaining firms”), equilibrium prices, and consumer welfare.

Early strategy works, for example Steiner (1975), postulate that mergers should increase aggregate profits of the merging firms, even in the absence of any cost efficiencies resulting from economies of scope or scale. It was also conjectured that horizontal mergers should result in an increase in equilibrium prices, for *all* of the products offered by the industry. This was assumed, for example, in the classical paper by Williamson (1968). However, early attempts to substantiate these conjectures on the basis of industrial organization models failed. For example, Szidarovszky and Yakowitz (1982), Salant et al. (1983), and Davidson and Deneckere (1984) all concluded from their analyses that aggregate profits of merging firms actually *decline*, unless accompanied with significant cost efficiencies due to synergies or economies of scope. At first sight, this appears counterintuitive, since the merged firm always has the option to maintain the (quantity) decisions pertaining to the pre-merger equilibrium and improve on these to achieve higher aggregate profits. However, the dynamics in the competition models analyzed by the above authors are such that the new post-merger equilibrium is associated with *lower* aggregate profits.

A seminal step toward resolving this enigma was provided by Deneckere and Davidson (1985). These authors explained that the counterintuitive findings in the prior literature were the result of analyzing the question in the context of Cournot competition models, in which firms select sales quantities or targets as opposed to prices. Davidson and Deneckere proceeded to show that, under price (Bertrand) competition, the anticipated effects can be demonstrated: *In the absence of cost efficiencies resulting from a merger*, aggregate profits of the merging firms increase as do equilibrium prices. Even the equilibrium profits of the *remaining* firms increase, while the consumer ends up holding the bag, *i.e.*, consumer welfare declines. Their analysis is based on a model with completely symmetric firms and

linear demand and cost functions. (This model was first proposed by Shubik and Levitan (1980)). In their appendix, Deneckere and Davidson extend the results to competition models with non-linear demand functions satisfying five assumptions, the most important of which is that the industry is *symmetrically* differentiated, *i.e.*, all firms share the same constant marginal procurement cost rate and the demand for any pair of products is identical when both its own price and those of the competitors are the same.

In this paper, we establish the above conjectures for general *price* competition models with general non-linear demand and cost functions as long as the models are supermodular, with two additional structural conditions: (i) each firm's profit function is strictly quasi-concave in its own price(s), and (ii) markets are competitive, *i.e.*, in the pre-merger industry, each firm's profits increase when any of his competitors increases his price, unilaterally. In particular, we show that in the absence of cost synergies, both the component-wise smallest and largest post-merger price equilibrium are larger than their pre-merger counterparts, implying reduced consumer welfare. (The existence of a component-wise smallest and largest equilibrium follows from the fact that both the pre-merger and post-merger competition models are supermodular.) In addition, the post-merger equilibrium profit of the merged firm exceeds the pre-merger aggregate equilibrium profits of the merging firms. Perhaps most surprisingly, even the equilibrium profits of the remaining firms increase. (In the case of multiple equilibria, all of these comparison results pertain both to the largest and smallest equilibrium.) These results have been conjectured to hold for general supermodular price competition models, see for example the influential survey chapters by Whinston (2006, 2007).

Mergers often result in cost synergies and such synergies may induce price *reduction*, to counter the balance the price *increases* that result from the increase market concentration. We show, however, in subsection 4.2 that the post-merger equilibrium depends in a significant way on the specific *structural form* by which cost synergies impact the products' cost functions. As described in subsection 4.2, after any merger plan is announced, typically, high level operational consultants are retained to characterize and quantify these synergies. In spite of the enormous impact these synergy assessment projects have, little attention has been devoted to this topic in the operations literature. Noted exceptions are Gupta and Gerchak (2002) and Iyer and Jain (2004).

We establish our results, at first, for settings where each firm in the industry offers a single product; we then generalize them to industries with general multi-product firms. We also derive conditions under which cost synergies, by themselves, result in *lower* equilibrium prices and discuss how the combined effect of increased market concentration and cost synergies can be assessed efficiently.

The results in this paper are of interest to individual firms competing in an oligopoly and either considering a merger or wishing to evaluate the consequences of a potential merger by others. They are also of interest to government agencies such as the Antitrust Division of the Department of Justice (DOJ) and the Federal Trade Commission (FTC). As discussed in detail, in Section 5, these agencies are charged with the task of evaluating more than a thousand proposals annually. The DOJ and FTC initially focused on relatively simple market concentration measures such as the (post-merger) Herfindahl-Hirschman Index (HHI) to assess the potential for a "substantial lessening of competition in the industry," the standard prescribed by the Clayton and Hart-Scott-Rodino Act. (The HHI is defined as the sum of the squares of the firms' market shares.) However, economists have pointed out, repeatedly, that, in particular in the case of differentiated products, these market concentration measures are poor surrogates for the actual changes in equilibrium prices, consumer welfare, and firm profits. As a consequence, the DOJ and FTC have, since the early nineties, conducted *merger simulations* when mergers in differentiated product markets are proposed. Here, an oligopoly model for the industry is estimated and a pre- and post-merger equilibrium computed, see Werden and Froeb (1994, 1996), and Werden (1997) and the above mentioned discussion of the use of the UPP measure. The merger simulation approach was also widely adopted by other economists (, outside of the DOJ and FTC,) for example Baker and Bresnahan (1985), Berry and Pakes (1993), Hausman et al. (1994), Hausman and Leonard (1999), Nevo (2000), Dube (2005), and Thomadsen (2005).

The results in our paper provide important support for merger simulations. In the above described class of supermodular price competition models, we are able to show that, if the pre-merger industry has a unique equilibrium, the *post-merger* smallest equilibrium can be computed by applying a simple *tatōnnement* scheme with the *pre-merger* price vector as the starting point. Moreover, this scheme generates an *increasing* sequence of price vectors which converges to the post-merger equilibrium. (The post-merger smallest and largest equilibrium can also be computed by applying a *tatōnnement* scheme, with the smallest and largest feasible price vector as its starting point, respectively; these schemes are monotone as well.)

The remainder of this paper is organized as follows. In Section 2, we give a brief literature review. Section 3 presents our general model with some preliminaries. The fundamental comparison results are obtained in Section 4, while section 5 describes the relationship between our results. Recently, a fundamental discussion has emerged led by Farrell and Shapiro (2010a) as well as Schmalensee (2009), ??? simple measures such as discussion of the Upward

Pricing Pressure (UPP) measure can, or should, be used as a proxy for the actual changes in equilibrium prices.¹

2 Literature Review

Few topics in industrial organization economic theory have been driven as intensively by policy, legislative, and legal debates and innovation as the impact of mergers and acquisitions have on market equilibria. As mentioned, Williamson (1968) appears to have been the first contribution to the literature in this area. The author demonstrated, with a simple model, that even if a merger results in price increases, (as he conjectured would often happen,) these may be accompanied with reductions in marginal costs due to synergies. The combined effect on aggregate surplus or welfare may therefore be positive in spite of universal price increases in the industry. As elementary as this observation is in 2011, Williamson's insights directly challenged prior criteria in the U.S. courts, attempting to apply anti-trust laws such as the Clayton Act. For example, in the 1962 case of *Brown Shoe vs. United States*, the court refused to entertain the argument that cost efficiencies arising from the merger could result in increased welfare. In 1967, the Supreme Court went even further when evaluating Procter and Gamble's acquisition of Clorox. It argued that such cost synergies should actually be viewed as an additional argument *against* the merger, in as much as they result in additional profit and cash flow enhancements for the merged enterprise.

Prior to the eighties, the strategy literature posited that aggregate profits of merging firms should increase even in the absence of any cost synergies, see *e.g.*, Steiner (1975) chapters 2 and 3. As mentioned in the Introduction, the first attempts to establish this result in a formal oligopoly model are due to Salant et al. (1983), Szidarovszky and Yakowitz (1982), and Davidson and Deneckere (1984). However, these papers found that aggregate profits of merging firms may, in fact, decline. All three papers analyzed mergers in the context of Cournot competition for a homogeneous good. Perry and Porter (1985) countered that the enigmatic outcome in, for example, Salant et al. (1983), is due to the authors ignoring cost synergies resulting from a merger in their homogeneous Cournot model. These authors show that the aggregate profits of merging firms are guaranteed to increase, if the cost synergies are sufficiently large.

Continuing to address Cournot oligopolies with homogeneous goods, Farrell and Shapiro (1990) expanded the discussion to the impact mergers have on the *equilibrium price*. (In a model with homogeneous goods, all products are sold for the same price.) Farrell and Shapiro show that the equilibrium *price* increases under linear cost structures and in the absence of cost synergies. These authors also derive a necessary and sufficient condition for a price increase under certain classes of *non-linear* cost functions and possible cost synergies².

Deneckere and Davidson (1985) made a seminal contribution to the discussion, showing that all of the anticipated effects can be guaranteed in specific classes of (Bertrand-) *price* competition models with differentiated goods: in the absence of cost synergies, equilibrium prices are guaranteed to increase, the equilibrium profits of a merged enterprise exceed the aggregate of the pre-merger profits of the merging firms while the equilibrium profits of all other firms increase as well. The authors established these results in a symmetric model with linear demand and cost functions. (As mentioned in our Introduction, in their appendix, the authors extend these results to non-linear demand functions, under five conditions, the most important of which is that the industry is symmetrically differentiated, see *ibid.*)

Thereafter, a few contributions have been made to generalize the Deneckere and Davidson (1985) results to models allowing for asymmetry among the firms or general non-linear demand functions. Zhao and Howe (2010) generalize the Deneckere and Davidson results to models with linear demand and cost functions such that in each product's demand function the coefficient in front of the product's own price is product specific but a single uniform coefficient applies to all cross terms in all demand functions.

Many oligopoly models employ affine demand and cost functions because of the ensuing analytical simplifications. Examples include Singh and Vives (1984) and Hackner (2000) in the economics literature and Farahat and Perakis (2007), Adida and DeMiguel (2011) and Perakis and Sien in operations management. The affine structure is clearly restrictive; indeed Jaffe and Weyl recently established that with more than two firms, no discrete choice model can generate linear demand functions. Farshal and Perakis showed that the linear structure involves other difficulties in the multi-product modes, these authors develop a "non-negative restriction" of affine demand functions and, among other contributions, Davidson and Deneckere results for this variant of the affine structure, see Proposition 5.1.

Werden and Froeb (1994) established the Deneckere and Davidson results for a model with multinomial logit demands and linear costs; these authors applied the model to the U.S. market of long-distance carriers, calculating the impact of various potential mergers. Levy and Reitzes (1992) established the above results in a model where all

¹Carl Shapiro is chief economist of the Antitrust Division of the Department of Justice (DOJ) and Joseph Farrell is head of the Bureau of Economics at the Federal Trade Commission (FTC).

²Unfortunately, the necessary and sufficient condition is stated in terms of the pre-merger and post-merger equilibria outcomes rather than the primitives of the model

consumers and all n firms are located on a circle: each consumer patronizes the firm whose full price, consisting of a direct price plus a travel cost proportional to the distance to the firm, is lowest. As mentioned in the Introduction, the influential survey chapters by Whinston (2006, 2007) *conjecture* that the results in Deneckere and Davidson should apply to general supermodular price competition models, a conjecture our paper confirms the above mentioned two additional conditions.

As discussed, merger analysis has become a standard tool to evaluate the impact of potential mergers, a trend stimulated by the development of effective structural econometric methods for oligopoly models. Here, the demand and cost functions in a price competition model are estimated. Thereafter, a counterfactual study is undertaken to estimate the price-, market share- and profit implications of a potential merger. Examples include Werden and Froeb (1994) for the market for long distance carriers, Nevo (2000) for the ready-to-eat cereal industry, Dube (2005) for the soft drink industry and Thomadsen (2005) for the fast-food drive thru industry in Santa Clara County. See Berry and Pakes (1993) for a general discussion of the use of the above econometric methods to enable merger simulations and Baker and Bresnahan (1985) for an early application based on more elementary estimation methods.

All of these merger simulation studies expect and confirm the above mentioned phenomena in terms of increases in equilibrium prices and profits. We refer the reader to section 5 for a review of the literature discussing alternatives to merger simulation, as tools to approximate the impact of mergers and acquisitions.

3 The Model

We initially consider an industry with N firms, each offering a single product to the market. (In subsection 4.3 we generalize our results to industries with general multi-product firms.) The expected demand volumes for these products depend on all product prices according to a general system of demand equations. Each firm selects its price level from a given, closed, price interval. The cost incurred by each firm depends on its sales volume according to a given, possibly nonlinear, cost function. We characterize the impact of a merger of several of the firms, without loss of generality the first I firms, with $2 \leq I \leq N$. Thus, for each firm i , $i = 1, \dots, N$, let

p_i = the price selected;

p_i^{min} (p_i^{max}) = the minimum (maximum) feasible price;

$d_i(p) = d_i(p_1, \dots, p_N)$ = the expected sales volume;

$C_i(d_i)$ = the total cost incurred by firm i , specified as a differentiable function of its sales volume.

We use the common notation, p_{-i} , to denote the $(N-1)$ -dimensional vector of prices pertaining to all but firm i 's prices. Similarly, we denote by p_{-I} , the $(N-I)$ -dimensional price vector (p_{I+1}, \dots, p_N) .

We consider fully general differentiable demand functions, merely assuming, without loss of generality, that:

$$\frac{\partial d_i}{\partial p_i} \leq 0 \quad \forall i = 1, \dots, N \quad \text{and} \quad \frac{\partial d_i}{\partial p_j} \geq 0 \quad \forall i \neq j \quad (1)$$

i.e., each product's demand function is downward sloping in its own price and nondecreasing in any of the competing products' prices.

As to the price bounds, $\{p_i^{min}\}$ and $\{p_i^{max}\}$ we assume that they are set loosely enough as to be non-binding whenever a firm determines the best response to a given set of choices by the competitors. We impose these bounds merely to ensure that the feasible price range for each product is a compact set.

The expected profit function of each firm when operating by itself, is thus given by

$$\pi_i(p) = p_i d_i(p) - C_i(d_i(p)), \quad i = 1, \dots, N \quad (2)$$

In this section, we assume that when the first I firms merge, this merger *does not* result in any cost savings, *i.e.*, the I products continue to be procured in the pre-merger way, so that the cost function of the merged firm is given by $C^m(p) = \sum_{i=1}^I C_i(d_i(p))$. The post-merger profit function for the merged firm is thus given by

$$\pi^m(p) \equiv \sum_{i=1}^I p_i d_i(p) - C^m(p) = \sum_{i=1}^I \{p_i d_i(p) - C_i(d_i(p))\} = \sum_{i=1}^I \pi_i(p). \quad (3)$$

We assume that the profit functions exhibit the following two properties:

(Q) (Quasi-Concavity) Each firm i 's profit function $\pi_i(p)$ is strictly quasi-concave in its own price variable p_i , $i = 1, \dots, N$.

(S) (Strategic Complementarity)

- (i) For all $i = 1, \dots, N$ the profit function $\pi_i(p_i, p_{-i})$ is supermodular in every price pair (p_i, p_j) with $j \neq i$.
- (ii) The profit function of the merged firm $\pi^m(p_1, \dots, p_N)$ is supermodular in each price pair (p_i, p_j) with $i = 1, \dots, I, j = 1, \dots, N$, and $i \neq j$.

Condition (S) has been used, with regularity, in the literature. See, for example, Cabral and Villas-Boas (2005). Vives (1985, 1990) identified broad sufficient conditions in terms of the demand functions and cost structures which guarantee that conditions (Q) and (S.i) are satisfied simultaneously: Assume, the demand functions are *twice* differentiable and that, for example, each firm has an increasing and convex cost function $C_i(\cdot)$, $d_i(p)$ is log-concave while $\frac{\partial^2 \log d_i}{\partial p_i \partial p_j} \geq 0, \forall j \neq i, i.e., d_i$ is log-supermodular in every price pair (p_i, p_j) with $j \neq i$. (See remark 2 on p. 156 of Vives (2001).)

It is harder to identify sufficient conditions for property (S.ii), *i.e.*, for the supermodularity of the profit function of a merged firm in terms of simple structural properties of the individual products' cost and demand functions. However, condition (S.ii) is easily verified directly. This applies, in particular, when the profit functions are *twice* differentiable, in which case (S.ii) is equivalent to

$$\frac{\partial^2 \pi^m}{\partial p_i \partial p_j} \geq 0 \quad \forall i = 1, \dots, I \text{ and } \forall j \neq i, \text{ on the price cube } X_{l=1}^N [p_l^{min}, p_l^{max}]. \quad (4)$$

As shown in Section 4.1, one special, but frequently applied, case in which conditions (Q) and (S) can be guaranteed upfront is when all demand and cost functions are affine.

4 Pre- and Post-Merger Comparison

In this section, we describe our main results. In particular we show that in the absence of cost synergies, both the component-wise smallest and largest post-merger price equilibria are larger than their pre-merger counterparts. This implies that consumer welfare declines due to the merger. In addition, all firms' equilibrium profits increase, with the understanding that we compare the profits of the newly merged firm with the aggregate of their pre-merger profits. We distinguish between the following two games, describing the competition in the industry before and after the merger:

$[\Gamma^{pre}] \{A_1 \dots A_I; [\pi_1, \pi_2, \dots, \pi_N]; i = 1, \dots, N\}$: The pre-merger N firm competition game in which each of the N firms operates as an independent competitor; firm $i = 1, \dots, N$ selects its price from the interval $A_i = [p_i^{min}, p_i^{max}]$ and faces the profit function π_i .

$[\Gamma^{post}] \{X_{l \in I} A_l, A_I + 1, \dots, A_N; [\pi^m, \pi_{I+1}, \dots, \pi_N]; i = m, I + 1, \dots, N\}$: The post-merger game, with (N-I+1) players, the merged firm m and firms $I + 1, \dots, N$. The merged firm selects its I-dimensional price vector from the price cube $X_{l=1}^I A_l$ and faces the profit function π^m . The remaining firms $i = I + 1, \dots, N$ have the same feasible action space and profit functions as in the pre-merger game.

In addition, we define the following set of *restricted* games for any given vector of prices $p_{-I}^\circ = (p_{I+1}^\circ, \dots, p_N^\circ)$, pertaining to the firms *not* involved in the merger:

$$\Gamma^{res}(p_{-I}^\circ) = \{A_1, \dots, A_I; \pi_i^{res}(p_1, \dots, p_I | p_{-I}^\circ); i = 1, \dots, I\}, \text{ where} \\ \pi_i^{res}(p_1, \dots, p_I | p_{-I}^\circ) \equiv \pi_i(p_1, \dots, p_I, p_{-I}^\circ), \quad i = 1, \dots, I \quad (5)$$

These games have the first I firms as independent players, each with his feasible price interval as his action space and a profit function obtained from the profit function in the unrestricted pre-merger game the prices of the remaining firms $I + 1, \dots, N$ at the levels in the vector p_{-I}° . We first need the following lemma:

- Lemma 4.1** (a) *The games Γ^{pre} , Γ^{post} , and $\Gamma^{res}(p_{-I}^\circ)$, for any price vector $p_{-I}^\circ \in X_{l=I+1}^N A_l$ are all supermodular and have a component-wise smallest equilibrium, which we denote by $\underline{p}^*(pre)$, $\underline{p}^*(post)$, and $\underline{p}^*(p_{-I}^\circ)$ respectively. They also have a component-wise largest equilibrium, denoted by $\bar{p}^*(pre)$, $\bar{p}^*(post)$, and $\bar{p}^*(p_{-I}^\circ)$.*
- (b) *For each firm $i = 1, \dots, N$, there exists a unique best response $\Psi_i(p_{-i})$ for any feasible price vector $p_{-i} \in X_{l \neq i} A_l$.*
- (c) *In the post-merger game, Γ^{post} , the merged firm m has a component-wise smallest [largest] best response function price vector, $\underline{\Psi}^m(p_{-I})$ [$\bar{\Psi}^m(p_{-I})$] = (p_1, \dots, p_I) , for any price vector $p_{-I} \in X_{l=I+1}^N A_l$.*

Proof: (a) All of the considered games have continuous profit functions and action spaces that are lattices, either simple closed intervals or, for the merged firm in the game Γ^{post} , the cube $X_{I=1}^I A_I$. To establish the supermodularity of the various games, it therefore suffices to verify that the players' profit functions have the required supermodularity properties. For the games Γ^{pre} and $\Gamma^{res}(p_{-I}^o)$ this is immediate from condition (S.i). In the game Γ^{post} , each firm $i = I+1, \dots, N$ has the same profit function π_i as in Γ^{pre} and this profit function is supermodular in (p_i, p_j) for all $j \neq i$. Finally, in the game Γ^{post} , the merged firm m has profit function π^m which is supermodular in (p_i, p_j) for all $i = 1, \dots, I$ and all $j = 1, \dots, N$ with $j \neq i$ by condition (S.ii). Since the games are supermodular, it follows that they have a component-wise smallest and a component-wise largest equilibrium, see e.g., Theorem 4.2.1 in Topkis (1998).

(b) This result follows from the strict quasi-concavity of each profit function π_i in its own-price variable p_i , $i = 1, \dots, N$, see condition (Q).

(c) Part (c) follows from Lemma 4.2.2 (c) in Topkis (1998) and the supermodularity of Γ^{post} by part (a). ■

As is well known, one of the implications of a game being supermodular is that its component-wise smallest equilibrium can be computed by a simple tatōnnement scheme which starts with the vector p^{min} , the component-wise smallest element of the feasible price space³. In such tatōnnement schemes, the players iteratively determine best responses to choices made by their competitors in earlier iterations of the scheme. There is considerable flexibility in terms of the sequence in which best response updates are made. Topkis (1998) and Vives (2001) focus on the so-called simultaneous optimization and Round-Robin versions. In the former, all players determine (simultaneously) in each iteration, their best responses to the choices made in the prior iteration with a specific rule determining *which* best response is selected when the best response fails to be unique. In the Round-Robin version, one chooses a particular permutation of the players; following this permutation, each player is sequentially offered the opportunity to adopt his best response to the most recent choices made by all competitors.

Our first main result is to show that the component-wise smallest and largest equilibrium in the post-merger game are (component-wise) *larger* than the corresponding equilibria in the pre-merger game. Our proof is based on identifying pairs of specific tatōnnement schemes one of which pertains to the post-merger game and one to the pre-merger game, such that in each iteration the price vector determined in the post-merger tatōnnement scheme dominates that obtained in the pre-merger scheme, while the pre-merger scheme converges to a specific equilibrium in the pre-merger game, and the post-merger scheme converges to its counterpart in the post-merger game. This proof technique is reminiscent of that employed in Allon and Federgruen (2007).

To show that $\underline{p}^*(post) \geq \underline{p}^*(pre)$ and $\bar{p}^*(post) \geq \bar{p}^*(pre)$, we use the following pairs of schemes, respectively:

• **Pre-Merger Increasing Scheme**

- Step 0: $p^{(0)} := p^{min}$; $k = 1$
- Step 1: For $i = 1, \dots, I$, set $(p_1^{(k)}, \dots, p_I^{(k)}) = \underline{p}^*(p_{-I}^{(k-1)})$ the smallest equilibrium of the game $\Gamma^{res}(p_{-I}^{o(k-1)})$
- For $i = I+1, I+2, \dots, N$, set $p_i^{(k)} = \Psi_i(p_{-i}^{(k-1)})$; $k = k+1$ and repeat Step 1.

• **Post-Merger Increasing Scheme**

- Step 0: $q^{(0)} := p^{min}$; $k = 1$
- Step 1: For $i = 1, \dots, I$, set $(q_1^{(k)}, \dots, q_I^{(k)}) = \underline{\Psi}^m(q_{-I}^{(k-1)})$;
- For $i = I+1, I+2, \dots, N$, set $q_i^{(k)} = \Psi(q_{-i}^{(k-1)})$; $k = k+1$ and repeat Step 1.

• **Pre-Merger Decreasing Scheme**

- Step 0: $\tilde{p}^{(0)} := p^{max}$; $k = 1$
- Step 1: For $i = 1, \dots, I$, set $(\tilde{p}_1^{(k)}, \dots, \tilde{p}_I^{(k)}) = \bar{p}^*(\tilde{p}_{-I}^{(k-1)})$ the largest equilibrium of the game $\Gamma^{res}(\tilde{p}_{-I}^{o(k-1)})$
- For $i = I+1, I+2, \dots, N$, set $\tilde{p}_i^{(k)} = \Psi_i(\tilde{p}_{-i}^{(k-1)})$; $k := k+1$ and repeat Step 1.

• **Post-Merger Decreasing Scheme**

- Step 0: $\tilde{q}^{(0)} := p^{max}$; $k = 1$
- Step 1: For $i = 1, \dots, I$, set $(\tilde{q}_1^{(k)}, \dots, \tilde{q}_I^{(k)}) = \bar{\Psi}^m(\tilde{q}_{-I}^{(k-1)})$;

³The same property applies to the component-wise largest equilibrium, starting the tatōnnement scheme at the largest feasible price vector p^{max} .

– For $i = I+1, I+2, \dots, N$, set $\hat{q}_i^{(k)} = \Psi(\hat{q}_{-I}^{(k-1)})$; $k = k+1$ and repeat Step 1.

Lemma 4.2 (a) The sequence $\{p^{(k)}\}_{k=1}^{\infty}$ increases monotonically to \underline{p}^* (pre).

(b) The sequence $\{q^{(k)}\}_{k=1}^{\infty}$ increases monotonically to \underline{p}^* (post).

(c) The sequence $\{\hat{p}^{(k)}\}_{k=1}^{\infty}$ decreases monotonically to \bar{p}^* (pre).

(d) The sequence $\{\hat{q}^{(k)}\}_{k=1}^{\infty}$ decreases monotonically to \bar{p}^* (post).

Proof: (a) We first show, by induction, that the sequence $\{p^{(k)}\}_{k=1}^{\infty}$ is monotonically increasing. Clearly $p^{(1)} \geq p^{(0)} = p^{min}$. Assume $p^{(k-1)} \geq p^{(k-2)}$ for some $k \geq 2$. For $i = I+1, \dots, N$, we have

$$p_i^{(k)} = \Psi_i(p_i^{(k-1)}) \geq \Psi_i(p_i^{(k-2)}) = p_i^{(k-1)} \quad (6)$$

where the inequality follows from the induction assumption and the fact that in a supermodular game the $\Psi_i(\cdot)$ operator is increasing, for all i , see Lemma 4.2.2 in Topkis (1998). Moreover, for the merging firms $i = 1, \dots, I$, the Pre-Merger Increasing Scheme specifies that

$$(p_1^{(k)}, \dots, p_I^{(k)}) = \underline{p}^*(p_{-I}^{(k-1)}) \geq \underline{p}^*(p_{-I}^{(k-2)}) = (p_1^{(k-1)}, \dots, p_I^{(k-1)}), \quad (7)$$

where the inequality follows again from the induction assumption, as well as from the fact that the smallest equilibrium in the supermodular, restricted game $\Gamma^{res}(p_{-I}^o)$ is an increasing function of any of the parameters in p_{-I}^o since, for all $i = 1, \dots, I$, each firm i 's payoff function in these restricted games is continuous and supermodular in (p_i, p_j) for all $j = I+1, \dots, N$ (see Theorem 4.2.2 in Topkis (1998)).

(6) and (7) together establish that $p^{(k)} \geq p^{(k-1)}$, thus completing the induction proof for the monotonicity of scheme $\{p^{(k)}\}$ which is bounded from above by p^{max} and hence converges to a limit vector p^* . By the continuity of the profit functions, p^* is a fixed point of the joint best response operator in the pre-merger game, i.e., p^* is an equilibrium of the game Γ^{pre} .

It remains to be shown that $p^* = \underline{p}^*(pre)$, the component-wise *smallest* equilibrium of Γ^{pre} , i.e., $p^* \leq \underline{p}^*(pre)$. To prove this inequality, consider for any precision $\epsilon > 0$, the following ϵ -approximation of the Pre-Merger Increasing Scheme:

Approximate Pre-Merger Increasing Scheme (APMIS):

• **Step 0:** $x^{(0)} := p^{min}$; $l:=1$

• **Step 1:** (Best response for firms $I+1, \dots, N$)

For $i = 1, \dots, I$, $x_i^{(l)} := p_i^{min}$;

For $i = I+1, \dots, N$, $x_i^{(l)} := \Psi_i(x^{(l-1)})$

$l = l+1$

• **Step 2:** (Best response for firms $1, \dots, I$)

For $i = 1, \dots, I$, $x_i^{(l)} := \Psi_i(x^{(l-1)})$;

for $i = I+1, \dots, N$, $x_i^{(l)} := x_i^{(l-1)}$

$l = l+1$

If $\|x^{(l)} - x^{(l-1)}\|_{\infty} \leq \epsilon$, go to Step1, otherwise, go to Step 2.

Note that when APMIS executes a batch of consecutive Step 2 iterations, an (ϵ -approximation) of the smallest equilibrium in the restricted game $\Gamma^{res}(\cdot)$ is being computed, given the most recently updated prices for the firms $I+1, \dots, N$. Thus, modulo the ϵ -approximation in the stopping criterion of Step 2, each time APMIS reenters Step 1, a new element of the sequence $\{p^{(k)}\}$ in the Pre-Merger Increasing Scheme is being generated. Thus, the scheme $\{x^{(l)}\}_{l=1}^{\infty}$ converges to an ϵ -approximation $x^*(\epsilon)$ of the limit vector p^* of the scheme $\{p^{(k)}\}_{k=1}^{\infty}$. Moreover, by the continuity of the profit functions, $\lim_{\epsilon \downarrow 0} x^*(\epsilon) = p^*$.

To show that $p^* \leq \underline{p}^*(pre)$, it thus suffices to show that $x^*(\epsilon) \leq \underline{p}^*(pre)$. This inequality follows by comparing the sequence $\{x^{(l)}\}$ with $\{y^{(l)}\}_{l=1}^{\infty}$, the scheme generated by the ‘‘simultaneous optimization’’ variant of the tatonnement scheme in the pre-merger game Γ^{pre} , when, like $\{x^{(l)}\}$, starting at the smallest feasible price vector p^{min} . As mentioned in the proof of part (a), since all profit functions in the pre-merger game are continuous and the firm’s feasible action sets compact, it follows from Theorem 4.3.4 in Topkis (1998) that this simultaneous optimization tatonnement scheme $\{y^{(l)}\}_{l=1}^{\infty}$ converges to $\underline{p}^*(pre)$. Moreover, $x^{(l)} \leq y^{(l)}$ for all $l = 1, 2, \dots$, since the

simultaneous optimization tatônnement scheme $\{y^{(l)}\}_{l=1}^{\infty}$, executes, in each iteration, a version of Step 1 or Step 2 of APMIS in which *all* firms are permitted to update their price to a *larger* best response value, as opposed to APMIS, where only firms $I + 1, \dots, N$ [Step 1 [Step 2]] are permitted to do so while the remaining firms $1, \dots, I$ [$I + 1, \dots, N$] forced to set their price level at the minimum [previous] level. Thus, the inequalities $x^{(l)} \leq y^{(l)}$, $l = 1, 2, \dots$ follow by complete induction, employing the fact that the profit functions are supermodular.

(b) The sequence $\{q^{(k)}\}_{k=1}^{\infty}$ is the sequence generated by the “simultaneous optimization” variant of the tatônnement scheme, applied to the game Γ^{post} and starting at p^{min} . By Lemma 1(a), the game Γ^{post} is supermodular. Since the payoff functions in this game are continuous and the action sets of all players compact, it follows from Theorem 4.3.4 in Topkis (1998) that the scheme $\{q^{(k)}\}_{k=1}^{\infty}$ converges to $\underline{p}^*(post)$.

(c) and (d): The proofs of parts (c) and (d) are analogous to those of parts (a) and (b), respectively. ■

In addition to the quasi-concavity and strategic complementarity conditions **(Q)** and **(S)** we need one additional assumption to allow for comparison of pre- and post-merger prices:

(MP) (**Marginal Profitability for the Merged Firm**) For any set of prices p_{-I}° selected by firms $I + 1, \dots, N$, the best response $\underline{\Psi}^m(p_{-I}^{\circ}) = (p_1, \dots, p_I)$ employs price levels that are larger than the products’ marginal costs, *i.e.*, $p_i \geq C'_i(d_i)$ for $i = 1, \dots, I$.

This condition is entirely innocuous when the cost functions are affine: in this case, we may, without loss of generality select $p^{min} \geq c$, the constant marginal cost rate vector. When the cost functions are non-linear, the Marginal Profitability condition (MP) may be somewhat restrictive, but can still be argued to apply in most settings.

Indeed, the following is a frequently used sufficient condition for the Marginal Profitability condition (MP):

(CM) (**Competitive Markets**): For all $i = 1, \dots, N$, $\pi_i(p_i, p_{-i})$ is increasing in p_{-i} .

The (CM) condition has been postulated, for example by Milgrom and Roberts (1990) as well as Cabral and Villas-Boas (2005). The former pointed out that under the supermodularity condition (S.i), (CM) reduces to assuming that for all $i = 1, \dots, N$, firm i ’s profit $\pi_i(p_i^{min}, p_{-i})$ is increasing in competitors’ prices, when charging at its minimum price level. (Since π_i is supermodular in (p_i, p_j) for all $j \neq i$, it has increasing differences in every such price pair, *i.e.*, $\pi_i(p_i, p'_j) - \pi_i(p_i, p_j)$ is increasing in p_i , for any pair of prices $p_j < p'_j$. Thus $\pi_i(p_i^{min}, p'_j) - \pi_i(p_i^{min}, p_j) \geq 0 \Rightarrow \pi_i(p_i, p'_j) - \pi_i(p_i, p_j) \geq 0$ for all $p_i \geq p_i^{min}$.)

Lemma 4.3 *Under condition (S.i), (CM) \Rightarrow (MP).*

Proof: By (CM) we have, in every price point $p \in X_{i=1}^N A_i$, for all $i \neq j$ that $\frac{\partial \pi_i}{\partial p_j} = (p_i - C'(d_i)) \frac{\partial d_i}{\partial p_j} \geq 0$. In view of (1) this implies that for all $i = 1, \dots, N$ $p_i \geq C'_i(d_i)$, with $d_i = d_i(p)$ for any feasible price vector p , and, in particular, when the prices of the first I products are selected as best responses. ■

We now derive our first main result, *i.e.*, we show that both the largest and smallest equilibria in the post-merger game dominate, component-wise, their counterparts in the pre-merger game.

Theorem 4.4 (*Pre- and Post-Merger Price Comparisons*) *Assume conditions (Q), (S), and (CM) apply.*

The post-merger equilibrium $\underline{p}^(post)$ [$\bar{p}^*(post)$] is component-wise larger than the pre-merger equilibrium $\underline{p}^*(pre)$ [$\bar{p}^*(pre)$], *i.e.*, $\underline{p}^*(post) \geq \underline{p}^*(pre)$ and $\bar{p}^*(post) \geq \bar{p}^*(pre)$.*

Proof: We show that $\underline{p}^*(post) \geq \underline{p}^*(pre)$; the comparison proof for the largest equilibrium in the post- and pre-merger game is entirely analogous. In view of Lemma 4.2, it suffices to show that in each iteration $k = 0, 1, \dots$, $q^{(k)} \geq p^{(k)}$. We prove this by induction. The starting conditions of the two schemes have $q^{(0)} = p^{(0)}$, so that the statement holds for $k=0$. Assume it holds after the $(k-1)$ st iteration, *i.e.*, $q^{(k-1)} \geq p^{(k-1)}$. For firms $i = I + 1, \dots, N$, $q_i^{(k)} = \Psi_i(q^{(k-1)}) \geq \Psi_i(p^{(k-1)}) = p_i^{(k)}$ is immediate from the supermodularity condition (S.i), see, for example, Lemma 4.2.2c in Topkis (1998).

It thus remains to be shown that $q_i^{(k)} \geq p_i^{(k)}$ for $i = 1, \dots, I$. Since p^{min} and p^{max} are selected so as not to impact on the best response price choices in either the pre- or post-merger industry, we have that the price vector $(q_1^{(k)}, \dots, q_I^{(k)})$ is an interior point of the price space and therefore satisfies the following First Order Conditions. For all $i = 1, \dots, I$:

$$\begin{aligned} 0 &= \frac{\partial \pi^m(q_1^{(k)}, \dots, q_I^{(k)}, q_{I+1}^{(k-1)}, \dots, q_N^{(k-1)})}{\partial p_i} = \sum_{l=1}^I \frac{\partial \pi_l(q_1^{(k)}, \dots, q_I^{(k)}, q_{I+1}^{(k-1)}, \dots, q_N^{(k-1)})}{\partial p_i} \\ &= \frac{\partial \pi_i(q_1^{(k)}, \dots, q_I^{(k)}, q_{I+1}^{(k-1)}, \dots, q_N^{(k-1)})}{\partial p_i} + \sum_{l \neq i; l=1}^I [q_l^{(k)} - C'_l(d_l)] \frac{\partial d_l}{\partial p_i}, \end{aligned} \quad (8)$$

with $d_i \equiv d_i(q_1^{(k)}, \dots, q_I^{(k)}, q_{I+1}^{(k-1)}, \dots, q_N^{(k-1)})$. By the marginal profitability condition (MP), we have for all $l = 1, \dots, i-1, i, i+1, \dots, I$ that $q_l^{(k)} \geq C_l'(d_l)$, while $\frac{\partial d_l}{\partial p_i} \geq 0$, see (1). Thus, for all $i = 1, \dots, I$,

$$\frac{\partial \pi_i(q_1^{(k)}, \dots, q_I^{(k)}, q_{I+1}^{(k-1)}, \dots, q_N^{(k-1)})}{\partial p_i} \leq 0. \quad (9)$$

By the strict quasi-concavity of the profit functions $\{\pi_i, i = 1, \dots, I\}$ it follows that

$$\Psi_i(q_1^{(k)}, \dots, q_I^{(k)}, q_{I+1}^{(k-1)}, \dots, q_N^{(k-1)}) \leq q_i^{(k)}, \quad i = 1, \dots, I \quad (10)$$

Consider now the restricted game $\Gamma^{res}(q_{-I}^{(k-1)})$. This game is supermodular by Lemma 4.1. Let $\Psi^{res}(x)$ denote the I-dimensional joint best response vector of the I competing firms in the game $\Gamma^{res}(q_{-I}^{(k-1)})$ to an assumed price vector $x = (x_1, \dots, x_I)$:

$$\Psi_i^{res}(x) \equiv \Psi_i(x_1, \dots, x_I, q_{I+1}^{(k-1)}, \dots, q_N^{(k-1)}), \quad i = 1, \dots, I \quad (11)$$

In addition, for all $n = 1, 2, \dots$ let $\Psi^{res(n)}(\cdot)$ denote the n-fold application of the best response operator Ψ^{res} . (10) implies that $\Psi^{res}(q_1^{(k)}, \dots, q_I^{(k)}) \leq [q_1^{(k)}, \dots, q_I^{(k)}]$. Since the game $\Gamma^{res}(q_{-I}^{(k-1)})$ is supermodular, it follows that the best response operator $\Psi^{res}(\cdot)$ is monotonically increasing. Hence,

$$p^{min} \leq \Psi^{res(n)}(q_1^{(k)}, \dots, q_I^{(k)}) \leq \Psi^{res(n-1)}(q_1^{(k)}, \dots, q_I^{(k)}) \leq \Psi^{res}(q_1^{(k)}, \dots, q_I^{(k)}) \leq [q_1^{(k)}, \dots, q_I^{(k)}]. \quad (12)$$

It follows that the monotonically decreasing and bounded sequence $\{\Psi^{res(n)}(q_1^{(k)}, \dots, q_I^{(k)})\}_{n=1}^{\infty}$ converges to a limit vector q^* , with

$$[p_1^{(k)}, \dots, p_I^{(k)}] = p^* (p_{-I}^{(k-1)}) \leq p^* (q_{-I}^{(k-1)}) \leq q^* \leq [q_1^{(k)}, \dots, q_I^{(k)}], \quad (13)$$

thus completing the induction proof. The first inequality in (13) follows from the fact that the smallest equilibrium in a supermodular game is a monotonically increasing vector-function of any parameter (string) such that each player's payoff function in the game is continuous and supermodular in the player's action variable and the parameter, see Theorem 4.2.2 in Topkis (1998). This supermodularity property follows from condition (S.i).

Thus, to complete the verification of the string of inequalities in (13) only the second inequality remains to be substantiated. However, this inequality follows from the fact that $q^* = \lim_{n \rightarrow \infty} \Psi^{res(n)}(q_1^{(k)}, \dots, q_I^{(k)})$ is an equilibrium of the game $\Gamma^{res}(q_{-I}^{(k-1)})$ and hence dominates $\underline{p}^*(q_{-I}^{(k-1)})$ the component-wise *smallest* equilibrium of this game. The fact that q^* is an equilibrium of this game follows from Theorem 2.10 in Vives (2001) since the game $\Gamma^{res}(q_{-I}^{(k-1)})$ is supermodular with continuous payoff functions. ■

We now show that, beyond generating higher equilibrium prices, the merger also results in equilibrium profits for the merged firm that are larger than the aggregate of the pre-merger profits among the I merging firms. Moreover, and perhaps most surprisingly, the remaining (N-I) firms also earn a higher expected profit *after* the merger. We establish these results under the (CM) condition, (the stronger version of (MP) as shown in Lemma 4.3).

We show that these profit comparison results apply, both to the largest and smallest equilibria in the pre-merger and post-merger games. The comparison results are, in particular, important for the *largest* equilibrium, since it is well known from Theorem 7 in Milgrom and Roberts (1990), that under conditions (S) and (CM) the component-wise *largest* equilibrium is *simultaneously* preferred by all firms in the industry. Thus, if multiple equilibria exist, it is most plausible that the *largest* equilibrium will be adopted.

Define for all $i = 1, \dots, N$, $\bar{\pi}_i(pre)$ [$\underline{\pi}_i(pre)$] as the corresponding equilibrium profit values in the pre-merger game. For all firms $i = I+1, \dots, N$, let $\bar{\pi}_i(post)$ [$\underline{\pi}_i(post)$] denote the equilibrium profit in the post-merger game under the largest [smallest] equilibrium. In addition, let $\bar{\pi}^m(post)$ [$\underline{\pi}^m(post)$] denote the merged firm's equilibrium profit in the post-merger game under the largest [smallest] equilibrium.

Theorem 4.5 (*Profit Comparison Before and After the Merger*)

Assume conditions (Q), (S), and (CM) apply.

$$(a) \quad \underline{\pi}^m(post) \geq \sum_{l=1}^I \underline{\pi}_l(pre) \quad (14)$$

$$\underline{\pi}_i(post) \geq \underline{\pi}_i(pre), \quad i = I+1, \dots, N \quad (15)$$

$$(b) \quad \bar{\pi}^m(post) \geq \sum_{l=1}^I \bar{\pi}_l(pre) \quad (16)$$

$$\bar{\pi}_i(post) \geq \bar{\pi}_i(pre), \quad i = I+1, \dots, N \quad (17)$$

Proof: (a) For $i = I + 1, \dots, N$: $\pi_i(p_{-i}^*(pre)) = \pi_i(p_{-i}^*(pre)) \leq \pi_i(p_{-i}^*(pre), p_{-i}^*(post)) \leq \pi_i(p_{-i}^*(post), p_{-i}^*(post)) = \pi_i(p_{-i}^*(post))$ thus proving (15).

The first inequality follows from $p_{-i}^*(post) \geq p_{-i}^*(pre)$, see Theorem 4.4, and the (CM) condition; the second equality follows from $p_{-i}^*(post)$ being a best response to $p_{-i}^*(post)$, since $p_{-i}^*(post)$ is an equilibrium in the post-merger game.

Similarly, $\sum_{i=1}^I \pi_i(p_{-i}^*(pre)) = \sum_{i=1}^I \pi_i(p_{-i}^*(pre)) \leq \sum_{i=1}^I \pi_i(p_{-i}^*(pre), p_{-i}^*(post)) \leq \sum_{i=1}^I \pi_i(p_{-i}^*(post), p_{-i}^*(post)) = \sum_{i=1}^I \pi_i(p_{-i}^*(post))$, verifying (14). The first inequality follows again from $p_{-i}^*(post) \geq p_{-i}^*(pre)$, see Theorem 4.4(a), and the (CM) condition; the second inequality follows from the fact that the vector $(p_{-1}^*(post), \dots, p_{-I}^*(post))$ is a best response price vector for the merged firm to prices $p_{-I}^*(post)$ selected by the remaining firms $I + 1, \dots, N$.

(b) The proof of part (b) is analogous to that of part (a). ■

Going back to industries which start out with N single product firms, theorem ??? implies that the aggregate profits of a coalition are superadditive in the ??? of the coalition. Thus, assume an industry offering a collection of PN products is partitioned into m coalitions of firms $\{B_1, B_2, \dots, B_m\}$ parts by the result of prior merger. Firm j sells l_j products, $j = 1, \dots, m$ where $N = \sum_{j=1}^m l_j$. Consider now an additional merger of two of these coalitions, without loss of generality, B_1 and B_2 . Under the conditions of Theorem ???, we obtain the following corollary which generalizes the corollary on page ??? of Deneckere and Davidson (1985)

Assuming condition (Q_m) let $\pi_i(B_{??})$ and $\pi_i(\bar{B}_i)$ denote the aggregate equilibrium profits of firms $i = 1, \dots, m$ in the pre-merger industry, under the component-wise smallest and largest equilibrium, respectively. Finally, let $\pi(B_1 \cup B_2)$ and $\bar{\pi}(B_1 \cup B_2)$ denote the aggregate profits of the firms arising from the merger of firms 1 and 2, again under the componentwise smallest and largest equilibrium, respectively.

Corollary 4.6 (Aggregate Profits are Supermodular in the Size of the Coalition) Assume conditions (Q_m) , (S_m) and (CM) hold.

$$\pi(B_1 \cup B_2) \geq \pi_1(B_1) + \pi_2(B_2) \quad (18)$$

$$\bar{\pi}(B_1 \cup B_2) \geq \bar{\pi}_1(B_1) + \bar{\pi}_2(B_2). \quad (19)$$

Proof: Immediate from the proof of Theorem 4.8. ■

4.1 Affine Demand and Cost Functions

In this subsection we apply our results to the special case where both the demand and cost functions are affine, but otherwise general, *i.e.*,

$$d_i(p) = a_i - b_i p_i + \sum_{j \neq i} \beta_{ij} p_j, \quad i = 1, \dots, N \quad (20)$$

$$C_i(d_i) = c_i d_i + e_i, \quad i = 1, \dots, N \quad (21)$$

where $\{a_i, b_i, \beta_{ij}, c_i, e_i\}$ are given parameters with $b_i, \beta_{ij} \geq 0$. As explained in the Introduction, this structure is used in many applications. Without loss of generality, assume each product is priced at or above its marginal cost value, *i.e.*, $p_i^{min} = c_i$ for all $i = 1, \dots, N$. It is easily verified that all three of the conditions (Q), (S), and (MC) are satisfied. This implies that both the pre-merger and post-merger games are supermodular. Assume, in addition, that the price sensitivity coefficients satisfy the well known dominant diagonality conditions:

$$(D1) \quad -\frac{\partial d_i}{\partial p_i} \geq \sum_{j \neq i} \frac{\partial d_i}{\partial p_j} \Leftrightarrow b_i \geq \sum_{j \neq i} \beta_{ij}, \quad i = 1, \dots, N$$

$$(D2) \quad \sum_{j=1}^N \frac{\partial d_j}{\partial p_i} \leq 0 \Leftrightarrow b_i \geq \sum_{j \neq i} \beta_{ji}, \quad i = 1, \dots, N$$

These conditions are very intuitive: (D1) states that a uniform price increase for all firms cannot result in an increase of any product's sales volume; (D2) states that if *any* product's price is increased, unilaterally, aggregate sales in the industry do not increase. Under the dominant diagonality conditions, we have that both the pre-merger and post-merger games have a *unique* equilibrium $p^*(pre)$ and $p^*(post)$, respectively. This follows from the fact that the Jacobian of the system of First Order Conditions is a dominant diagonal matrix, see Vives (2001) and Gabay and Moulin (1980). The following corollary is therefore immediate from Theorems 4.4 and 4.5.

Corollary 4.7 Consider an industry with affine demand and cost functions (21, 20). Assume in addition that the dominant diagonality conditions (D1, D2) hold.

(a) There exists a unique equilibrium $p^*(pre)$ in the pre-merger game, and a unique equilibrium $p^*(post)$ in the

post-merger game with $p^*(pre) \leq p^*(post)$.

(b) The equilibrium profits of the merged firm exceed the aggregate equilibrium pre-merger profits of the merging firms. Similarly, the equilibrium profits of all remaining firms increase because of the merger:

$$\begin{aligned}\pi^m(post) &\equiv \pi^m(p^*(post)) \geq \sum_{i=1}^I \pi_i(p^*(pre)) \equiv \sum_{i=1}^I \pi_i(pre) \\ \pi_i(post) &\equiv \pi_i(p^*(post)) \geq \pi_i(p^*(pre)) \equiv \pi_i(pre), \quad i = I + 1, \dots, N\end{aligned}$$

4.2 Cost Efficiencies Resulting from the Merger

Thus far, we assumed that the merger does not affect the cost structure of the products offered by merging firms. Frequently, mergers result in significant cost synergies. Indeed, such synergies are often the driving force, or one of the principal impetuses, behind a merger. For example, in 2005 Proctor & Gamble announced the largest acquisition in its history, agreeing to buy Gillette in a \$57 billion stock deal. The acquisition presented P&G with the opportunity to become the leader in the household and personal care market. The merging firms had reported 2004 profits of \$6.5 billion and \$1.6 billion, respectively. In the traditional “freeze” period following the merger proposal, AT Kearney was retained to assess the cost synergies. (Such assessments by independent consulting firms are routinely undertaken in any significant merger proposal.) The firm estimated the cost synergies at approximately \$1 billion per year; over half the size of the total pre-merger profits of Gillette.

While reduced competition results in price increases, see Theorem 4.4, it is generally believed that cost synergies have the opposite effect. In actuality, whether this can be guaranteed or not depends on the specific way the synergies impact on the cost functions of the products being merged. The simplest synergy model assumes that every *marginal* cost function is shifted by the *same* constant $\sigma > 0$, i.e.,

$$C_i^{post}(d_i) = C_i(d_i) - \sigma d_i, \quad i = 1, \dots, I \quad (22)$$

The pre-merger and post-merger cost functions may be viewed as special cases of a parameterized set of functions $C_i(d_i, \sigma) = C_i(d_i) - \sigma d_i$, with the pre-merger [post-merger] cost function corresponding with $\sigma = 0$ [$\sigma = 1$].

Proposition 4.8 *Assume the merger induces synergies for the cost structures of products $i = 1, \dots, I$, as described by (22). Assume, in addition, that the dominant diagonal condition (D1) applies. These synergies result in price decreases for the smallest and largest post-merger equilibrium compared to their levels in the absence of any cost synergies.*

Proof: By Theorem 4.2.2 in Topkis (1998), it suffices to show that $\frac{\partial^2 \pi^m}{\partial p_i \partial \sigma} \geq 0$ for all $i = 1, \dots, I$ while for all $i = I + 1, \dots, N$ $\frac{\partial^2 \pi_i}{\partial p_i \partial \sigma} = 0$. Note that $\frac{\partial^2 \pi^m}{\partial p_i \partial \sigma} = -\sum_{l=1}^I \frac{\partial C_l(d_l, \sigma)}{\partial p_i \partial \sigma} = -\left\{ \frac{\partial d_i}{\partial p_i} - \sum_{l \neq i} \frac{\partial d_l}{\partial p_i} \right\} > 0$ by the dominant diagonal condition (D1). ■

Thus, a merger associated with cost synergies described by a *uniform* marginal cost reduction as in (22), induces two opposite effects. The “increased market concentration”, by itself, *increases* the price equilibrium; however, the cost synergies induce *decreases* in all equilibrium prices. Which of the two effects dominates, depends on the magnitude of σ .

It should be noted that the price effects described by Proposition 4.7 are specific to a *uniform* reduction of the marginal cost functions of products $1, \dots, I$ by the same constant. If the marginal cost reduction is *product specific*, i.e., $C_i^{post}(d_i) = C_i(d_i) - \sigma_i d_i$, or if it fails to be constant, i.e., $C_i^{post}(d_i) = C_i(d_i) - \sigma(d_i)$ for some non-linear function $\sigma(\cdot)$, it does not appear to be possible to guarantee a reduction of the equilibrium prices as compared to a post-merger equilibrium without such synergies⁴.

Returning to the synergy structure (22), focusing on the largest equilibrium, for example, two values of interest are:

$$\begin{aligned}\sigma^+ &= \min\{\sigma : \bar{p}^*(post|\sigma) \leq \bar{p}^*(pre)\}, \text{ and} \\ \sigma^- &= \min\{\sigma : \bar{p}_i^*(post|\sigma) < \bar{p}_i^*(pre) \text{ for some } i = 1, \dots, N\} \\ &= \min\{\sigma : \bar{p}_i^*(post|\sigma) < \bar{p}_i^*(pre) \text{ for some } i = 1, \dots, I\} \leq \sigma^+, \end{aligned} \quad (23)$$

⁴Farrell and Shapiro (2010a) for example assume that the cost synergies result in a uniform *percentage* reduction of the marginal costs. Their Proposition 1, confining itself to a merger of *two* firms, indeed states as an *assumption* “Suppose that the price charged by the merged firm for each product and non-decreasing in the marginal cost of the other product”.

where $\bar{p}^*(post|\sigma)$ denotes the largest equilibrium in the post-merger game under a given marginal cost savings σ . In other words, σ^+ (σ^-) denotes the minimum cost savings such that all (at least one) of the equilibrium prices decreases after the merger. (The second equality in (23) follows from the fact that if $\bar{p}^*(post|\sigma) \geq \bar{p}^*(pre)$ for all $i = 1, \dots, I$, then ranking applies to the prices of the remaining firms, since these are best responses to the prices selected by the merging firms.)

In assessing whether the proposed merger is likely to “lessen competition” one may then evaluate whether the magnitude of σ^- (σ^+) is a realistic possibility. (Both σ^- and σ^+ can easily be computed by embedding the tatônnement scheme in a bi-section search for the “break even” value of σ .)

Alternatively, one may assume that the merger results in a marginal cost reduction of *one* of the products of the merging firms only, and calculate σ^+ on this basis. This approach was followed, for example, by Nevo (2000) for the ready-to-eat cereal industry. After carefully estimating the demand functions of the different ready-to-eat cereal products, Nevo (2000) simulates various potential pairwise mergers among the six major national competitors. In his Table 5, the author reports the price increases that result from various potential mergers, assuming that the cost functions remain unaltered. Table 6 proceeds to report what marginal cost reductions for individual products would restore the equilibrium prices to levels at or below the pre-merger values. These “break even” values are then discussed to evaluate whether the net effect of the merger on prices is likely to be positive or negative.

4.3 Mergers of multi-product firms

In our base model, we consider a (pre-merger) industry in which each of the products is sold by an independent company. In this subsection we extend our results to the more prevalent case where some or all of the existing firms sell more than one product. (We continue to assume that each product is sold by a single firm.) We characterize the equilibrium consequences of a merger between *two* of these firms.

Assume there are n firms in the industry, numbered $i = 1, \dots, n$ with firm i offering $l_i \geq 1$ products to the market and $N = \sum_{i=1}^m l_i$. We thus use a double index to differentiate among the various products, with product (i, j) referring to the j -th product offered by firm i , $i = 1, \dots, n$ and $j = 1, \dots, l_i$. For firm i , let $p_i = (p_{i1}, \dots, p_{il_i})$ denote the firm’s price vector and let p denote the N -dimensional vector containing all prices for all N products. The profit function for firm i is thus given by:

$$\pi_i(p) = \sum_{j=1}^{l_i} [p_{ij} d_{ij}(p) - c_{ij}(d_{ij}(p))]. \quad (24)$$

Without loss of generality, assume firms 1 and 2 merge to create a new merged firm m with profit function

$$\pi^m(p) = \pi_1(p) + \pi_2(p) \quad (25)$$

(As in the base model, we initially assume that the merger leaves all cost functions unaltered.) To ensure that both the pre-merger Γ^{pre} and the post-merger game Γ^{post} are supermodular we need a variant of condition (S):

(S_m) : (Strategic Complementarity)

- (i) For all $i = 1, \dots, n$, the profit function $\pi_i(p_i, p_{-i})$ is a supermodular function of the vector p_i and has increasing differences with respect to p_{-i} .
- (ii) The profit function of the merged firm, $\pi^m(p_1, \dots, p_n)$ is a supermodular function of (p_1, p_2) and has increasing differences with respect to the remaining prices $p_{-\{1,2\}} = (p_3, \dots, p_n)$.

Along with the fact that all n firms in the pre-merger game Γ^{pre} and all $n - 1$ firms in the post-merger game Γ^{post} have action spaces that are compact lattices, condition (S_m) guarantees that both games are supermodular. Similarly, we need a slight variant of the quasi-concavity condition (Q):

(Q_m) : The profit functions $\pi_i(p_1, p_2, \dots, p_n)$ are strictly quasi-concave functions of firm i ’s price vector p_i , $i = 1, \dots, n$.

As in the base model, we need to consider restricted versions of the pre-merger game in which only firms 1 and 2 are able to vary their price vectors, under *given* price choices $p_{-\{1,2\}} \equiv (p_3, \dots, p_n)$ for the remaining firms. We refer to this restricted duopoly as $\Gamma^{res}(p_{-\{1,2\}})$.

In view of the strict quasi-concavity condition (Q_m) , each firm $i = 1, \dots, N$ has a *unique* best response $\Psi_i(p_{-i})$ to any given choice of prices by the remaining firms. In view of the supermodularity condition (S_m) , the merged firm has a component-wise smallest [largest] best response $\underline{\Psi}^m(p_{-\{1,2\}})$ [$\bar{\Psi}^m(p_{-\{1,2\}})$] to any given price vector $p_{-\{1,2\}}$ of the remaining firms $i = 1, \dots, N$.

As before, let $\underline{p}^*(pre)$ and $\bar{p}^*(pre)$ [$\underline{p}^*(post)$ and $\bar{p}^*(post)$] denote the component-wise smallest and largest equilibrium in the pre-merger [post-merger] game. Let $\underline{\pi}_i(pre)$, $\bar{\pi}_i(pre)$ ($i = 1, \dots, n$) and $\underline{\pi}_i(post)$, $\bar{\pi}_i(post)$ ($i = 1, \dots, n$) ($i = 3, \dots, n$) denote the associated profit values. Finally, $\underline{\pi}^m(post)$ and $\bar{\pi}^m(post)$ denote the profit values of the merged firm in these two equilibria of Γ^{post} .

Theorem 4.9 (Price and Profit Comparisons for Mergers of Multi-Product Firms)

Assume conditions (Q_m) , (S_m) , and (CM) hold.

(a) $\underline{p}^*(pre) \leq \underline{p}^*(post)$ and $\bar{p}^*(pre) \leq \bar{p}^*(post)$

(b) $\underline{\pi}^m(post) \geq \underline{\pi}_1(pre) + \underline{\pi}_2(pre)$

$\bar{\pi}^m(post) \geq \bar{\pi}_1(pre) + \bar{\pi}_2(pre)$

$\underline{\pi}_i(post) \geq \underline{\pi}_i(pre)$, $i = 3, \dots, n$

$\bar{\pi}_i(post) \geq \bar{\pi}_i(pre)$, $i = 3, \dots, n$

Proof: (a) We show $\underline{p}^*(pre) \leq \underline{p}^*(post)$, the comparison of the pair of *largest* price equilibrium being analogous. The proof goes along the lines of those of Theorem 4.3 and 4.4. Define the following pair of tailored tâtonnement schemes:

Pre-Merger Increasing Scheme:

Step 0: $p^{(0)} := p^{min}$, $k = 1$

Step 1: For $i = 1, 2$ set $(p_1^{(k)}, p_2^{(k)}) = \underline{p}^*(p_{-\{1,2\}}^{(k-1)})$, the smallest equilibrium of the restricted duopoly game $\Gamma^{res}(p_{-\{1,2\}}^{(k-1)})$, under a fixed price vector $p_{-\{1,2\}}^{(k-1)}$ for the remaining firms;

For $i = 3, \dots, n$, set $p_i^{(k)} = \Psi_i(p_{-i}^{(k-1)})$; $k=k+1$.

Post-Merger Increasing Scheme:

Step 0: $q^{(0)} := p^{min}$, $k = 1$

Step 1: For $i = 1, 2$ set $(q_1^{(k)}, q_2^{(k)}) = \underline{\Psi}^m(q_{-\{1,2\}}^{(k-1)})$;

For firms $i = 3, \dots, n$, set $q_i^{(k)} = \Psi_i(q_{-i}^{(k-1)})$; $k=k+1$.

A straightforward extension of the proof of Lemma 4.2 establishes that, once again,

$$\begin{aligned} p^{(0)} \leq p^{(1)} \leq \dots \leq p^{(k)} \leq \lim_{k \rightarrow \infty} p^{(k)} &= \underline{p}^*(pre) \\ q^{(0)} \leq q^{(1)} \leq \dots \leq q^{(k)} \leq \lim_{k \rightarrow \infty} q^{(k)} &= \underline{p}^*(post) \end{aligned} \quad (26)$$

It thus suffices to prove that $q^{(k)} \geq p^{(k)}$. The proof proceeds, once again, by induction. $p^{min} = q^{(0)} \geq p^{(0)} = p^{min}$. Assume, therefore, that $q^{(k-1)} \geq p^{(k-1)}$ for some $k \geq 1$. Since p^{min} and p^{max} are selected so as not to impact on the best response choices in either the pre- or post-merger industry, we have that the price vector $(q_1^{(k)}, q_2^{(k)})$ is an interior point of the price space and therefore satisfies the following First Order Conditions:

$$\begin{aligned} 0 &= \frac{\partial \pi^m(q_1^{(k)}, q_2^{(k)}, q_3^{(k-1)}, \dots, q_n^{(k-1)})}{\partial p_{1j}} = \frac{\partial \pi^1(q_1^{(k)}, q_2^{(k)}, q_3^{(k-1)}, \dots, q_n^{(k-1)})}{\partial p_{1j}} + \frac{\partial \pi^2(q_1^{(k)}, q_2^{(k)}, q_3^{(k-1)}, \dots, q_n^{(k-1)})}{\partial p_{1j}} \\ &= \frac{\partial \pi^1(q_1^{(k)}, q_2^{(k)}, q_3^{(k-1)}, \dots, q_n^{(k-1)})}{\partial p_{1j}} + \sum_{r=1}^{l_2} \{q_{2r}^{(k)} - c'_{2r}(d_{2r})\} \frac{\partial d_{2r}}{\partial p_{1j}}, \quad \forall j = 1, \dots, l_1. \end{aligned} \quad (27)$$

$$\begin{aligned} 0 &= \frac{\partial \pi^m(q_1^{(k)}, q_2^{(k)}, q_3^{(k-1)}, \dots, q_n^{(k-1)})}{\partial p_{2j}} \\ &= \frac{\partial \pi^2(q_1^{(k)}, q_2^{(k)}, q_3^{(k-1)}, \dots, q_n^{(k-1)})}{\partial p_{2j}} + \sum_{r=1}^{l_1} \{q_{1r}^{(k)} - c'_{1r}(d_{1r})\} \frac{\partial d_{1r}}{\partial p_{2j}}, \quad \forall j = 1, \dots, l_2. \end{aligned} \quad (28)$$

It follows from the (CM) condition and (1) that the second term to the far right of (27) [(28)] is non-negative so that

$$\frac{\partial \pi^i(q_1^{(k)}, q_2^{(k)}, q_3^{(k-1)}, \dots, q_n^{(k-1)})}{\partial p_{ij}} \leq 0, \quad \forall i = 1, 2 \text{ and } j = 1, \dots, l_i \quad (29)$$

We first show that (29), along with conditions (Q_m) and (S_m) imply that

$$\Psi_1(q_2^{(k)}, q_3^{(k-1)}, \dots, q_n^{(k-1)}) \leq q_1^{(k)}, \text{ and} \quad (30)$$

$$\Psi_2(q_1^{(k)}, q_3^{(k-1)}, \dots, q_n^{(k-1)}) \leq q_2^{(k)}. \quad (31)$$

Let Ψ^{res} denote the joint best response operator in the restricted duopoly $\Gamma^{res}(q_{-\{1,2\}}^{(k-1)})$, which is, again, a supermodular game by condition (S_m). (By the strict quasi-concavity condition Ψ^{res} is uniquely defined.) Let $\Psi^{res(r)}$ denote the r -fold application of this operator, $r = 1, 2, \dots$. Since Ψ^{res} is a non-increasing operator, we obtain, by induction that

$$[p_1^{(k)}, p_2^{(k)}] = \underline{p}^*(p_{-\{1,2\}}^{(k-1)}) \leq \underline{p}^*(q_{-\{1,2\}}^{(k-1)}) \leq q^* \leq \Psi^{res(n)}([q_1^{(k)}, q_2^{(k)}]) \leq \Psi^{res(n-1)}([q_1^{(k)}, q_2^{(k)}]) \leq \dots \leq [q_1^{(k)}, q_2^{(k)}] \quad (32)$$

where $q^* = \lim_{n \rightarrow \infty} \Psi^{res(n)}([q_1^{(k)}, q_2^{(k)}])$ is an equilibrium of the restricted duopoly and hence component-wise larger than $\underline{p}^*(q_{-\{1,2\}}^{(k-1)})$, the smallest equilibrium in this game, thus verifying the second inequality in (32). The first inequality follows from the induction assumption $q^{(k-1)} \geq p^{(k-1)}$ and in particular $q_{-\{1,2\}}^{(k-1)} \geq p_{-\{1,2\}}^{(k-1)}$, as well as the fact that the smallest equilibrium in a supermodular game is a monotonically increasing vector function of any parameter (string) such that each player's payoff function in the game is continuous and supermodular in the player's action variables and the parameters, see Theorem 4.2.2 in Topkis (1998). In addition to (32), we have:

$$p_i^{(k)} = \Psi_i(p^{(k-1)}) \leq \Psi_i(q^{(k-1)}) = q^{(k)}, \quad i = 3, \dots, n \quad (33)$$

where the inequality follows from the induction assumption and the fact that the best response operator Ψ_i in the supermodular pre-merger game is non-increasing. (32) and (33) together complete the induction step, *i.e.*, they verify that $q^{(k)} \geq p^{(k)}$.

It remains to be shown that (27) and (28) hold. We prove (27), the proof of (28) being analogous. Since the profit function π_1 is strictly quasi-concave in firm 1's price vector p_1 , it has a *unique* local maximum for any given price vectors p_2, \dots, p_n of the remaining firms. The (unique) best response $\Psi_1(q_2^{(k)}, q_3^{(k-1)}, \dots, q_n^{(k-1)})$ can be obtained as the limit of a sequence $\{p_1(t)\}_{t=1}^{\infty}$ which optimizes the individual prices $(p_{11}, p_{12}, \dots, p_{1l_1})$ of firm 1 in a Round Robin way. Thus, to prove (27) it suffices to show that $p_1(t) \leq q_1^{(k)}$ for all $t = 0, 1, 2, \dots$. We prove this by induction. Clearly $p_1(0) = q_1^{(k)}$. Assume $p_1(t) \leq q_1^{(k)}$ for some $t \geq 0$. Let $r \in \{1, \dots, l_1\}$ denote the index of the product whose price is being optimized in the $(t+1)$ -st iteration. Note that $\operatorname{argmax}_{p_{1r}} \{\pi_1(p_{11}(t), \dots, p_{1r}, \dots, p_{1l_1}(t), q_2^{(k)}, q_3^{(k-1)}, \dots, q_n^{(k-1)})\} \leq \operatorname{argmax}_{p_{1r}} \{\pi_1(q_{11}^{(k)}, \dots, p_{1r}, \dots, q_{1l_1}^{(k)}, q_2^{(k)}, q_3^{(k-1)}, \dots, q_n^{(k-1)})\} \leq q_{1r}^{(k)}$, thus verifying $p_1(t+1) \leq q_1^{(k)}$ and completing the induction proof. (The first inequality follows from the supermodularity condition (S.i), while the second inequality follows from $\frac{\partial \pi_1(q_1^{(k)}, q_2^{(k)}, q_3^{(k-1)}, \dots, q_n^{(k-1)})}{\partial p_{1r}} \leq 0$ (see (24),) and the quasi-concavity of the single variable function $\pi_1(q_{11}^{(k)}, \dots, q_{1l_1}^{(k)}, q_2^{(k)}, q_3^{(k-1)}, \dots, q_n^{(k-1)})$.)

(b) The proof of part (b) is analogous to that of Theorem 4.5, using part (a) of this theorem. ■

5 Connection with Antitrust Policy

Farrell and Shapiro (2010a) discuss the difficulty government agencies such as the Antitrust Division of the DOJ and the FTC face in determining which proposed mergers to scrutinize to evaluate whether the merger will “substantially lessen the competition in the industry,” the phrase used in Section 7 of the 1914 Clayton Act, as modified in 1950. Since 2000, firms with a “transaction value” above \$50 million, who wish to engage in a merger, are required to notify the DOJ and FTC of their plans, an outgrowth of the 1976 Hart-Scott-Rodino Act. (The transaction value is defined as the aggregate capital value of the merging firms.) As a consequence, the DOJ and FTC reviewed, in 2008 alone, 1,726 proposed mergers and acquisitions with an aggregate value of more than \$1 trillion, see Farrell and Shapiro (2010a).

The Clayton Act requires the government agency to “prove” in court that a proposed merger would result in a “substantial lessening of the competition in the industry,” based on a comprehensive industry study. There is, therefore, a strong need for a fairly simple “pre-screening” test to identify which of the thousands of merger proposals are most likely to result in the greatest reductions of the competitive dynamics in an industry and a commensurate reduction in consumer welfare.

Traditionally, the government has used simple market concentration measures as their “litmus” test, in particular the so-called Herfindahl-Hirschman Index (HHI), defined as the sum of the squares of the (anticipated) post-merger market shares in the industry. Many economists have argued that this HHI-measure is a relatively limited predictor in the particular case of differentiated products, and have proposed alternatives instead. Building on ideas developed by O'Brien and Salop (2000) and Werden (1996), Farrell and Shapiro (2010a) have advocated the use of so-called Upward Pricing Pressure (UPP) measures instead of HHI. Based on the pre-merger equilibrium $p^*(pre)$, UPP increases

for firms $i = 1, \dots, I$ are defined in Farrell and Shapiro (2010a, equation (0)) as follows:

$$T_i \equiv \sum_{l=1, l \neq i}^I (p_l^* - C_l) \delta_{il} > 0^5, \quad (34)$$

where $p^* = p^*(pre)$ and $\delta_{il} \equiv \frac{\partial d_l(p^*)}{\partial p_i} / \left| \frac{\partial d_i(p^*)}{\partial p_i} \right|$ denotes the *diversion ratio* from product i to product l , a term coined by Farrell and Shapiro (2010a). Let ϵ_{ii} and ϵ_{il} respectively denote product i 's own and cross-price elasticity with respect to product l , measured at the pre-merger equilibrium p^* , *i.e.*,

$$\epsilon_{ii} = \frac{\partial d_i(p^*)}{\partial p_i} \frac{p_i^*}{d_i(p^*)} \text{ and } \epsilon_{il} = \frac{\partial d_l(p^*)}{\partial p_i} \frac{p_i^*}{d_l(p^*)} \quad (35)$$

Clearly,

$$\delta_{il} = \frac{-\epsilon_{il}}{\epsilon_{ii}} \frac{d_i(p^*)}{d_l(p^*)}, \quad i \neq l \quad (36)$$

In other words, the diversion ratio from product i to product l equals the ratio of the own- and cross-price elasticity multiplied by the ratio of product l and i 's (pre-merger) sales volume.

Note that

$$-T_i \frac{\partial d_i(p^*)}{\partial p_i} = \frac{\partial \pi^m(p^*)}{\partial p_i}, \quad i = 1, \dots, I \quad (37)$$

To verify this identity, recall from (3) that

$$\frac{\partial \pi^m(p^*)}{\partial p_i} = \frac{\partial \pi_i(p^*)}{\partial p_i} - \frac{T_i \partial d_i(p^*)}{\partial p_i} = \frac{-T_i \partial d_i(p^*)}{\partial p_i} > 0, \quad i = 1, \dots, I,$$

since $T_i > 0$ and $\frac{\partial d_i}{\partial p_i} < 0$, see (1). Assume, for example, that the pre-merger industry had settled on the smallest equilibrium $\underline{p}^*(pre)$. Then, the larger the UPP-measures $\{T_1, \dots, T_I\}$ are, the larger the price increases for the products of the merging firms in their best response to the pre-merger equilibrium $\underline{p}^*(pre)$, *i.e.*, the larger $|\underline{\Psi}^m(\underline{p}^*(pre)) - \underline{p}^*|$.

While unstated in Farrell and Shapiro (2010a), the UPP-measures may thus be viewed as proxies for the ultimate measure of interest:

$$|\underline{p}^*(post) - \underline{p}^*(pre)|_\infty \geq |\underline{\Psi}^m(\underline{p}^*(pre)) - \underline{p}^*|_\infty. \quad (38)$$

The inequality (38) follows from the fact that $\underline{p}^*(pre) \leq \underline{p}^*(post)$, see Theorem 4.4 (a). This implies the simultaneous optimization variant of the tâtonnement scheme in the post-merger game Γ^{post} , which starts at $\underline{p}^*(pre) \leq \underline{p}^*(post)$, generates an *increasing* sequence of price vectors which converges to $\underline{p}^*(post)$:

$$\underline{p}^*(pre) \leq \underline{\Psi}^m(\underline{p}^*(pre)) \leq \underline{\Psi}^{m(n)}(\underline{p}^*(pre)) \leq \underline{\Psi}^{m(n+1)}(\underline{p}^*(pre)) \leq \underline{p}^*(post). \quad (39)$$

Farrell and Shapiro (2010a) describe the above iterative scheme when motivating the use of the UPP-measures, however without monotonicity or convergence proofs.

Farrell and Shapiro (2010a,b) also argue that it is considerably easier to evaluate or estimate UPP measures as compared to conducting a full blown merger simulation. Indeed, they argue that diversion ratios can often be estimated or approximated without having to estimate the industry's complete set of demand functions⁶. This has been debated by various authors such as Epstein and Rubinfeld (2010) and Schmalensee (2009).

One of Farrell and Shapiro's arguments is that, in contrast to traditional market concentration measures or a full blown merger simulation, the UPP measures do not require an upfront specification of the boundaries of the market being considered. This is always a difficult question to resolve. As an example, the DOJ, when litigating to prevent the merger between Oracle and Peoplesoft, identified the relevant market of human relations and financial management systems as consisting of these two firms and SAP. However, the court rejected the DOJ's argument, identifying other suppliers of related software and faulting the DOJ for an inadequate specification of the relevant product market.

It is, of course, true that in most model specifications, estimates of own and cross-price elasticities among the products of the merging firms depend on which set of firms and products are included in the market model. However, it could be argued that these elasticities are relatively insensitive to the market boundary choice. In fact, the DOJ/FTC

⁵Farrell and Shapiro implicitly assume that the UPP-measures T_i are non-negative. As discussed above, this is guaranteed to hold under the (CM) condition.

⁶Farrell and Shapiro (2010b) states: "for example, horizontal or documentary evidence from win/loss reports, discount approval processes, or customer switching patterns can be highly informative about the diversion ratio."

define the relevant market precisely by adding firms and products to the market of interest until the price elasticities of the firms are insensitive to the further addition of new products.

As to Farrell and Shapiro (2010a,b)'s argument that UPP measures are easier to evaluate than complete merger simulations, it should be noted that merger simulations may be reduced to implementing, say, the "simultaneous optimization" variant of the tâtonnement scheme in the post-merger game, starting from the current (pre-merger) equilibrium price vector, see Lemma 4.1 and Theorem 4.4.

Schmalensee (2009), while praising Farrell and Shapiro (2010a) for "having made a significant contribution that has the potential to improve merger enforcement," takes issue with their recommendation to use the UPP measure as the indicator by which to rank different merger proposals as the "quantity is unrelated to any measure of customer harm". Instead, Schmalensee (2009) argues for the use of an approximate estimate of post-merger *price changes* and proposes Price Change Assuming Linearity (PCAL) as an alternative to UPP. PCAL calculates the post-merger equilibrium assuming all cost-functions are linear and all demand functions for the products of the merging firms are *linear* as well. An additional major assumption is that the demand functions of the products of the merging firms do not depend on the prices of the other firms in the industry, effectively assuming that the merged firm can operate as a monopolist. The results in this paper show that the post-merger equilibrium can be calculated as the limit vector of an increasing sequence of best response price vectors to the pre-merger (observed) equilibrium. To compute this sequence, one needs to postulate a system of demand functions. If the above linear functions - without dependence on prices of non-merging firms - is deemed adequate, this can be used to generate the price change estimates. However, if other specifications (that result in supermodular profit functions) seem more reasonable, these can be evaluated with little effort as well, on the basis of the above simple tâtonnement scheme.

References

- Adida, E., V. DeMiguel. 2011. Supply Chain Competition with Multiple Manufacturers and Retailers. *Operations Research* 49(6) 1095–1100.
- Allon, G., A. Federgruen. 2007. Competition in service industries. *Operations Research* 55(1).
- Baker, J.B., T.F. Bresnahan. 1985. The gains from merger or collusion in product-differentiated industries. *The Journal of Industrial Economics* 33(4) 427–444.
- Berry, S., A. Pakes. 1993. Some applications and limitations of recent advances in empirical industrial organization: Merger analysis. *The American Economic Review* 83(2) 247–252.
- Cabral, LMB, J.M. Villas-Boas. 2005. Bertrand Supertraps. *Management Science* 51 599–613.
- Davidson, C., R. Deneckere. 1984. Horizontal mergers and collusive behavior. *International Journal of Industrial Organization* 2(2) 117–132.
- Deneckere, R., C. Davidson. 1985. Incentives to form coalitions with Bertrand competition. *The RAND Journal of Economics* 16(4) 473–486.
- Dube, J.P. 2005. Product Differentiation and Mergers in the Carbonated Soft Drink Industry. *Journal of Economics and Management Strategy* 14(4).
- Epstein, R.J., D.L. Rubinfeld. 2010. Understanding UPP. *The BE Journal of Theoretical Economics* 10(1) 21.
- Farahat, A., G. Perakis. 2007. A comparison of Bertrand and Cournot profit in oligopolies with differentiated products. Tech. rep., Working paper.
- Farrell, J., C. Shapiro. 1990. Horizontal mergers: An equilibrium analysis. *The American Economic Review* 80(1) 107–126.
- Farrell, J., C. Shapiro. 2010a. Antitrust Evaluation of Horizontal Mergers: An Economic Alternative to Market Definition. *The B.E. Journal of Theoretical Economics* 10(1).
- Farrell, J., C. Shapiro. 2010b. Upward Pricing Pressure in Horizontal Merger Analysis: Reply to Epstein and Rubinfeld. *The BE Journal of Theoretical Economics* 10(1) 41.
- Gabay, D., H. Moulin. 1980. On the uniqueness and stability of Nash equilibria in noncooperative games. *Applied stochastic control in econometrics and management science* 271–293.
- Gupta, D., Y. Gerchak. 2002. Quantifying operational synergies in a merger/acquisition. *Management Science* 48(4) 517–533.
- Hackner, J. 2000. A NOTE ON PRICE AND QUANTITY COMPETITION IN DIFFERENTIATED OLIGOPOLIES. *Journal of Economic Theory* 93(2) 233.

- Hausman, J., G. Leonard, J.D. Zona. 1994. Competitive analysis with differentiated products. *Annales d'Economie et de Statistique* 159–180.
- Hausman, J.A., G.K. Leonard. 1999. Efficiencies from the consumer viewpoint. *Geo. Mason L. Rev.* 7 707.
- Iyer, A.V., A. Jain. 2004. Modeling the impact of merging capacity in production-inventory systems. *Management Science* 1082–1094.
- Levy, D.T., J.D. Reitzes. 1992. Anticompetitive effects of mergers in markets with localized competition. *Journal of Law, Economics, and Organization* 8(2) 427.
- Milgrom, P., J. Roberts. 1990. Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica* 1255–1277.
- Nevo, A. 2000. Mergers with differentiated products: The case of the ready-to-eat cereal industry. *The RAND Journal of Economics* 31(3) 395–421.
- O'Brien, D.P., S.C. Salop. 2000. Competitive effects of partial ownership: Financial interest and corporate control. *Antitrust Law Journal* 67 559.
- Perry, M.K., R.H. Porter. 1985. Oligopoly and the incentive for horizontal merger. *The American Economic Review* 75(1) 219–227.
- Salant, S.W., S. Switzer, R.J. Reynolds. 1983. Losses from horizontal merger: the effects of an exogenous change in industry structure on Cournot-Nash equilibrium. *The Quarterly Journal of Economics* 98(2) 185–199.
- Schmalensee, R. 2009. Should New Merger Guidelines Give UPP Market Definition? *Global Competition Review* .
- Shubik, M., R. Levitan. 1980. *Market structure and behaviour*. Harvard University Press.
- Singh, N., X. Vives. 1984. Price and quantity competition in a differentiated duopoly. *The RAND Journal of Economics* 15(4) 546–554.
- Steiner, P.O. 1975. *Mergers: Motives, effects, policies*. University of Michigan Press Ann Arbor, MI.
- Szidarovszky, F., S. Yakowitz. 1982. Contributions to Cournot oligopoly theory. *Journal of Economic Theory* 28(1) 51–70.
- Thomadsen, R. 2005. The effect of ownership structure on prices in geographically differentiated industries. *The Rand Journal of Economics* 36(4).
- Topkis, D.M. 1998. *Supermodularity and complementarity*. Princeton University Press.
- Vives, X. 1985. On the efficiency of Bertrand and Cournot equilibria with product differentiation. *Journal of Economic Theory* 36(1) 166–175.
- Vives, X. 1990. Nash equilibrium with strategic complementarities. *Journal of Mathematical Economics* 19(3) 305–321.
- Vives, X. 2001. *Oligopoly pricing: Old ideas and new tools*. The MIT Press.
- Werden, G.J. 1996. A robust test for consumer welfare enhancing mergers among sellers of differentiated products. *The Journal of Industrial Economics* 44(4) 409–413.
- Werden, G.J. 1997. Simulating the Effects of Differentiated Products Mergers: A Practical Alternative to Structural Merger Policy. *Geo. Mason L. Rev.* 5 363.
- Werden, G.J., L.M. Froeb. 1994. The effects of mergers in differentiated products industries: Logit demand and merger policy. *Journal of Law, Economics, and Organization* 10(2) 407.
- Werden, G.J., L.M. Froeb. 1996. Simulation as an alternative to structural merger policy in differentiated products industries. M.B. Coate, A.N. Kleit, eds., *The economics of the antitrust process*. Springer, 65–88.
- Whinston, M. 2006. *Lectures on Antitrust Economics*. Cambridge, The MIT Press.
- Whinston, M.D. 2007. Antitrust Policy toward Horizontal Mergers. M. Armstrong, R. Porter, eds., *Handbook of Industrial Organization*, vol. 3, chap. 36. North Holland, 2369–2440.
- Williamson, O.E. 1968. Economies as an antitrust defense: The welfare tradeoffs. *The American Economic Review* 58(1) 18–36.
- Zhao, J., E. Howe. 2010. Inverse matrices and merger incentives from Bertrand competition. *Working Paper, Department of Economics, University of Saskatchewan* .