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Abstract

In this paper, we postulate a general class of price competition models with Mixed Multinomial Logit demand functions under affine cost functions. We first characterize the equilibrium behavior of this class of models in the case where each product in the market is sold by a separate, independent firm and customers share a common income level. We identify a simple and very broadly satisfied condition under which a Nash equilibrium exists while the set of Nash equilibria coincides with the solutions of the system of First Order Condition equations, a property of essential importance to empirical studies. This condition specifies that in every market segment, each firm captures less than 50% of the *potential* customer population when pricing at a level which, under the condition, can be shown to be an upper bound for a rational price choice for the firm irrespective of the prices chosen by its competitors. We show that under a somewhat stronger, but still broadly satisfied version of the above condition, a *unique* equilibrium exists. We complete the picture, establishing the existence of a Nash equilibrium, indeed a *unique* Nash equilibrium, for markets with an *arbitrary* degree of concentration; under sufficiently tight price bounds. We then discuss two extensions of our model: unequal customer income and a continuum of customer types. A discussion of the multi product case is included in the appendix. The paper concludes with a discussion of implications for structural estimation methods.

1 Introduction and Summary

Our primary goal in this paper is to characterize the equilibrium behavior of price competition models with Mixed Multinomial Logit (MMNL) demand functions under affine cost structures. In such models, the market is partitioned into a finite set or a continuous spectrum of customer segments, differentiated by, for example, demographic attributes, income level, and/or geographic location. In each market segment, the firms' sales volumes are given by a Multinomial Logit Model (MNL). In spite of the huge popularity of MMNL models in both the theoretical and empirical literature, it is not known, in general, whether a Nash equilibrium (in pure strategies[1]) of prices exists, and whether the equilibria can be uniquely characterized as

[1]Henceforth, 'equilibrium' will refer to pure strategy equilibrium unless otherwise stated.

the solutions to the system of First Order Condition (FOC) equations. (This system of equations is obtained by specifying that all firms' marginal profit values equal zero.) Indeed, as elaborated on in the next section, there are many elementary price competition models in which either *no* or a *multiplicity* of Nash equilibrium exist.

Characterization of the equilibrium behavior in price competition models with MMNL demand functions has remained a formidable challenge because the firms' profit functions fail, in general, to have any of the standard structural properties under which the existence of an equilibrium can be established. For example, the profit functions fail to be quasi-concave. (When firms offer multiple products, this quasi-concavity property is absent, even in a *pure* rather than a *mixed* MNL model, as shown by Hanson and Martin (1996), exhibiting a counterexample in a 3-product monopolist model with logit demand functions.)

Consider, for example, the seminal paper by Berry et al. (1995) studying market shares in the United States automobile industry which introduced, at least in the empirical industrial organization literature, a new estimation methodology to circumvent the problem that prices, as explanatory variables of sales volumes, are typically endogenously determined. The paper postulates a MMNL model for the industry. One of the empirical methods developed in the paper is based on estimating the model parameters as those under which the observed price vector satisfies the FOC equations. The authors acknowledge (in their footnote 12) that it is unclear whether their model possesses an equilibrium, let alone a unique equilibrium. Even if these questions can be answered in the affirmative, so that the observed price vector can be viewed as the unique price equilibrium, it is unclear whether it is necessarily identified by the FOC equations which the estimation method relies on[2].

In a more recent example, Thomadsen (2005b) pointed out that in many empirical studies the distance between the consumer and each of the competing product outlets or service providers is naturally and essentially added to the specification of the utility value. (Examples following this practice include Manuszak (2000), Dube et al. (2002), Bradlow et al. (2005), Thomadsen (2005a), Davis (2006) and Allon et al. (2011).) Distance attributes depend jointly on the firm and the consumer. Such geography-dependent utility functions can be cast as special cases of the general model in Caplin and Nalebuff (1991) the most frequently employed foundation for the existence of an equilibrium. However, Thomadsen (2005b) points out that the conditions in Caplin and Nalebuff (1991) that guarantee the existence of an equilibrium do not apply to such specifications, except for very restrictive geographical distributions of the (potential) consumer base. Similar difficulties in the application of the Caplin-Nalebuff existence conditions arise when the utility functions involve other attributes that depend jointly on the firm/customer type combination, for example brand loyalty characteristics; see Section 3.

We identify a simple and very broadly satisfied condition under which a Nash equilibrium exists and the set of Nash equilibria coincides with the solutions of the system of FOC equations, a property of essential importance to empirical studies. Our existence condition merely requires that any single product captures less than a majority of the *potential* customer population in any of the market segments; moreover, this market share restriction only needs to hold when the product is priced at a level, which under the condition, is shown to be an *upper bound* for a rational price choice, irrespective of the prices charged for the competing

[2]In their footnote 12, Berry et al. (1995) wrote "We assume that a Nash equilibrium to this pricing game exists, and that the equilibrium prices are in the interior of the firms' strategy sets (the positive orthant). While Caplin and Nalebuff (1991) provide a set of conditions for the existence of equilibrium for related models of single product firms, their theorems do not easily generalize to the multi-product case. However, we are able to check numerically whether our final estimates are consistent with the existence of an equilibrium. Note that none of the properties of the estimates require uniqueness of equilibrium, although without uniqueness it is not clear how to use our estimates to examine the effects of policy and environmental changes.". We explain several of the reasons why the conditions in Caplin and Nalebuff (1991) fail to apply to the BLP model, beyond it's multi-product feature in Section 6.1 footnote 10.

products. To guarantee *uniqueness* of the Nash equilibrium a second condition is needed, restricting any given product's market share to a *third* of the *potential* market. No restrictions, whatsoever, are required with respect to the distribution of population sizes across the different market segments.

We develop our theory, first, under the assumption that the market is partitioned into a finite set of segments, such that in each segment market shares are determined on the basis of a pure MNL model. Many empirical studies follow this approach, segmenting the market geographically and/or on the basis of demographic attributes (e.g., gender, race, age, and income bracket). Other empirical models consider a *continuum* of customer types, by treating some of the parameters in the consumer choice model as continuous random variables. While we show that all of our results carry over to such settings, it is more difficult to verify our existence and uniqueness conditions since the market share restrition has to apply for each of the market segments or customer types. Indeed, the condition may, sometimes, fail to hold when the modeler assumes that some of the coefficients in the utility functions have distributions with infinite support, thus allows for rare customer types with arbitrary relative weights for different attributes. However, recent studies, *e.g.,* Nevo (2001), have observed that little is gained in terms of market specification by allowing for such arbitrary random shocks in the utility function coefficients, see Section 4 for a more complete discussion.

Our results differ from those in the seminal paper by Caplin and Nalebuff (1991) in three ways: (i) the model specification, (ii) the conditions guaranteeing existence of a Nash equilibrium, and (iii) the analytical approach.

In terms of the *model specification*, our class of MMNL models generalizes that of Caplin and Nalebuff (1991), itself a generalization of many existing models in the industrial organization literature. In particular, along with a similar utility measure for the outside option, our MMNL model is based on postulating a utility function for each product and market segment which consists of three parts: the first component is an arbitrary function of the product's non-price attributes and the non-income related customer characteristics in the given market segment. The second term captures the joint impact of the product's price and the customer's income level via a general product dependent price-income sensitivity function of both, merely assumed to be concave and decreasing in the price variable. The third and final term denotes a random utility component with an extreme value distribution as in standard Multinomial Logit models. Our structure generalizes that in Caplin and Nalebuff (1991) in two ways: First, Caplin and Nalebuff specify the first term in the utility functions as a weighted average of the non-price related product attributes, with each customer type or market segment characterized by a unique vector of weights. Our model specification allows for an arbitrary structural dependence on the customer types. Second, their price-income sensitivity function is specified as a concave function of the difference of the customer's income and the product's price, as opposed to our general function of income and price.

In terms of the *existence conditions*, beyond operating within a narrower class of consumer choice models, Caplin and Nalebuff require that the distribution of population sizes across the different customer types satisfies specific (ρ-concavity) properties which, as mentioned above are violated in many applications. We impose *no* restrictions on this distribution. At the same time we require the above condition precluding any *single* product from capturing a majority of the potential market. No such market concentration condition is required in Caplin and Nalebuff (1991).

Finally, in terms of the *analysis approach*, Caplin and Nalebuff (1991) identify a set of conditions under which each firm's profit function is quasi-concave in its price variable over the *complete* price space; this represents the standard approach in equilibrium analyses of competition models, establishing desired structural properties on the full strategy space. Our existence condition does *not* imply that the firms' profit functions have any of these structural properties on the *full* price space. Instead, our approach is to (i) identify a compact region in the feasible price space on which the profit functions *are* quasi-concave in the firm's own price(s), or one in which they possess the so-called single point crossing property, discussed in Section 4; this guarantees the existence of an equilibrium in the restricted price region. We then establish that (ii) the equilibria identified with respect to the restricted region continue to be equilibria in the full price region and (iii) that no equilibria exist outside the identified restricted price region.

While the above results characterize the equilibrium behavior for all but heavily concentrated markets we complete the picture, giving a condition for the existence of a Nash equilibrium, indeed a *unique* Nash equilibrium, for markets with an *arbitrary* degree of concentration: The condition specifies that, the maximum feasible price vector falls below a given upper bound. In other words, to guarantee that a market with an arbitrary degree of concentration has a (unique) Nash equilibrium, sufficiently tight exogenous price limits must prevail while no such limits are needed when (one of) the above market concentration condition(s) applies. Another important distinction is that under the price limit condition, the equilibrium may reside at the boundary of the feasible price region and therefore fails to satisfy the FOC equations. A counterexample shows that if neither a very high level of market concentration can be excluded, nor the feasible price region sufficiently confined, *no* Nash equilibrium may exist for the price competition model. We also discuss the implications of these results for econometricians both in settings where a specific price vector is *observed* and *assumed* to be the (or an) equilibrium, and those where neither the model parameters nor a price equilibrium is observed.

As in Caplin and Nalebuff (1991), we assume that each firm sells a single product and start with the assumption that all customers share the same income level. Settings with an arbitrary income distribution and those where the firms offer an arbitrary number of products are covered in our Extensions Section 6. (As far as the former is concerned, we show that our results continue to apply as long as the price-income sensitivity functions are separable into an income and a price-dependent function.)

The remainder of the paper is organized as follows: Section 2 provides a review of the relevant literature. Section 3 introduces the consumer choice model. Our equilibrium existence and uniqueness results are presented in Section 4. Section 5 develops the example showing that a Nash equilibrium may fail to exist in the absence of any conditions precluding highly concentrated markets or, alternatively, enforcing sufficiently tight price limits. Section 6 discusses extensions of our base model which allow for a continuous specification of customer types, firms offering multiple products, and settings with a general income distribution. Our final section 7 describes the implications of these results for the econometrician attempting to estimate the model parameters.

2 Literature Review

There has been a plethora of price competition models for industries with differentiated products or services, beginning with the seminal paper by Bertrand (1883). One important class of such competition models employs demand functions based on a MNL discrete choice model. This model was proposed by McFadden (1976), a contribution later awarded with the 2000 Nobel Price in Economics. As explained in the Introduction, the model may be derived from an underlying random utility model, see (1) in Section 3, with homogeneous coefficients, *i.e.*, the special case where the customer population does not need to be segmented. Luce and Suppes (1965) attribute this derivation to an unpublished manuscript by Holman and Marley. The MNL model has been widely used in the economics, marketing, and operations management literature, among many other fields, see, for example Ben-Akiva and Lerman (1993), Anderson et al. (2001), and Talluri and Van Ryzin (2005). The MNL model satisfies the so-called Independence of Irrelevant Alternatives (IIA) axiom according to which the *ratio* of any pair of firms' market shares is independent of the set of other alternatives that are offered to the consumers. This axiom was first postulated by Luce (1959) but Debreu (1960) pointed out that the IIA property is highly restrictive, as illustrated by his famous red bus-blue bus example: the relative market share of an alternative is, in general, significantly affected if a close substitute to this alternative is added to the choice set.

To remedy this problem, Ben-Akiva (1973) introduced the so-called *nested logit* model, where the choice process is modeled as a two-stage nested process: the consumer first selects among broad classes of alternatives (, *e.g.*, air versus ground transportation) and subsequently a specific variant among the selected class of alternatives (*e.g.*, a specific flight). This approach still ignores systematic differences in the way different customer segments trade off relevant attributes of the various products or services. To address the issue of systematic customer heterogeneity, the *mixed multinomial logit model* (MMNL) was introduced, apparently first by Boyd and Mellman (1980) and Cardell and Dunbar (1980); earlier papers in the seventies, for example Westin (1974), had derived a similar model by treating, in a single segment model, the attribute vector as random with a given distribution. The properties of the MMNL model have been extensively studied in the economics and marketing literature, see *e.g.*, Train et al. (1987), Steckel and Vanhonacker (1988), Gonul and Srinivasan (1993), Berry (1994), Jain et al. (1994). More recently, McFadden and Train (2000) show that, under mild conditions, any discrete choice model derived from random utility maximization generates choice probabilities that can be approximated, arbitrarily closely, by a MMNL model. Moreover, these authors show that MMNL models enjoy numerical and estimation advantages beyond other discrete choice models. (It would be of considerable interest to extend our results to the general class of choice models considered by McFadden and Train (2000).)

Whether or not a Nash equilibrium exists in a Bertrand price competition model depends fundamentally on the structure of the demand functions as well as the cost structure. The same applies to the uniqueness of the equilibrium. Milgrom and Roberts (1990) and Topkis (1998) identified broad classes of demand functions under which the resulting price-competition model is supermodular, a property guaranteeing the existence of a Nash equilibrium.

More specifically, for the *pure* MNL model with a cost structure that is affine in the sales volume, Anderson et al. (2001) established the existence of a (unique) Nash equilibrium in the special case where all firms are symmetric, *i.e.*, have identical characteristics. Bernstein and Federgruen (2004) extended this result for the case of general asymmetric firms, and a generalization of MNL models referred to as attraction models. For the same model, Gallego et al. (2006) provide sufficient conditions for the existence of a unique equilibrium, under cost structures which depend on the firm's sales volume according to an increasing convex function. Konovalov and Sándor (2009) recently showed that the existence of a unique equilibrium can be guaranteed in the multi-product generalization of a pure MNL-price competition model.

Seemingly minor variants of the pure MNL model may result in a fundamentally different equilibrium behavior of the associated price competition model. For example, Cachon and Harker (2003) report that under a simple piecewise linear transformation of the MNL demand functions, and a cost function that is proportional to the square root of the sales volume, the model may have no, one or multiple equilibria as a single parameter is varied. (This is demonstrated with an example involving two symmetric firms.) Similar erratic behavior was demonstrated by Chen and Wan (2003) for what is, arguably, the seminal price competition model for service competition, due to Luski (1976) and Levhari and Luski (1978).

For price competition models with nested logit demand functions, Liu (2006) recently established the existence of a unique Nash equilibrium. As mentioned in the Introduction, Caplin and Nalebuff (1991) is the seminal paper establishing sufficient conditions for the existence of a price equilibrium when the demand functions are based on a broad class of MMNL models. They also show that, under these conditions, a unique price equilibrium exists in the case of a *duopoly or* when products are characterized by their price and a *single*, one-dimensional attribute, while the density of the customer type distribution is log-concave

(See Dierker (1991) for an alternative treatment). As mentioned by many authors, *e.g.*, Berry et al. (1995) and Thomadsen (2005b), these sufficient conditions are often not satisfied in many industry-based models.

Peitz (2000, 2002) have shown that a price equilibrium exists in certain variants of the Caplin and Nalebuff model, allowing for settings where customers maximize their utility functions subject to a budget constraint or when they may purchase an arbitrary amount of each of the products in the market, as opposed to a single unit. Unfortunately, the utility functions in Peitz do not depend on the product prices, so that the firms' incentive to mitigate price levels arises purely from the customers' budget constraints. Mizuno (2003) establishes the existence of a unique price equilibrium for certain classes of models (*e.g.,* logit, nested logit) in which the demand functions *are* log-supermodular. As we show at the end of Section 4 this property fails to apply in general MMNL models.

The emerging literature on combined assortment planning and pricing competition models, usually describes demand as arising from a pure MNL model, see *e.g.,* Misra (2008), Hopp and Xu (2008) and Besbes and Saure (2010). This, in spite of the above mentioned limitations of the pure MNL model, discussed by Kök et al. (2009) in the context of assortment planning.

As explained in the Introduction, our model assumptions generalize those made in Caplin and Nalebuff (1991). Our paper also builds on results in Thomadsen (2005b) which provide a sufficient condition for the existence of a price equilibrium - but not its uniqueness - when the demand functions arise from a general MMNL model; his condition relates the firm's variable cost rate to the value of the non-price related variables in the utility measures, see (1) below. It is difficult to assess how widely applicable the condition is.

3 The Price Competition Model

Consider an industry with J competing single-product firms each selling a specific good or service. The firms differentiate themselves via an arbitrary series of observable product characteristics as well as their price. Each firm faces a cost structure which is affine in the expected sales volume. Customers are assumed to purchase only one unit and can be segmented into K distinct groups, each with a known population size. (In Section 6, we discuss models with a continuum of customer types or market segments. All of the results obtained in Section 4, for the case of a finite set of market segments, continue to apply there.) If the potential buyers in the model represent consumers, the different segments may, for example, represent different geographical areas, in combination with socioeconomic attributes, such as age, gender, race, income level, number of years of formal education, occupational and marital status, etc. In the case of Business to Business (B2B) markets, the different segments may again represent different geographical regions, industry sub-sectors (government agencies, educational institutions, for profit companies) and firm size levels[3]. When modeling, for example, an industry of automobile part suppliers, each automobile manufacturer may represent a segment by itself. The chosen segmentation should reflect the various observable factors which may impact how different product attributes are traded off by the potential buyers. We use the following

[3]Firm size may, for example, be defined as the firm's annual revenues or its capital value.

notation for all firms $j = 1, \ldots, J$ and customer segments $k = 1, \ldots, K$:

$$\begin{aligned}
x_j &= \text{an L-dimensional vector of observable } \textit{non-price} \text{ attributes for firm} j; \\
c_j &= \text{the variable cost rate for firm } j; \\
p_j &= \text{the price selected by firm } j;\ p_j \in [p_j^{min}, p_j^{max}] \text{ with } 0 \le p_j^{min} < c_j \le p_j^{max}; \\
h_k &= \text{the population size of customer segment } k; \\
S_{jk} &= \text{expected sales volume for firm } j \text{ among customers in segment } k; \\
S_j &= \text{expected aggregate sales volume for firm } j \text{ across all customer segments}; \\
\pi_{jk} &= \text{expected profit for firm } j \text{ derived from sales to customers in segment } k; \\
\pi_j &= \text{expected aggregate profits for firm } j.
\end{aligned}$$

We thus assume that each firm selects its price from a given closed interval of feasible prices. To our knowledge, compact feasible price ranges are required for any of the known approaches to establish the existence of a Nash equilibrium[4]. At the same time, the restriction is without loss of essential generality. Consider first p_j^{min}. In the absence of other considerations, we may set $p_j^{min} = 0$[5]. As for p_j^{max}, price limits may result from a variety of sources, *e.g.,* government regulation, maximum price levels specified by suppliers or franchisers, limits set by industry organizations, or branding considerations. In other settings, where no such exogenous price limits prevail, one can always select unrestrictive upper bounds for p_j^{max} which are well above reasonable price choices. (For example, no fast food meal will be priced beyond \$20 and no subcompact car beyond the \$25,000 level.) Moreover, we will show that under a widely applicable condition and p^{max} sufficiently large, the choice of p^{max} has no impact on the price equilibrium.
Market shares within each customer segment may be derived from a standard random utility model, as follows. First, let

$$u_{ijk} = U_{jk}(x_j) + G_j(Y_i, p_j) + \epsilon_{ijk}, j = 1, \ldots, J; k = 1, \ldots, K; \text{and } \ i = 1, 2, \ldots\ , \tag{1}$$

denote the utility attributed to product j by the ith customer in segment k, with income or firm size Y_i. Recall that x_j is a vector of *observable* product attributes. Conversely, ϵ_{ijk} denotes a random *unobserved* component of customer utility. The functions $\{U_{jk}, j = 0, \ldots, J\}$ are completely general. The second term $G_j(\cdot, p_j)$ which we refer to as the *price-income sensitivity function* reflects how the utility of each product depends on its price, where marginal price sensitivity may vary with income level. Similarly, the utility associated with the no-purchase option is given by

$$u_{i0k} = U_{0k}(x_1, \ldots, x_J) + \epsilon_{i0k}, k = 1, \ldots, K; i = 1, 2, \ldots\ . \tag{2}$$

As we exclude Veblen goods, $G_j(\cdot, p_j)$ is decreasing in the price level p_j. We assume that the G_j functions in (1) are twice differentiable and concave in p_j, with $lim_{p_j \uparrow \infty} G_j(\cdot, p_j) = -\infty$. Thus, let

$$g_j(\cdot, p_j) = |\frac{\partial G_j(\cdot, p_j)}{\partial p_j}| = \frac{-\partial G_j(\cdot, p_j)}{\partial p_j} > 0, j = 1, \ldots, J \ \ and \ \ k = 1, \ldots, K, \tag{3}$$

denote the (absolute value of the) marginal change in the utility value of product j due to a marginal change in its price. (Below, we discuss an alternative interpretation of the g_j-functions.) In view of the concavity of

[4] Caplin and Nalebuff (1991), for example, assume that prices are selected from a closed interval $[p^{min}, p^{max}]$ with $p_j^{min} = c_j$ and $p_j^{max} = Y$, the consumer's income level. We make no upfront specification for these limits, allowing $0 \le p_j^{min} < c_j$ and $p_j^{max} \neq Y$ Indeed, for certain durable or investment goods and certain income levels, p_j may be in excess of Y.

[5] We assume $p^{min} < c$ to ensure, under our existence conditions, that any Nash equilibrium $p^* > p^{min}$, see Lemma 4.1 below.

the G_j-functions, we have $\forall j = 1, \ldots, J$ and $k = 1, \ldots, K$

$$g_j(\cdot, p_j) \text{ is increasing in the price level } p_j. \tag{4}$$

Many model specifications in the literature employ a $G(\cdot, \cdot)$ function *common* to all products $j = 1, \ldots, J$, see, *e.g.,* the various models listed in Section 3 of Caplin and Nalebuff (1991). However, in some applications, even the marginal utility shift due to a price increase may differ among the different competing products[6]. No assumptions are needed with respect to the dependence of the price sensitivity function $g_j(Y_i, p_j)$ on Y_i, even though one would typically have that it is decreasing in Y_i. As in Caplin and Nalebuff (1991), we initially assume that all consumers share the same income level or firm size Y (hereafter referred to as 'income' alone). Extensions to the general model, with varying income levels, are developed in Section 6.

To complete the specification of utility functions (1) and (2), $\{\epsilon_{ijk}, j = 0, 1, \ldots, N\}$ is an i.i.d. sequence of random variables, for all $i = 1, 2, \ldots$ and $k = 1, \ldots, K$. We further assume that the random components ϵ_{ijk} follow a type 1 - extreme value or Gumbel distribution:

$$Pr\left[\epsilon_{ijk} \leq z\right] = e^{-e^{-(z/\delta+\gamma)}}, j = 0, \ldots, J; k = 1, \ldots, K; i = 1, 2, \ldots, \tag{5}$$

where γ is Euler's constant (0.5772) and δ is a scale parameter. The mean and variance of the random term $\{\epsilon_{ijk}\}$ are $E[\epsilon_{ijk}] = 0$ and $var[\epsilon_{ijk}] = \delta^2\pi^2/6$. Without loss of generality, we scale, for each customer segment k=1,...,K, the units in which the utility values are measured such that $\delta = 1$. This random utility model results in the well known MNL model for demand for product j among customers of segment k.

$$S_{jk} = h_k \frac{e^{U_{jk}(x_j)+G_j(Y,p_j)}}{\left[e^{U_{0k}(x_1,\ldots,x_J)} + \sum_{m=1}^{J} e^{U_{mk}(x_m)+G_m(Y,p_m)}\right]}; j = 1 \ldots, J; k = 1, \ldots, K. \tag{6}$$

Aggregating the sales volume for individual customer segments in (6) over all segments results in the following expected sales functions for each product:

$$S_j = \sum_{k=1}^{K} S_{jk} = \sum_{k=1}^{K} h_k \frac{e^{[U_{jk}(x_j)+G_j(Y,p_j)]}}{\left[e^{U_{0k}(x_1,\ldots,x_J)} + \sum_{m=1}^{J} e^{U_{mk}(x_m)+G_m(Y,p_m)}\right]}. \tag{7}$$

An alternative foundation for the sales volume formula (6) is to assume that among potential customers in segment k, each firm j and the no-purchase option have a so-called attraction value given by:

$$a_{jk} = e^{U_{jk}(x_j)+G_j(Y,p_j)}, \;\; j = 1, ..., J, \;\; k = 1, ..., K, \tag{8}$$

$$a_{0k} = e^{U_{0k}(x_1,\ldots,x_J)}, \;\; k = 1, ..., K. \tag{9}$$

Under the four intuitive axioms specified in Bell et al. (1975), this uniquely gives rise to the demand volumes specified in (6).

The above consumer choice model thus distinguishes between two types of customer heterogeneity: (i) heterogeneity that is attributable to observable customer attributes such as their geographical location or socio-economic profile, and (ii) intrinsic heterogeneity not explained by any systematic or observable customer

[6]Caplin and Nalebuff (1991) already recognized the value of allowing for product-dependent price-income sensitivity functions. As explained below, see (15), they confine themselves to the case when these functions differ by a proportionality constant only, thus assuming that for any pair of products, the *ratio* of the marginal utility changes due to a $1 price increase remains *constant*, irrespective of the products' price levels.

attributes. This model specification covers most random utility models in the literature. As an example, consider the following general specification used in Berry (1994).

$$u_{ij} = x_j\beta_i + \xi_j - \alpha p_j + \epsilon_{ij}, j = 1, ..., J; i = 1, 2, ... \tag{10}$$

$$\beta_{il} = \beta_l + \sigma_l\zeta_{il}, \;\; l = 1, ..., L \;\text{ and }\; i = 1, 2, ..., \tag{11}$$

Here, $\{\epsilon_{ij}\}$ is again a sequence of unobservable random noise terms which is i.i.d. The vector $[\alpha, \beta, \sigma]$ is a 2L+1 dimensional string of parameters. Finally, the sequences $\{\zeta_{il}\}$ and $\{\xi_j\}$ are random sequences with zero mean, which may, or may not, be observable. Often, ξ_j is used to represent an *unobservable* utility component which reflects attributes of firm j unobserved by the modeler but with common value among the customers.
To verify that the general structure in Berry (1994) can be treated as a special case of (1) - (5), assume the $\{\zeta_{il}\}$ distributions are discrete and segment the customer population such that all customers in any segment k, share the same ζ_{il}-value for each of the L observable product attributes, i.e., $\zeta_{il} = \hat{\zeta_{kl}}$ for all customers i in segment k. Specifying $U_{jk}(x_j) = \xi_j + \sum_{l=1}^{L} x_{jl}[\beta_l + \theta\hat{\zeta_{kl}}]$ and $G_j(Y_i, p_j) = -\alpha p_j$, we note that the general Berry-model arises as a special linear specification of our structure. A restriction inherent in the Berry model is the assumption that α, the marginal disutility for firm j's product due to a marginal price increase, is uniform across all products and all price and income levels. In many practical applications, price sensitivity may vary significantly along any one of these dimensions.
Other MMNL consumer choice models employ one or more measurable attributes which depend on the specific firm and customer segment combination. For example, if the customer segmentation is in part based on the customer's geographic location, a measure d_{jk} for the distance between customer segment k and firm j may be added to the specification in (1) as follows:

$$U_{jk}(x_j) = F_j(x_j, d_{jk}) + \xi_{jk}, \tag{12}$$

with ξ_{jk}, again, an unobservable component in firm j's utility measure that is common among all customers of segment k. See, for example, the discussion of Thomadsen (2005a) below.

In other applications, the distance measure d_{jk} refers to a measure of a-priori affinity. For example, if, on the basis of nationalistic sentiments, customers have a propensity to buy from a domestic provider, this may be modeled by basing the segmentation in part on the consumer's nationality and defining the distance $d_{jk} = 0$ if segment k represents the same nationality as firm j, and $d_{jk} > 0$, otherwise. Alternatively, the a-priori affinity may be based on past purchasing behavior. Both the economics and the marketing literature have addressed the fact that customers tend to be inert or firm/brand loyal; *i.e.*, because of explicit or psychological switching costs, customers tend to stay with their current provider or brand, even if they would otherwise be more attracted by a competitor. Chintagunta et al. (2005), for example, model this as a MMNL model, segmenting customers, in part, on the basis of the firm most recently patronized; a distance measure d_{jk} is added to the utility measure where $d_{jk} = 0$ if customers of segment k used to buy from firm j and $d_{jk} = 1$ otherwise.

Another general model was introduced in the seminal paper by Caplin and Nalebuff (1991) with the specific objective of establishing the existence of a price equilibrium for a broad class of consumer choice models. This general model assumes that each potential customer i is characterized by a weight vector $\alpha_i \in \Re^L$ as well as an income level Y, such that

$$u_{ij} = \sum_{l=1}^{L} \alpha_{il} h_l(x_{jl}) + \beta_j \Gamma(Y - p_j), \;\; j = 1, ..., J \;\text{ and }\; i = 1, ..., N, \tag{13}$$

for given functions $\Gamma(\cdot)$ and $h_l(\cdot)$, with $\Gamma(\cdot)$ concave and increasing, and for given constants $\beta_j > 0,\ \ j = 1, \ldots, J$[7]. In other words, the Caplin-Nalebuff model assumes that customers characterize each product j in terms of a transformed attribute vector x'_j, the l-th component of which is given by $x'_{jl} \equiv h_l(x_{jl}),\ j = 1, \ldots, J$ and $l = 1, \ldots, L$. Customers then aggregate the (transformed) attribute values via a linear aggregate measure with different customers applying a different weight vector α to the attribute values. Assuming the distribution of α is discrete we obtain the Caplin-Nalebuff structure as a special case of our random utility model (1) - (5), as follows: segment the customer population into segments such that all customers in a segment share the same α values. In other words, for all customers in segment k, $\alpha_{il} = \alpha_l^{(k)}$. The Caplin-Nalebuff model (13) thus arises as a special case of our model with

$$U_{kj}(x_j) = \sum_{l=1}^{L} \alpha_l^{(k)} h_l(x_{jl}),\ \ \forall j = 1, ..., J \text{ and } \ k = 1, ..., K; \tag{14}$$

$$G_j(Y_i, P_j) = \beta_j \Gamma(Y_i - p_j), \tag{15}$$

while the scale parameter δ of the $\{\epsilon_{jk}\}$-variables are chosen such that $\delta = 0$[8]. Alternatively, the L-dimensional attribute vector x_j may be partitioned into an observable and an unobservable part: $x = [x', x'']$ with x' an L'-dimensional vector of observable attributes and x'' a J-dimensional vector of product indicator variables, *i.e.,* $x_{j,L'+j} = 1$ and $x_{j,L'+m} = 0\ \forall m \neq j$. If the weights $\{\alpha_l : l = L'+1, \ldots, L'+J\}$ follow independent Gumbel distributions, denoting (unobserved) utility components while each point $(\alpha_1, \ldots, \alpha_{L'})$ constitutes a separate market segment, we retrieve a (specific type of) MMNL models where the mixture is over the given distribution of $(\alpha_1, \ldots, \alpha_{L'})$ only. To obtain the existence of a Nash equilibrium in this price competition model, the authors assume, further, that the probability density function $f(\alpha)$ of the consumer attribute vector α is ρ-concave for a specific value of ρ, i.e. for any pair of points $\alpha^{(0)}$ and $\alpha^{(1)}$ in the convex support of the distributions, and any scalar $0 < \lambda < 1$:

$$f(\lambda\alpha^{(0)} + (1-\lambda)\alpha^{(1)}) \geq [\lambda f(\alpha^{(0)})^{\rho} + (1-\lambda) f(\alpha^{(1)})^{\rho}]^{1/\rho} \text{ and } \ \rho = -1/(L+1). \tag{16}$$

Thomadsen (2005a) has shown that geographic distance measures can be incorporated in this specification by appending an indicator vector for each of the J firm locations. However, the author also shows that the requirement of a ρ-concave probability density function for the customer attribute vector α precludes all but the most restrictive geographic customer distributions. In addition, under the Caplin-Nalebuff model, the price income sensitivity function G_j for the different products $j = 1, \ldots, J$ differ from each other only in the proportionality constant β_j. Moreover, the customer's income and the product's price impact the product's utility value only via their difference. This represents a significant restriction, in particular when dealing with items or services, the unit price of which constitutes a negligible fraction of a typical customer's income.

Caplin and Nalebuff (1991) showed that many of the existing consumer choice models arise as a special case of their model, including the classical models by Hotelling (1929) and Lancaster Kelvin (1966), Perloff and Salop (1985), Jaskold Gabszewicz and Thisse (1979), Shaked and Sutton (1982), Economides (1989), Christensen et al. (1975), and Anderson et al. (2001). All of these models specify $\Gamma(Y_i - p_j) = Y_i - p_j$

[7]Caplin and Nalebuff (1991) consider, in addition, a generalization of (13) in which the L-dimensional vector of product attributes x is first transformed into a L'-dimensional vector of utility benefits t(x). Instead of (13), the utility value of firm j for customer i is then specified as $u_{ij} = \sum_{l=1}^{L'} \alpha_{il} t_l(x_j) + \beta_j g(Y_i - p_j)$. This specification can also be shown to be a special case of our model. The authors state, however, that in most applications, preferences take the simpler form of (13).

[8]Caplin and Nalebuff (1991) represent the proportionality constant β_j as the $(n+1)$-st utility benefit measure associated with the product, *i.e.,* $\beta_j = t_{n+1}(x_j)$.

or $\Gamma(Y_i - p_j) = log(Y_i - p_j)$. This includes the consumer choice model in the later, seminal Berry et al. (1995) paper; where $G_j(Y_i, p_j) = \beta\Gamma(Y_i - p_j) = \beta log(Y_i - p_j)$.
We conclude this section with a few preliminary results related to our model. It is easily verified that, in each market segment, the price sensitivity of each firm's demand with respect to its own price is given by

$$\frac{\partial S_{jk}}{\partial p_j} = -g_j(Y, p_j) S_{jk}(1 - \frac{S_{jk}}{h_k}), \quad j = 1, \ldots, J; k = 1, \ldots, K, \text{ so that} \tag{17}$$

$$g_j(Y, p_j) = \frac{-\partial log S_{jk}}{\partial p_j} / (1 - \frac{S_{jk}}{h_k}), \quad j = 1, \ldots, J; k = 1, \ldots, K. \tag{18}$$

In other words, $g_j(Y, p_j)$ may be interpreted as the percentage increase in firm j's market share, due to a unit price decrease, expressed as a fraction of the percentage of market segment k, not yet captured by the firm. We therefore refer to $g_j(\cdot, \cdot)$ as the *price penetration rate*. Similarly, the price sensitivity of firm j's demand with respect to the competitor's price is given by

$$\frac{\partial S_{jk}}{\partial p_m} = g_m(Y, p_m) S_{mk} S_{jk} / h_k, \quad m \neq j. \tag{19}$$

We assume, without loss of essential generality, that for all market segments $k = 1, \ldots, K$:

$$|\frac{\partial S_{jk}}{\partial p_j}| > \sum_{m \neq j} \frac{\partial S_{jk}}{\partial p_m}, \;\; j = 1, \ldots, J. \tag{20}$$

This condition is a classical dominant-diagonal condition (see e.g. Vives (2001)) and merely precludes that a uniform price *increase* by *all* J firms would result in an *increase* of any of the firms' expected sales volume.

4 The equilibrium behavior in the price competition model

In this section, we provide a sufficient condition under which the price competition model permits a Nash equilibrium and a second, somewhat stronger, condition under which this equilibrium is unique. These conditions merely preclude a very high degree of market concentration and are easily verified on the basis of the model primitives only. We conclude the section with a sufficient condition for a (unique) Nash equilibrium that applies to markets with an arbitrary degree of market concentration. Unlike, for example, the existence conditions in Caplin and Nalebuff (1991), our conditions allow for arbitrary distributions of the population sizes $\{h_k : k = 1, \ldots, K\}$ in the various customer segments.
Recall that, for any of the K market segments, $g_j(Y, p_j)$ may be interpreted as the percentage increase in firm j's market share - expressed as a function of the percentage of the market segment not yet captured by the firm - due to a *unit* decrease in the firm's prices. Similarly, let

$$\omega_j(Y, p_j) = (p_j - c_j) g_j(Y, p_j), \tag{21}$$

denote a *dimensionless* elasticity, *i.e.,* for any of the K market segments the *percentage* increase in firm j's market share - expressed as a function of the percentage of the market not yet captured by the firm - due to a one *percent* decrease in the *variable profit margin*. As the product of two continuous functions $\omega_j(Y, p_j)$ is continuous, with $\omega_j(Y, c_j) = 0$ and $\lim_{p_j \uparrow \infty} \omega_j(Y, p_j) = \infty$. By the intermediate value theorem, we conclude that, for any critical elasticity level $\eta > 0$ there exists a price level $\bar{p_j}(\eta) > c_j$, with $\omega_j(Y, \bar{p_j}(\eta)) = \eta$. Moreover, ω_j is strictly increasing as the product of an increasing and a strictly increasing function, implying the existence of a *unique* price level $\bar{p_j}(\eta)$ such that for all (p_j^1, p_j^2) with $p_j^1 \leq \bar{p_j}(\eta) \leq p_j^2$,

$$\omega_j(Y, p_j^1) \leq \omega_j(Y, \bar{p_j}(\eta)) = (\bar{p_j}(\eta) - c_j) g_j(Y, \bar{p_j}(\eta)) = \eta \leq \omega_j(Y, p_j^2). \tag{22}$$

Our main condition for the existence of a Nash equilibrium in the interior of the price region, or even a *unique* such equilibrium, consists of excluding the possibility of *excessive market concentration*. In particular, existence of a Nash equilibrium can be guaranteed if any single firm captures less than 50% of the potential market in any customer segment when pricing at a level which, under the condition, will be shown to be an *upper bound* for the firm's equilibrium price choice. Similarly, if every single firm captures less than *one third* of the potential market in each segment (- again when pricing at a level which, under the condition, is shown to be an upper bound for his price choice -), a *unique* Nash equilibrium can be guaranteed. Thus, for any maximum market share $0 < \mu < 1$ among all *potential* customers in each segment, we define the following condition:

C(μ) In each market segment $k = 1, \ldots, K$, each firm j captures less than μ of the market among all potential customers when pricing at the level $\bar{p_j}((1-\mu)^{-1})(j = 1, \ldots, J)$, (irrespective of what prices the competitors choose within the feasible price range).

Note that each firm j's market share, in each market segment k, can be evaluated in closed form using (6). As mentioned, the critical maximal market shares μ of importance in the results below are $\mu = 1/2$ and $\mu = 1/3$.

The following lemma shows that, under $C(\mu)$, any firm j's relevant price region may be restricted to $[c_j, \bar{p_j}((1-\mu)^{-1})]$.

Lemma 4.1 *Fix* $\mu > 0$. *Under condition* $C(\mu)$*, the best response of any firm* j *to any given feasible price vector* p_{-j} *is a price* $c_j \leq p_j^*(p_{-j}) \leq \bar{p_j}((1-\mu)^{-1})$.

Proof: Fix $j = 1, \ldots, J$. Clearly, $\pi_j(p_j, p_{-j}) < 0 = \pi_j(c_j, p_{-j})$ for any $p_j < c_j$ so that $c_j \leq p_j^*(p_{-j})$. Note that $\pi_j = \sum_{k=1}^{K} \pi_{jk} = \sum_{k=1}^{K} (p_j - c_j) S_{jk}$. Invoking (17) we obtain

$$\frac{\partial \pi_j}{\partial p_j} = \sum_{k=1}^{K} S_{jk}[1 - (p_j - c_j) g_j(Y, p_j)(1 - \frac{S_{jk}}{h_k})] = \sum_{k=1}^{K} S_{jk}[1 - \omega_j(Y, p_j)(1 - \frac{S_{jk}}{h_k})] < 0, \tag{23}$$

for any price $p_j \geq \bar{p_j}((1-\mu)^{-1})$. To verify (23), note from (22) that for $p_j \geq \bar{p_j}((1-\mu)^{-1})$, $\omega_j(Y, p_j) \geq (1-\mu)^{-1}$, while $(1 - \frac{S_{jk}}{h_k}) > (1-\mu)$. The later inequality follows from S_{jk} being decreasing in p_j, see (17), while $S_{jk}/h_k < \mu$ when $p_j = \bar{p_j}((1-\mu)^{-1})$, see $C(\mu)$. ■

Thus, the market concentration test $C(\mu)$ is conducted while setting each firm's price level *above* what (, under the condition,) is rational. Therefore, as rational firms will price below $\bar{p}((1-\mu)^{-1})$, condition $C(\mu)$ does not preclude that, *in equilibrium*, a firm captures a share *above* μ in some or all market segments. Below, we show that condiditon C(μ) may, for example, be verified by evaluating JK market shares only, employing the closed form market share expression given by (6).

For $\mu = 1/2$, condition $C(1/2)$ is easily satisfied in the applications we are familiar with as is its stronger version $C'(1/2)$, see (34) below[9]. In these industrial organization studies, no single firm captures the majority of the *potential* market, (in particular when pricing at a most unfavorable price level). For example, in the drive-thru fast food industry studied by Allon et al. (2011), the largest (estimated) market share of any outlet in any market segment is less than 20%, even if it reduces its price to its variable cost c and all competing firms charge infinitely large prices. In other words, even condition C(1/3) is easily satisfied.

[9]The following is another sufficient condition for $C(1/2)$: The no-purchase option captures the majority of the consumer population, in each market segment, even when all firms select p^{min}. While much stronger than $C(1/2)$, it is clearly satisfied in many industries.

Thomadsen (2005a) studies the drive-thru fast food industry in Santa Clara where a similar dispersion of market shares, in each of the market segments considered, can be assumed.

As a last example, consider the ready-to-eat cereal industry, which is widely characterized as one with high concentration, high price-cost margins, a quote from the opening sentence in Nevo (2001); see Schmalensee (1978) and Scherer(‘1982) for similar characterizations. In this industry, each of the competing manufacturers offers a series of cereals, so an adequate representation of this industry requires a multi-product competition model as in Section 6.3.(Indeed, Nevo(2001) has estimated such a multi-product MMNL model for the industry.) In spite of this industry being viewed as one of high concentration, the aggregate market share of Kellogg, the largest competitor, varied between 41.2% in the first quarter of 1988 and 32.6% in the last quarter of 1992, with market shares calculated among all cereal consumers as opposed to the potential consumer population.

We now establish that, under condition $C(1/2)$, a Nash equilibrium exists and that the set of Nash equilibria coincides exactly with the solutions to the system of FOC equations.

Theorem 4.2 *Assume condition $C(1/2)$ applies and $\bar{p}(2) \in [p^{min}, p^{max}]$*[10].
(a) The Price Competition Model has a Nash equilibrium.
(b) Every Nash equilibrium p^* is a solution to the First Order Conditions (FOC):

$$\frac{\partial \pi_j}{\partial p_j} = \sum_{k=1}^{K} S_{jk}[1 - (p_j - c_j)g_j(Y, p_j)(1 - \frac{S_{jk}}{h_k})] = 0, \; \forall\, j = 1, \ldots, J, \tag{24}$$

and has $c < p^* < \bar{p}(2)$.
(c) Every solution to the FOC is a Nash equilibrium.

Proof: To simplify the notation, we write $\bar{p}$ as shorthand for $\bar{p}(2)$.
(a) In order to prove the result on the full price cube, we first establish the existence of a Nash equilibrium p^* in the *interior* of the restricted price cube $X_{j=1}^{J}[p_j^{min}, \bar{p_j}]$. This follows from the Nash-Debreu theorem as each firm's feasible action set $[p_j^{min}, \bar{p_j}]$ is a compact, convex set and as the profit function $\pi_j(p)$ is concave in p_j on the complete price cube $X_{j=1}^{J}[p_j^{min}, \bar{p_j}]$. Concavity follows by differentiating (23) with respect to p_j as follows:

$$\begin{aligned}\frac{\partial^2 \pi_j}{\partial p_j^2} &= \sum_{k=1}^{K} \Big\{ -S_{jk} g_j(Y, p_j)(1 - \frac{S_{jk}}{h_k})[1 - (p_j - c_j)g_j(Y, p_j)(1 - \frac{S_{jk}}{h_k})] - S_{jk} g_j(Y, p_j)(1 - \frac{S_{jk}}{h_k}) \\ &\quad + \frac{S_{jk}}{h_k}(p_j - c_j) g_j(Y, p_j) S_{jk}(-g_j(Y, p_j))(1 - \frac{S_{jk}}{h_k}) \Big\} - \frac{\partial g_j(Y, p_j)}{\partial p_j}(p_j - c_j) \sum_{k=1}^{K} S_{jk}(1 - \frac{S_{jk}}{h_k}), \\ &= \sum_{k=1}^{K} h_k g(Y, p_j)(\frac{S_{jk}}{h_k})(1 - \frac{S_{jk}}{h_k})[-2 + g_j(Y, p_j)(p_j - c_j)(1 - 2\frac{S_{jk}}{h_k})] - \frac{\partial g_j(Y, p_j)}{\partial p_j}(p_j - c_j) \sum_{k=1}^{K} S_{jk}(1 - \frac{S_{jk}}{h_k}) \\ &< 0. \end{aligned} \tag{25}$$

To verify the inequality, note that the second term to the right of (25) is negative since $g_j(Y, p_j)$ is increasing

[10]If $\bar{p}(2) \notin [p^{min}, p^{max}]$, it is still possible to establish the existence of a Nash equilibrium, however one in which some or all of the price levels are at the boundary of the feasible price region, *i.e.,* parts (b) and (c) of the theorem fail to apply in this case.

in p_j (see (4)). As to the first term, it follows from (22) that $g_j(Y, p_j)(p_j - c_j) < 2$ for all $p_j < \bar{p_j}$. Thus, since $S_{jk}/h_k \geq 0$,

$$[-2 + g_j(Y, p_j)(p_j - c_j)(1 - 2\frac{S_{jk}}{h_k})] < 0, \qquad k = 1, \ldots, K. \tag{26}$$

We have shown that a price vector p^* exists which is a Nash equilibrium on the restricted price cube $X_{j=1}^{J}[p_j^{min}, \bar{p_j}]$. To show that p^* is a Nash equilibrium on the full price range $X_{j=1}^{J}[p_j^{min}, p_j^{max}]$ as well, is suffices to show that $\pi_j(p_j, p_{-j}^*) < \pi_j(\bar{p_j}, p_{-j}^*) \leq \pi_j(p_j^*, p_{-j}^*) \; \forall \, p_j > \bar{p_j}$.
The first inequality follows from Lemma 4.1, while the second inequality follows from the fact that p^* is a Nash equilibrium on the price vector $X_{j=1}^{J}[p_j^{min}, \bar{p_j}]$.
(b) In view of Lemma 4.1 and since $p^{min} \leq c$, any price equilibrium $p^* \in X_{j=1}^{J}[p_j^{min}, \bar{p_j}]$. To show that it is, in fact, an *interior* point of $X_{j=1}^{J}[c_j, \bar{p_j}]$, and hence a solution of the FOC (24), note that $\frac{\partial \pi_j(c_j, p_{-j}^*)}{\partial p_j} = \sum_{k=1}^{K} S_{jk} = S_j > 0$ while $\frac{\partial \pi_j(\bar{p_j}, p_{-j}^*)}{\partial p_j} < 0$, by Lemma 4.1.
(c) Consider a solution, p^* of the FOC (24). It follows from (23) that $p_j^* < \bar{p_j} \; \forall \; j = 1, \ldots, J$. In view of the concavity of $\pi_j(p_j, p_{-j})$ in p_j on the price cube $X_{j=1}^{J}[p_j^{min}, \bar{p_j}]$, p^* is a Nash equilibrium on this price cube and, by the proof of part (a) on the full price range $X_{j=1}^{J}[p_j^{min}, p_j^{max}]$ as well. ■

The following theorem establishes that a *unique* Nash equilibrium can be guaranteed under the slightly stronger condition $C(1/3)$.

Theorem 4.3 *Assume condition $C(1/3)$ applies and $\bar{p}(3/2) \in [p^{min}, p^{max}]$.*
(a) The price competition model has a unique Nash equilibrium $p^* < \bar{p}(3/2)$ which satisfies the FOC equations (24).
(b) The FOC equations (24) have p^* as their unique solution.

Proof: Following the proof of Theorem 4.2, replacing $\mu = 1/2$ by $\mu = 1/3$, we obtain the existence of a Nash equilibrium $c < p^* < \bar{p}(3/2)$, which is a solution to the FOC equations (24), and, vice versa, every solution to this system of equations is a Nash equilibrium. Moreover, without loss of generality, the price region may be restricted to $P = X_{j=1}^{J}[c_j, \tilde{p}_j(3/2)]$. It therefore suffices to show that the equilibrium is unique. We establish this by showing that on the price region P:

$$|\frac{\partial \pi_j^2}{\partial p_j^2}| > \sum_{m \neq j} |\frac{\partial^2 \pi_j}{\partial p_j \partial p_m}| \quad , \; j = 1, \ldots, J. \tag{27}$$

This inequality is a sufficient condition for the best response function to be a contraction mapping, see Vives (2001). Fix $j = 1, \ldots, J$. By the definition of $\bar{p}_j(3/2) = \bar{p}_j((1 - 1/3)^{-1})$ and (22), we have

$$g_j(Y, p_j)(p_j - c_j) < 3/2 \;\; \forall \, p_j < \bar{p}_j(3/2), \;\; \text{so that}$$

$$1/2 > 1 - g_j(Y, p_j)(p_j - c_j)(1 - 2\frac{S_{jk}}{h_k}) > -0.5, \quad \forall \, p_j < \bar{p}_j(\frac{3}{2}), \;\; \forall k = 1, \ldots, K, \tag{28}$$

$$2 - g_j(Y, p_j)(p_j - c_j)(1 - 2\frac{S_{jk}}{h_k}) > +0.5, \quad \forall \, p_j < \bar{p}_j(\frac{3}{2}), \;\; \forall k = 1, \ldots, K, \tag{29}$$

by the mere fact that $1/3 \geq S_{jk}/h_k > 0$. In particular, for all $p_j < \bar{p}_j(\frac{3}{2})$, and all $k = 1, \ldots, K$:

$$2 - g_j(Y, p_j)(p_j - c_j)(1 - 2\frac{S_{jk}}{h_k}) > |1 - g_j(Y, p_j)(p_j - c_j)(1 - 2\frac{S_{jk}}{h_k})|. \tag{30}$$

Multiplying (20) with (30) and summing over all $k = 1, \ldots, K$, we obtain:

$$\begin{aligned}
|\frac{\partial^2 \pi_j}{\partial p_j^2}| &= \sum_{k=1}^{K} |\frac{\partial S_{jk}}{\partial p_j}|[2 - g_j(Y, p_j)(p_j - c_j)(1 - 2\frac{S_{jk}}{h_k})] + \frac{\partial g_j(Y, p_j)}{\partial p_j}(p_j - c_j)\sum_{k=1}^{K} S_{jk}(1 - \frac{S_{jk}}{h_k}) \\
&> \sum_{k=1}^{K} |\frac{\partial S_{jk}}{\partial p_j}|[2 - g_j(Y, p_j)(p_j - c_j)(1 - 2\frac{S_{jk}}{h_k})] \\
&> \sum_{k=1}^{K}\sum_{m \neq j} \frac{\partial S_{jk}}{\partial p_m}|[1 - g_j(Y, p_j)(p_j - c_j)(1 - 2\frac{S_{jk}}{h_k})]| \\
&> \sum_{m \neq j} |\sum_{k=1}^{K} \frac{\partial S_{jk}}{\partial p_m}[1 - g_j(Y, p_j)(p_j - c_j)(1 - 2\frac{S_{jk}}{h_k})]| = \sum_{m \neq j} |\frac{\partial^2 \pi_j}{\partial p_j p_m}|,
\end{aligned} \tag{31}$$

where the first inequality follows from (4), thus completing the verification of (27). ■

The conditions needed for existence and uniqueness, $C(\mu)$, bear a remarkable relation to standard policy criteria used to define "moderately" or "highly concentrated" markets. The Department of Justice (DOJ) and the Federal Trade Commission (FTC) measure the degree of concentration in a market via the Herfindahl-Hirschman Index (HHI), defined as the sum of the squares of the market shares represented as percentages. (This index has the maximum value of 10,000 in case of a monopoly and approaches zero if the market is divided among a very large number of competitors with an equal market share.) The DOJ-FTC 1992 Horizontal Merger Guidelines define a market with an HHI below 1,000 as "unconcentrated", one between 1,000 and 1,800 as "moderately concentrated", and those with an HHI above 1,800 as "highly concentrated". Interestingly, when C(1/3) is violated the minimum possible HHI equals 1,111, and 2,500 when C(1/2) is violated[11][12]. (Thus, while it is unclear what the cut off value of 1,800 was based on, it corresponds with the average of the minimum HHI-values when C(1/2) and C(1/3) are violated.) These 1992 DOJ-FTC guidelines were updated in April 2010 and the new HHI cutoff level for a "highly concentrated market" has been increased from 1,800 to 2,500, the minimal value when C(1/2) is violated.

We have not yet addressed whether and under what conditions a price equilibrium exists in the few industries where a very high level of concentration does arise, in some of the market segments, and a single firm captures the majority of the potential customer population. Allon et al. (2010) show that such an existence guarantee can indeed be given but only in the presence of potentially restrictive exogenous price limits. (See Section 3 for a discussion of a variety of settings where firms operate with exogenous price limits.) Indeed, the following theorem, proven in Allon et al. (2010), shows that the price competition model is supermodular and has, in fact, a *unique* equilibrium when the feasible price range is such that any firm j's variable profit margin is at a level where the above defined elasticity $\omega(Y, p_j)$ is no larger than one (*i.e.*, $p^{max} \leq \bar{p}(1)$).

[11]These minima arise when a single firm captures one third or half of the market, respectively, with the remainder of the market being divided equally among infinitely many competitors.

[12]The FTC calculates the HHI based on the anticipated post-merger equilibrium, measuring market shares as a percentage of aggregate sales in the industry. Our $C(\mu)$ conditions put "market concentration" in a favorable light measuring each firm's market share as a percentage of the total *potential* customer population and under the assumption that the firm selects an above rational price level.

Theorem 4.4 *Assume* $p_j^{max} \leq \bar{p}(1), \;\; j = 1, \ldots, N.$
The Price Competition Model has a unique Nash equilibrium.

The proof of Theorem 4.4 is based on the fact that, under the condition $p^{max} \leq \bar{p}(1)$, the price competition game is supermodular; in particular the profit functions have the so-called "single crossing property" first introduced by Milgrom and Shannon (1994). Indeed, the single-crossing property can be shown to hold in general for the segment-by-segment profit functions $\{\pi_{jk} : j = 1, \ldots, J, \; k = 1, \ldots, K\}$ without *any* restrictions on the model parameters, see Lemma 9.1 in the Appendix. Unfortunately, the *aggregate* profit functions $\{\pi_j\}$ may fail to have this single crossing property on arbitrarily large price regions. However, in practice, the single-crossing property often carries over to the aggregate profit functions, even on very large price regions, so that existence of a (unique) Nash equilibrium is guaranteed even in markets that are highly concentrated, *i.e.,* where condition C(1/2) fails.

We conclude this section with a brief discussion of different ways in which condition C(μ) may be verified, efficiently. Since a firm's market share is maximized when all competitors adopt maximal prices, condition $C(\mu)$ is easily verified as follows:

$$\frac{e^{[U_{jk}(x_j)+G_j(Y,\bar{p_j})]}}{e^{U_{0k}(x_1,\ldots,x_J)} + e^{[U_{jk}(x_j)+G_j(Y,\bar{p_j})]} + \sum_{m\neq j} e^{[U_{mk}(x_m)+G_m(Y,p_m^{max})]}} \leq \mu, \; \forall j = 1, \ldots, J, \;\; k = 1, \ldots, K, \tag{32}$$

where $\bar{p_j}$, is shorthand notation for $\bar{p_j}((1-\mu)^{-1})$. Clearly, the larger the value chosen for p^{max}, the stronger condition $C(\mu)$ becomes. Therefore, if one is unwilling to specify p^{max} upfront, there are two alternative ways to proceed. First, one may determine, $\hat{p}(\mu)$ as the smallest of the JK unique roots of the equations in the single variable p:

$$\frac{e^{[U_{jk}(x_j)+G_j(Y,\bar{p_j})]}}{e^{U_{0k}(x_1,\ldots,x_J)} + e^{[U_{jk}(x_j)+G_j(Y,\bar{p_j})]} + \sum_{m\neq j} e^{[U_{mk}(x_m)+G_m(Y,p)]}} = \mu \;\; j = 1, \ldots, J \text{ and } k = 1, \ldots, K. \tag{33}$$

$C(\mu)$ is satisfied for any $p^{max} \leq \hat{p}(\mu)$. If $\hat{p}(\mu)$ is in excess of a reasonable upper bound for the products' prices, p^{max} may be set to $\hat{p}(\mu)$ without loss of generality and $C(\mu)$ may be assumed up front. Second, the following is a much stronger version of $C(\mu)$, which is obtained by letting $p^{max} \to \infty$ and is therefore independent of the boundary of the feasible region:

C'(μ) No individual firm j has, in any of the market segments, an expected utility measure larger than that of the no-purchase option, assuming the firm's product is priced at the level $\bar{p_j}((1-\mu)^{-1})$, *i.e.*,

$$U_{jk}(x_j) + G_j(Y, \bar{p}_j((1-\mu)^{-1}) + log(\mu^{-1} - 1) \leq U_{0j}(x_1, ..., x_J) \; \forall j = 1, \ldots, J, \;\; k = 1, \ldots, K. \tag{34}$$

The implication $C'(\mu) \Rightarrow C(\mu)$ follows, after some algebra, by observing that the left hand side of (32) is increasing in p^{max}, and letting $p^{max} \to \infty$.

5 Counter Example

The following counter example demonstrates that a condition like $C(1/2)$, broadly applicable as it is, is necessary for the existence of a Nash Equilibrium. Our counter example was inspired by Dube et al. (2008) who exhibit that multiple equilibria may arise in a price competition model with 2 firms (no outside good) and 3 customer segments, and a combination of linear and Constant Elasticity of Substitution (CES) demand functions. Consider a market with two firms and three consumer segments (*i.e.*, J = 2, K = 3) whose consumer utility functions are defined as follows:

$$\text{Firm 1: } U_{i11} = A - p_1 + \epsilon_{i11}; \qquad U_{i12} = \epsilon_{i12}; \qquad U_{i13} = B - p_1 + \epsilon_{i13};$$
$$\text{Firm 2: } U_{i21} = \epsilon_{i21}; \qquad U_{i22} = A - p_2 + \epsilon_{i22}; \qquad U_{i23} = B - p_2 + \epsilon_{i23}.$$

In this example, potential consumers in segment 1 (2) are entirely focused on firm 1 (2) and purchase the good or service as long as its price is below a consumer specific reservation value. In contrast, consumers in market segment 3 are potentially attracted by either firm. The purchase decisions of segment 3 customers are therefore based on both firms' pricing decisions. Following the derivation of (6) and (7), the demand functions for firm 1 and 2 are therefore given by

$$D_1 = N_1 \tfrac{e^{A-p_1}}{1+e^{A-p_1}} + N_2 \tfrac{1}{1+e^{A-p_2}} + N_3 \tfrac{e^{B-p_1}}{e^{B-p_1}+e^{B-p_2}}, \tag{35}$$
$$D_2 = N_1 \tfrac{1}{1+e^{A-p_1}} + N_2 \tfrac{e^{A-p_2}}{1+e^{A-p_2}} + N_3 \tfrac{e^{B-p_2}}{e^{B-p_1}+e^{B-p_2}}. \tag{36}$$

The profit for each firm is given by $\pi_j = (p_j - c_j)D_j$. The following set of parameters specify a game without a Nash equilibrium: A = 4, B = 2, $c_1 = c_2 = 1$, $N_1 = 1$, $N_2 = 2$, $N_3 = 3$ and $p_{j,max} = 10$. The following defines a cycle of best responses which is reached from any starting point in the feasible price region [1,10]x[1,10], where $br_1(p_2)$ denotes the best response of firm 1 to firm 2's price choice, p_2, and vice versa for $br_2(p_1)$:

$$br_1(7.08) = 10; \quad br_2(10) = 8.79; \quad br_1(8.79) = 8.11; \quad br_2(8.11) = 7.08.$$

Notice that the parameters specified above violate condition C(1/2), as well as the dominant diagonal condition specified in equation (20). However, the dominant diagonal condition, while necessary for the uniqueness of an equilibrium, is not necessary for its existence. The counterexample not only demonstrates the necessity of a condition like C(1/2), but also reinforces the fact that the existence of a (unique) equilibrium can not be taken for granted. With many structural estimation models relying on the existence of a (unique) equilibrium when estimating market parameters and evaluating policies, it is important to note that without an existence guarantee for an equilibrium, these methods may result in flawed estimates.

6 Extensions

In this section, we discuss two generalizations of the basic model. Section 10, in the Appendix, contains a brief discussion of extensions to settings where some or all of the competing firms sell multiple products.

6.1 Unequal Income or Firm Size Level

Thus far, we have assumed that all potential customers share the same income level Y. Our equilibrium results carry over to the case of general income distributions provided the price-income sensitivity functions $G_j(\cdot,\cdot)$ are separable, *i.e.,*

$$G_j(Y_i, p_j) = G_j^1(Y_i) + G_j^2(p_j), \tag{37}$$

with G_j^2 decreasing and concave. To model income or firm size heterogeneity, design the market segmentation to be based, in part, on the income level such that all potential customers in segment $k = 1, \ldots, K$ share the same income level Y_k. This case is easily handled under separable price-income sensitivity functions by replacing the term $U_{jk}(x_j)$ in (1) by $\bar{U}_{jk}(x_j) = U_{jk}(x_j) + G^1_j(Y_k)$, and $G_j(Y_i, p_j)$ by $\bar{G}_j(Y_i, p_j) = G^2(p_j)$ for all $j = 1, \ldots, J$ and $k = 1, \ldots, K$. Since $\bar{G}_j(\cdot, \cdot)$ does not depend on the income level, all results in Section 4 continue to apply. If a continuous income distribution is required, a model with a continuous rather than a finite set of customer types or segments is called for. See the next subsection for a treatment of this case.

In the presence of income heterogeneity, it would clearly be of interest to extend our results to settings where the price-income sensitivity functions fail to be separable, so that even the *marginal* utility functions $g_j(Y_j, p_j) = \frac{\partial u_{ijk}}{\partial p_j} \frac{\partial G_j(Y_i,p_j)}{\partial p_j}$ differ by income level. More generally, one would like to extend our results to settings where the price-income sensitivity functions themselves depend *both* on the firm and the customer segment in general ways, *i.e.,* not just via the customer's income level. In other words, one would like to generalize the consumer choice model (1) to allow for double-indexed price-income sensitivity functions $G_{jk}(Y_k, p_j)$, $j = 1, \ldots, J$ and $k = 1, \ldots, K$. Such general dependencies on the customer segment have thus far failed to be tractable, see, for example Caplin and Nalebuff (1991). Indeed, in Caplin and Nalebuff's treatment of the case of income heterogeneity, *i.e.,* section 8.1, only *linear* price-income sensitivity functions are allowed, see assumption A1 ibid[13].

6.2 A Continuum of Customer Types

In some applications, a *continuum* of customer types need to be considered in the consumer choice model. Our model is easily respecified to allow for a continuum of customer types $\theta \in \Theta$, with a density function $h(\theta)$. As before, assume first that all potential customers share the same income level Y; the generalization to arbitrary income distributions is handled as in Subsection 6.1. Let:

$$u_{ij}(\theta) = U_j(x_j|\theta) + G_j(Y, p_j) + \epsilon_{ij}(\theta), \;\; j = 1, \ldots, J \;\; and \;\; i = 1, 2, \ldots, \tag{38}$$

$$u_{i0}(\theta) = U_0(x_1, \ldots, x_J|\theta) + \epsilon_{i0}(\theta), \;\; i = 1, 2, \ldots. \tag{39}$$

Here, u_{ij} denotes the utility value attributed by the i-th customer of type $\theta \in \Theta$, to product j, $j = 0, \ldots, J$, and for all $j = 0, \ldots, J$ and types $\theta \in \Theta$, $\{\epsilon_{ij}(\theta)\}$ represents a sequence of independent random variables with Gumbel distributions. It is easily verified that the demand functions in (7) need to be replaced by:

$$S_j = \int_{\theta\in\Theta} S_{j\theta} d\theta = \int_{\theta\in\Theta} h(\theta) \frac{e^{[U_j(x_j|\theta)+G_j(Y,p_j)]}}{e^{[u_0(x_1,\ldots,x_J|\theta)]} + \sum_{m=1}^{J} e^{[U_m(x_m|\theta)+G_m(Y,p_m)]}} d\theta. \tag{40}$$

All of the results in Section 4 continue to apply; where the conditions C(μ) now are specified as:

C(μ) For each customer type $\theta \in \Theta$, each firm j captures less than μ of the market among all potential customers when pricing at the level $\bar{p_j}((c_j - \mu)^{-1})$ $(j = 1, \ldots, J)$, (irrespective of which prices the competitors choose from the feasible price range).

As mentioned in the Introduction, verification of condition C(μ) may be more involved in the case of a continuum of customer types. Starting with the Berry et al. (1995) paper, many empirical models add random shock terms to some of the parameters in the utility functions $U_{jk}(x_j)$ to add an additional level of

[13] Berry et al. (1995) appear, in the presence of income heterogeneity, to allow for a price-income sensitivity function that is non-separable, *i.e.,* $g_j(Y_i, p_j) = \alpha log(Y_i - p_j)$, see eq. (2-7a) ibid. As mentioned in the Introduction, their footnote 12 suggests that only the multi-product feature of their model precludes reliance on Caplin and Nalebuff (1991). In actuality, the choice of a *non-separable* price-income sensitivity function provides a second reason why the existence results in Caplin and Nalebuff do not apply to their model.

heterogeneity because of *unobservable* factors, beyond the additive unobservable heterogeneity included by the noise terms ϵ_{ijk}, and either ??? of or beyond heterogeneity due to *observable* customer characteristics, see (1). Often, these random shock terms are assumed to follow continuous distributions of a numerically ocnvenient type, for example a multivariate Normal distribution. Such specifications imply the existence of cutomer types who attribute an arbitrarily large weight to one of the product attributes and are hence, in vast marjority, attracted to a single product, irrespective of other product differences ?? the magnitude of price differences. The existence of such extreme customer types violate the market concentration restriction in condition C(μ), which needs to hold for every customer type, even those that are very rare and therefore hardly impact the structure of the firms' aggregate profit functions. If random shock terms with a parsimonious distributional description are deemed to be necessary in the model specification, this problem can be avoided by specifying distributions with a bounded support, for example uniform, triangular distributions or non-constrained beta distributions: Feenstra and Levinsohn (1995), for example, model the constumer's ??? attribute vectior as uniformly distributed on a finitecube in attribute space. In this case, it suffices to check the market share condition C(μ) for the corner points of the cube. In ??? ??? estimation procedures, the integrals in the sales volume functions (40) are evaluated by ??? samples from the distributions of customer types, a process which is as easily carried out for the above bounded support distributions as it is for Normals, say.

There is a lively and continuing debate as to the proper way to specify customer heterogeneity, see *e.g.,* Fiebig et al. (2010). Many recent authors, *e.g.,* Louviere et al. (1999, 2002, 2008), Louviere and Eagle (2006), and Louviere et al. (2007) have argued that the use of Normal random shocks results in "various misspeecifications". Fiebig et al. (2010) write: "choice modellers have ??? models that rely largely or exclusively on unobserved heterogeneity, largely abandoning attempts to explore heterogeneous tastes usting observables. To be fair, this is partly due to the rather limited set of consumer ??? ??? in most data sets and in choice modelling." Authors like *e.g.,* Hendel and Nevo (2006) have observed that the addition of random shocks to some of the coefficients in the utility functions, beyond the heterogeneity specified by a finite set of observable customer characteristics, ??? does not improve the model specification at all; conversely, specifying heterogeneity on the basis of a finite set of observable customer characteristic values, usually improves the ??? specification in a major way, see *e.g.* Harris and Keane (1999).

Other modellers avoid random shocks in the parameters of the utility functions altogether confining themselves to a *discrete* distribution of heterogeneity, *i.e.,* a finite segmentation of the market baed on observable cutomer characteristics, alone. This is referred to as the "??? class" model, see *e.g.,* Kamakura and Russell (1989), Thomadsen (2005a), Allon et al. (2011) and the textbook on market segmentation by Wedel and Kamakura (2000).

As with other specification challenges, the notional desire for broader structure of cutomer heterogeneity must be traded off against the risks of overspecification, for example the difficulty of estimating additional sets of parameters, and the ability to interpret the resulting market segments. The above observations indicate that an additional consideration is that overly refined segmentation may result in a valuation of condition C(μ) that leaves the modeler without a foundation to estimate parameters based on equilibrium conditions.

7 Structural Estimation Methods

In this section, we discuss the implications of our results for the econometrician desiring to estimate the parameters of a model with MMNL demand function. Very often, empiricists implicitly or explicitly "assume" that (I) the model possesses an equilibrium and (II) that any equilibrium, in particular the prevailing price vector (when observed), satisfies the system of FOC (24)[14]. The problems arising due to the potential existence of multiple equilibria or no equilibrium, have been featured prominently in recent papers, as well as the fact that a solution to the system of FOC equations may fail to be an equilibrium and vice versa. See, for example, Tamer (2003), Schmedders and Judd (2005), Ferris et al. (2006), Aguirregabiria and Mira (2007)[15], and Ciliberto and Tamer (2009). Theorem 4.2 shows that under condition $C(1/2)$ assumptions (I) and (II) indeed apply.

In most applications, very high degrees of market concentration can be ruled out on *a priori* grounds and condition C(1/2) may be assumed to hold upfront. One example is the aforementioned drive-thru fast food industry, the industry modeled with MMNL demand functions in both Thomadsen (2005a) and Allon et al. (2011): Elementary statistical studies reveal that even when aggregating across all chains, the fast food industry captures a minority of the potential market in any relevant demographic segment. Going forward, we distinguish between two types of estimation settings: estimation under an observed prices vector and estimation absent price observations.

7.1 Structural Estimation With a Given Observed Price Vector

As reviewed in Section 2, in many structural estimation studies a specific price vector p^* is observed. Assumptions (I) and (II), mentioned above, are essential for the methods to be used at all, because the structural estimation techniques rest on the assumption that the observed prices represent a price equilibrium which is the solution to the system of FOC equations (24). Condition $C(1/2)$ confirms both assumptions. Moreover, the model's parameters are, invariably, determined by solving a mathematical program. (This applies both to the General Method of Moments and Maximum Likelihood Estimation techniques, see *e.g.,* Nevo (2000).) As discussed above, high degrees of market concentration can, quite frequently, be ruled out on a priori grounds and therefore condition C(1/2) may be assumed to hold upfront. In such cases, in view of Lemma 4.1, the following constraints can be added to the mathematical program:

$$\omega_j(Y, p_j^*) = (p_j^* - c_j)g_j(Y, p_j^*) \leq 2 = \omega_j(Y, \bar{p}_j(2)), \quad j = 1, \ldots, J, \tag{41}$$

as they represent *necessary* conditions for an equilibrium when $C(1/2)$ holds, no less than the FOC equations (24) themselves[16]. This not only guarantees that the parameter estimates are consistent with the observed price vector being an equilibrium, it improves the estimation itself. Much is gained by restricting a numerically difficult search for optimal parameter values to its relevant region via the addition of known necessary conditions. As mentioned, typically, the price-income sensitivity functions $G_j(\cdot, \cdot)$ are specified within a given parametrized family of functions, *e.g.,* $G_j(Y, p_j) = g_j^1(Y) - \alpha_j p_j$, or $G_j(Y, p_j) = \alpha_j log(Y - p_j)$. In the former case, (41) reduces to $\alpha_j \leq \frac{2}{(p_j^* - c_j)}$ and to $\alpha_j \leq \frac{(Y - p_j^*)}{(p_j^* - c_j)}$ in the latter. The marginal cost vector c is sometimes known and sometimes part of the parameters that need to be estimated. In the latter case, (41) represents a joint constraint on the parameter(s) specifying the $G_j(\cdot, \cdot)$ function and the c_j values.

Conversely, if the constraints (41) are not added to the mathematical program they represent a useful test for the validity of obtained estimates: If some of the inequalities in (41) are violated for the computed parameter estimates while condition $C(1/2)$ holds, the observed price vector *fails* to be an equilibrium under the computed parameter estimates, see Lemma 4.1.

[14]See for example, the quote in the introduction of Berry et al. (1995)

[15]These authors note, for example: "The existence of multiple equilibria is a prevalent feature in most empirical games where best response functions are nonlinear in other players' actions. Models with multiple equilibria do not have a unique reduced form, and this incompleteness may pose practical and theoretical problems in the estimation of structural parameters."

[16]Note that we do *not* propose adding condition $C(1/2)$, via the inequalities (32), to the mathematical program since $C(1/2)$ represents a *sufficient* condition for existence only (albeit one that is very widely satisfied).

After the vector(s) of parameter estimates (and, if applicable, the estimate for c) are obtained, it is useful to double-check whether condition $C(1/2)$ is indeed satisfied. This test reduces to making the JK numerical comparisons in (32) with $\mu = 1/2$. If so, the observed price vector p^* is a (close approximation of a) Nash equilibrium under the estimated parameter vector. If positive, the same test (32), with $\mu = 1/2$ replaced by $\mu = 1/3$, guarantees that p^* is in fact the *unique* equilibrium. In the rare case where $C(1/2)$ is violated, p^* continues to be a Nash equilibrium, indeed the unique equilibrium, as long as $p_j^{max} \leq \bar{p}_j(1)$, see Theorem 4.4[17].

After the model parameters are estimated, most studies proceed to conduct counterfactual investigations. To predict changes in the price equilibrium and corresponding sales volumes resulting from a given change in one or several of the model's parameters it is important to know whether a *unique* equilibrium exists. The uniqueness conditions in Theorem 4.3 and Theorem 4.4 can again be used for this purpose: as mentioned, the former reduces to making the JK comparisons in (32) with $\mu = 1/3$ using the estimated parameters, while the latter reduces to the vector comparison $p^{max} \leq \bar{p}(1)$.

If condition $C(1/2)$ applies but condition $C(1/3)$ fails, one may still be able to establish that p^* is the unique equilibrium, based on an *ex post* numerical test. After all, under $C(1/2)$, in view of Theorem 4.2, it suffices to verify that the system of FOC (24) has the observed price vector p^* as its unique solution on the cube $X_{j=1}^{J}[p^{min}, p^{max}]$ under the parameter estimates by employing any of the known algorithms that identify all solutions to a system of equations. Thus, the characterization in parts (b) and (c) of Theorem 4.2 of the set of Nash equilibria as the solutions to (24) may be of great value in empirical studies.

An alternative ex post uniqueness test, under $C(1/2)$, is to verify that the *single* non-linear function given by the determinant of the Jacobian matrix associated with (24) has *no* root, *i.e.*,

$$\det \mathtt{J}(p) \neq 0 \quad \forall p \in X_{j=1}^{J}[\hat{c}_j, \bar{p}_j], \tag{42}$$

where $\mathtt{J}$(p) is an JxJ matrix with $\mathtt{J}(p_{mj}) = \partial^2 \pi_m / \partial p_m \partial p_j$. The validity of (42) follows from Kellogg (1976). (Recall, Theorem 4.2 (b) excludes the existence of equilibria on the boundary of the price region.)

7.2 Structural Estimation of the Game in the Absence of an Observed Price Vector

In other studies, the parameters of the price competition game need to be estimated in the absence of an observed price vector. This happens, for example, when estimating dynamic multi-stage games, see *e.g.*, Doraszelski and Pakes (2007). Most estimation methods consist of optimizing some objective $L(\theta, p(\theta))$ over all possible parameters vectors θ and all price vectors $p(\theta)$ that arise as a Nash equilibrium under θ. The objective may be a maximum likelihood function or pseudo-maximum likelihood function, see Aguirregabiria and Mira (2002, 2007). Alternatively it may be a (generalized) method-of-moments norm, see *e.g.* Pakes et al. (2004). The characterization of the equilibria $p(\theta)$ as the solutions to the FOC equations (24) helps, once again, enormously for any of these estimation methods: Traditional estimation methods, starting with Rust (1987)'s (nested) fixed point algorithmic approach, have projected the associated optimization problems onto the parameter space Θ; solving an optimization problem of the type:

$$min\{L(\theta, p(\theta) | \theta \in \Theta \text{ and } p(\theta) \text{ is an equilibrium under } \theta\}. \tag{43}$$

This means that a search is conducted through the parameter space and whenever a specific trial parameter vector $\hat{\theta} \in \Theta$ is evaluated, all associated price equilibria $p(\hat{\theta})$ are computed. As pointed out, for example

[17]Of course, even if condition $C(1/2)$ holds for the computed parameters $\hat{\theta}$, it is conceivable that, in the absence of constraints (41) a different parameter vector θ' would be found with a somewhat better GMM norm or maximum likelihood value and with some of the constraints (41) violated. This can only happen, in the rare case where condition $C(1/2)$ is violated under θ'. In this case there is no guarantee but it is possible that the observed price vector is an equilibrium under θ' as well as under θ.

by Aguirregabiria and Mira (2007), this approach may be infeasible even for simple models. A further complication is that even the computation of the equilibria $p(\theta)$, for any single parameter vector θ, may be very difficult. Many, have concluded that games in which multiple equilibria may exist can not be estimated, and have restricted themselves to model specifications in which uniqueness of the equilibrium can be guaranteed, at a minimum. An example of this approach is Bresnahan and Reiss (1990) in the context of empirical games of market entry. While an "ideal" model would specify the profit function of a firm to be dependent on the specific identity of the competitors, Bresnahan and Reiss addressed a specification where it depends only on the *number* of competitors in the market, thus ensuring the existence of a *unique* equilibrium, at the expense of ignoring the impact of heterogeneity. In the context of our class of price competition models, an analogous approach would be to suppress heterogeneity among customer preferences and to assume they all belong to a single (homogeneous) market segment. Fortunately, no such model restrictions are necessary. The prevalence of multiple equilibria can comfortably be dealt with, as long as the set of equilibria can be characterized as the solutions to a (closed form) set of equations like the FOC equations (24). Within the context of our class of price competition models, this characterization is obtained by Theorem 4.2. Instead of optimizing the projected unconstrained problem (43), Theorem 4.2 permits us to estimate the parameters by solving the *constrained* optimization problem:

$$min\{L(p,\theta) : \theta \in \Theta \text{ and } (24)\}. \tag{44}$$

As explained above, if $C(1/2)$ can be assumed on a priori grounds, in view of Lemma 4.1 constraints (41) could be added to (44); since these represent necessary conditions under $C(1/2)$,

$$min\{L(p,\theta) : \theta \in \Theta,\ (24) \text{ and } (41)\}. \tag{45}$$

We refer to Section 7.1 for a discussion of how uniqueness of an equilibrium can be guaranteed ex ante or confirmed ex post.

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9 Appendix: Single-Crossing Property for Segment-by-Segment Profit Functions

The single-crossing property discussed in Section 4 was first introduced by Milgrom and Shannon (1994) as a close variant to the preceding "Spence-Mirrlees" single-crossing condition, see Edlin and Shannon (1998). The Monotonicity Theorem (Theorem 4) in Milgrom and Shannon (1994) shows that this single crossing property is, in fact, equivalent to each of the best response functions being monotonically increasing.

Lemma 9.1 *Fix j = 1,...,N*
For each market segment k = 1,...,K the profit function $\pi_{jk}(p)$ *has the single-crossing property in* (p_j, p_{-j}) *i.e., for any* $p_j^1 < p_j^2$ *and* $p_{-j}^1 < p_{-j}^2$*:* $\pi_{jk}(p_j^1, p_{-j}^1) < \pi_{jk}(p_j^2, p_{-j}^1) \Rightarrow \pi_{jk}(p_j^1, p_{-j}^2) < \pi_{jk}(p_j^2, p_{-j}^2)$.

Proof: It suffices to show that

$$\frac{\partial \pi_{jk}}{\partial p_j} > 0 \Rightarrow \frac{\partial^2 \pi_{jk}}{\partial p_j \partial p_m} > 0 \;\; \forall m \neq j. \tag{46}$$

Note from (33) that for all $m \neq j$:

$$\frac{\partial^2 \pi_j}{\partial p_j \partial p_m} = \textstyle\sum_{k=1}^K \{ \frac{\partial S_{jk}}{\partial p_m} - (p_j - c_j) g_j(Y, p_j) [\frac{\partial S_{jk}}{\partial p_m} - 2 \frac{S_{jk}}{h_k} \frac{\partial S_{jk}}{\partial p_m}]\}, \tag{47}$$

$$= \textstyle\sum_{k=1}^K \frac{\partial S_{jk}}{\partial p_m} [1 - (p_j - c_j) g_j(Y, p_j)(1 - 2\frac{S_{jk}}{h_k})]\}. \tag{48}$$

However, by (23), for any $m \neq j$:

$$0 < \frac{\partial \pi_{jk}}{\partial p_j} = S_{jk}[1 - (p_j - c_j) g_j(Y, p_j)(1 - \frac{S_{jk}}{h_k})] \Leftrightarrow 1 - (p_j - c_j) g_j(Y, p_j)(1 - \frac{S_{jk}}{h_k}) > 0 \Rightarrow$$
$$1 - (p_j - c_j) g_j(Y, p_j)(1 - 2\frac{S_{jk}}{h_k}) > 0 \Leftrightarrow 0 < \frac{\partial S_{jk}}{\partial p_m}[1 - (p_j - c_j) g_j(Y, p_j)(1 - 2\frac{S_{jk}}{h_k})] = \frac{\partial^2 \pi_{jk}}{\partial p_j \partial p_m}. \tag{49}$$

(The second equivalence follows from $\frac{\partial S_{jk}}{\partial p_m} > 0$, see (19). The last identity follows from (48)). ■

10 Appendix: The Multi Product Case

In some settings, each firm j sells a series, say $n_j \geq 1$ of products in the market. Assuming the choice model described by equations (1) and (2) applies to each of the $\sum_{j=1}^J n_j$ products (and the no-purchase option), can simple and broadly applicable conditions, similar to condition $C(1/2)$, be identified under which the existence of a price equilibrium is guaranteed for the general "multi-product" price competition model?

To address this question, identify each product by a double index (j, r) with the first index denoting the product's firm identity. Thus, for the r-th product of firm j, append the double index (j, r) to each relevant variable and parameter. Analogous to (22), define $\bar{p}_{jr}(2)$ as the unique price level for product (j, r) for which

$$(p_{jr} - c_{jr})g_j(Y, p_{jr}) = 2, \;\; j = 1, \ldots, J, \;\; r = 1, \ldots, n_j. \tag{50}$$

It is, again, possible to show that no firm j would choose to set *all* of its products' prices at or above the $\{\bar{p}_{jr}(2) : r = 1, \ldots, n_j\}$ levels, under a generalization of condition $C(1/2)$ which states that no firm's *total* sales across *all* of its products exceeds 50% of the potential market in any one market segment, under such price choices. However, it is conceivable that a firm would choose *some* of its products' prices to exceed their $\bar{p}(2)$-levels. Therefore, the proof of Theorem 4.2 cannot be generalized in a direct way. At the same time, Allon et al. (2010) have shown that a Nash equilibrium, in fact a *unique* equilibrium, exists if the maximum prices $p^{max} \leq \bar{p}(1)$ with $\bar{p_{jr}}(1)$ the unique price level such that $(p_{jr} - c_{jr})g_j(Y, p_{jr}) = 1, \;\; j = 1, \ldots, J \;\; r = 1, \ldots, n_j$, see Theorem 6.1 therein.