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Working Paper

09-074

CPC/CPA Hybrid Bidding in a Second Price Auction

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December 17, 2008

We develop a model of online advertising in which each advertiser chooses from multiple advertising measurement metrics – paying either for each click on its ads (CPC), or for each purchase that follows an ad-click (CPA). Our analysis extends classic auction results by allowing players to make bids using two different pricing schemes, while the driving information for bidders' endogenous selection – the conversion rate – is hidden from the seller. We show that the advertisers with the most productive sites prefer to pay CPC, while advertisers with lower quality sites prefer to pay CPA – a result that may be viewed as counterintuitive since low quality sites cannot proudly tout their conversion rates. This result holds even if an ad platform's assessment of site quality is correct in expectation. We also show that by offering both CPC and CPA, an ad platform can weakly increase its revenues compared to offering either alternative alone.

Motivation

Modern online advertising platforms typically offer advertisers a variety of metrics to measure and pay for advertising deliveries. For example, advertisers may pay per impression (CPM), per click (CPC), per action (CPA), or in proportion of the dollar value of merchandise sold (ad valorem).

When advertisers and ad platforms evaluate payment metrics, it seems they currently consider effects on parties' incentives. For example, if an advertiser is paying per impression, an ad platform could increase the advertiser's cost (and hence the platform's revenue) by delivering a large quantity of worthless or low-value impressions (perhaps invisible ads or ads located in a portion of the page where few users look). Conversely, if an advertiser is paying per action, it could lower its cost (and hence reduce the platform's revenue) by claiming actions did not occur, when in fact the actions did occur. As advertisers and platforms negotiate measurement metrics, they therefore seek to balance parties' risks and responsibilities: An inapt metric could produce incentives that undermine the parties' relationship, and advertisers and platforms seek a metric that balances the various incentives.

But offering advertisers a choice of measurement metrics creates an additional complication we sense advertisers and ad platforms have not yet considered: The advertisers who choose to pay one way may differ, systematically, from those who choose to pay in some other way. Thus, averages that fail to condition on advertisers' choices may badly misestimate an advertiser's true characteristics – causing the platform to select ads that later prove to be ill-advised. In this paper, we present the problem in an algebraic model, then assess its import in practice.

Model

We begin with seven assumptions necessary for our analysis.

1. An ad platform offers two choices of advertising metrics: CPC and CPA.
2. There are N bidders competing for one indivisible unit of advertising.
3. The ad platform uses the ordinary second-price auction mechanism, charging the winning bidder a price given by the bid of the next-highest bidder.
4. Conversion valuations $v_i^a \in (0, \infty)$ have a continuous PDF f_V with finite first moment and with CDF F_V. Each bidder's valuation is drawn i.i.d. from this distribution for each advertiser $i \in I = \{1,2, \dots, N\}$.
5. Conversion rates $\varphi_i \in (0,1)$ give the proportion of clicks that lead to a conversion. Conversion rates have a continuous PDF f_φ with finite first moment, and with CDF F_φ. Each bidder's conversion rate is drawn i.i.d. from this distribution.
6. The ad unit receives α clicks per unit time, or $\alpha\varphi_i$ conversions per unit time.
7. No bidder knows the valuation or the conversion rate of any other; this information is also hidden from the ad platform. However, everyone is aware of the underlying distributions from which the values are drawn.

The value of a conversion is related to the value of a click through the following equivalence:

$$v_i^c = \varphi_i v_i^a$$

Similarly, a per-click bid equates to a per-conversion bid through the following equivalence:

$$b_i^c = \varphi_i b_i^a$$

The ad platform chooses an estimated conversion rate, θ, for use in assessing expected revenue of CPA bids relative to CPC bids. The ad platform combines all CPA bidders into a single pool, with a single θ, because the ad platform has limited information about particular bidders' valuations and conversion: The ad platform knows only that valuations and conversion rates are drawn i.i.d., but the ad platform does not know which bidders have which valuations or conversion rates. (In Theorem 2 we will confirm that the ad platform's estimate is correct in expectation.)

Terminology

We call an equilibrium *all-CPC* if all bidders pay by CPC. We call an equilibrium *all-CPA* if all bidders pay by CPA. We call an equilibrium *hybrid* if at least one bidder pays by CPC and at least one bidder pays by CPA.

Bidding Rules

Implementing a second-price auction, the ad platform allocates placement as follows:

1. If all bidders choose CPA or all bidders choose CPC, then the ad platform directly compares bids and selects the maximand. That is, bidder *j* wins if and only if $b_j^k = \max_{i \in I}\{b_i^k\}$ and pays $b_{(N-1)}^k = \max_{i \in I \setminus \{j\}}\{b_i^k\}$. Any other bidder receives no ad placement and earns a payoff of 0.
2. Otherwise, the ad platform deflates bids according to the ad platform's assessment of bidder conversion rates. In particular, the ad platform selects the bidder that maximizes the ad platform's expected revenue. That is, bidder *j* wins if and only if $r\left(b_j^k\right) = \max_{i \in I} r\left(b_i^{k_i}\right)$ where $r\left(b_i^{k_i}\right) = \begin{Bmatrix} \theta b_i^a & \text{if } k_i = a \\ b_i^c & \text{if } k_i = c \end{Bmatrix}$. If bidder *j* wins, then bidder *j* pays the bid of the next-highest bidder, converted (using the ad platform's estimated conversion rate θ) into the bidding metric that bidder *j* selected. That is, bidder *j* pays $p\left(b_{(N-1)}, k\right)$ where $b_{(N-1)} = \max_{i \in I \setminus \{j\}} r\left(b_i^{k_i}\right)$ and $p\left(b_{(N-1)}, k\right) = \begin{Bmatrix} \frac{b_{(N-1)}}{\theta} & \text{if } k = a \\ b_{(N-1)} & \text{if } k = c \end{Bmatrix}$. All other bidders receive no ad placement and earn a payoff of 0.

Profit Function

The per-period profit function of advertiser i bidding in $k_i \, \epsilon \, \{c = CPC, a = CPA\}$ is:

$$\pi_i^k = \begin{Bmatrix} s_i^k\left(v_i^k - b_{(N-1)}^k\right) & \text{if } i \text{ wins in all-CPC or all-CPA equilibrium.} \\ s_i^k\left(v_i^k - p\left(b_{(N-1)}, k\right)\right) & \text{if } i \text{ wins in hybrid equilibrium} \\ 0 & \text{otherwise} \end{Bmatrix} \quad \forall i$$

where s_i^k is the number of clicks or conversions received per unit time.

Lemma 1

Let assumptions 1 through 7 be true. Then it is a weakly dominant strategy for each bidder to reveal his true valuation of click or conversion, and bidding is truthful. That is, each advertiser bids $b_i^{k*} = v_i^k$ in all-CPC, all-CPA and hybrid equilibria.

Proof: See appendix.

Example: N=3, Conversion Rates and Valuations are Arbitrary

Consider the simplified case of three advertisers bidding on a single ad slot, with the ad platform using $\theta \in (0,1)$ to estimate conversion rates. With valuations and conversion rates as described below, equilibrium bids are as shown in the last two columns.

Bidder (i)	**Value of Action (v_i^a)**	**Conversion Rate (φ_i)**	**Bid in All-CPC (b_i^{c*}))**	**Bid in All-CPA (b_i^{a*})**
1	5	0.2	1.0	5
2	4	0.3	1.2	4
3	3	0.45	1.35	3

With three advertisers, there are 2^3=8 possible combinations of payment metrics. The following table summarizes possible payment metrics and associated bids (where bids match valuations, per Lemma 1).

Bidding metrics (C=CPC, A=CPA)	**Bids ($b_1^{k_1*}$, $b_2^{k_2*}$, $b_3^{k_3*}$)**
CCC	(1, 1.2, 1.35)
CCA	(1, 1.2, 3θ)
ACC	(5θ, 1.2, 1.35)
ACA	(5θ, 1.2, 3θ)
CAC	(1, 4θ, 1.35)
CAA	(1, 4θ, 3θ)
AAC	(5θ, 4θ, 1.35)
AAA	(5, 4, 3)

Taking the perspective of each bidder yields insight into plausible equilibria.

Begin with Bidder 1.

As Bidder 1 chooses its payment metric, he must consider four possible cases of others' choices:

Suppose bidders 2 and 3 both use CPC. Bidder 1 loses the auction by choosing C for any θ, whereas he wins the auction for $\theta \geq 0.27$ by choosing A. Thus he is weakly better-off by choosing A.

Suppose bidders 2 and 3 use CPC and CPA, respectively. Bidder 1 loses the auction by choosing C for any θ, whereas he wins the auction for $\theta \geq 0.24$ by choosing A. Thus he is weakly better-off by choosing A.

Suppose bidders 2 and 3 use CPA and CPC, respectively. Bidder 1 loses the auction by choosing C for any θ, whereas he wins the auction for $\theta \geq 0.27$ by choosing A. Thus he is weakly better-off by choosing A.

Suppose bidders 2 and 3 both use CPA. If $\theta > 0.25$, bidder 1 loses the auction by choosing C but wins by choosing A. Else, bidder 1 wins auction in both C and A, and receives payoff $\alpha(1-4\theta)$ in C and $\alpha \cdot 0.2(5-4)$ in A. Then we see that his payoff is weakly better in C for $\theta \leq 0.2$. Thus bidder 1 is weakly better-off by choosing C for $\theta \leq 0.2$.

Bidders 2 and 3 must similarly consider the possible cases of others' choices. The Appendix presents analysis from the perspectives of Bidders 2 and 3.

Iterated elimination of weakly dominated strategies yields Nash Equilibrium strategy profiles. Consider bidder 1's strategy. Suppose 1 chooses CPC in equilibrium. Then the preceding bid analysis indicates that $\theta \leq 0.2$, and bidders 2 and 3 must have chosen CPA. But bidder 3 would only choose CPA when both bidders 1 and 2 have chosen CPC, which is a contradiction. So bidder 1's choice in equilibrium is never CPC, which means any equilibrium must entail 1 bidding CPA. By similar reasoning, we can eliminate bidder 3 choosing CPA. This leaves two Nash strategy profiles: ACC and AAC.

The strategy profiles ACC and AAC are weakly Nash for any θ. Depending on the ad platform's choice of θ, some bidders may *strictly* prefer a particular pricing metric, yielding a more stable equilibrium. In particular, for $\theta \leq 0.27$, bidder 3 *strictly* prefers metric C; for $\theta \geq 0.27$, bidder 1 *strictly* prefers A. At $\theta = 0.27$, bidders 1 and 3 both strictly prefer A and C, respectively. Similar reasoning applies for AAC, where $\theta = 0.27$ again implies that bidders 1 and 3 strictly prefer A and C, respectively.

This example suggests that a low-performer (bidder 1) prefers CPA and a high-performer (bidder 3) prefers CPC. In the following section, we offer general conditions inducing hybrid equilibria.

Results

To compute equilibrium strategy profiles, we proceed as follow:

1. Analyze bid intervals to find the weakly dominant bid for each combination of competing advertisers: CPC vs. CPC, CPA vs. CPA and CPC vs. CPA.
2. Find the regions of θ in which both advertisers prefer to bid CPC, in which both advertisers prefer to bid CPA, and in which one advertiser prefers CPC while the other prefers CPA, all given that the ad platform's estimation of θ follows a Bayesian Nash Equilibrium.
3. Confirm that the ad platform's estimation of θ is correct in expectation.

We establish parts 1 and 2 in Theorem 1, and we establish part 3 in the subsequent Theorem 2.

Theorem 1

Let assumptions 1 through 7 be true. Let A denote set of indices of bidders bidding in CPA and C denote set of indices of bidders bidding in CPC.

1) *There exists exactly one all-CPC Bayesian Nash Equilibrium if and only if $\theta \leq \varphi_i \ \forall i$.*
2) *There exists exactly one all-CPA Bayesian Nash Equilibrium if and only if $\theta \geq \varphi_i \ \forall i$.*
3) *There exists exactly one hybrid Bayesian Nash Equilibrium if and only if $\varphi_a \leq \theta \leq \varphi_c \ \forall a \in A, c \in C$.*

Proof: See appendix.

Theorem 1 takes the ad platform's assessment of θ as given. In the Theorem 2, we show that if the ad platform makes its estimate endogenously, its estimate is correct in expectation.

Theorem 2

Let assumptions 1 through 7 be true. Then there exists some estimated conversion probability $\tilde{\theta}$ that is correct in expectation. That is, if the ad platform anticipates that CPA ad clicks will convert with probability $\tilde{\theta}$, the ad platform turns out to be right in expectation. Formally, there exists some $0 \leq \tilde{\theta} \leq 1$ such that $\tilde{\theta} = E_{k_i} E_{\varphi_i}\left[\varphi_i \middle| \{k_1 \times k_2 \times \ldots \times k_N, k_i \in \{c, a\}\}\right]$. Moreover, $\tilde{\theta} = E[\varphi]$.

Proof: See appendix.

At first glance, the result in Theorem 2 seems to mirror the Law of Iterated Expectations. But advertisers select their bidding metrics endogenously. That is, $k_1, k_2, \dots, k_N$ vary as a function of $\tilde{\theta}$. As a result, computation of $\tilde{\theta}$ involves solving for a fixed point. The following section, with N=2 and conversion rates and valuations distributed uniformly, illustrates Theorem 2 as a search for a fixed point.

Exposition: Results when N = 2 and Conversion Rates and Valuations are Uniform

We illustrate Theorem 2 using $N = 2$ with valuations and conversion rates drawn from uniform distributions:

1. $N = 2$
2. $v_i^a \sim Unif(0, \lambda), \lambda > 1$
3. $\varphi_i \sim Unif(0,1)$

We begin by computing the distribution of CPC valuations. Using a convolution, $z = v_i^a \varphi_i \sim \int_0^1 \int_z^\lambda \frac{1}{v} f_\varphi(w|v) f_v(v) dv dw = f_z(z) = \frac{1}{\lambda} ln \frac{\lambda}{z}$ with cdf $F_z(z) = \int_0^z \frac{1}{\lambda} ln \frac{\lambda}{z} dz = \frac{z}{\lambda}\left(ln \frac{\lambda}{z} + 1\right)$. The proof of Theorem 1 yields the following bounds in the four potential BNE, with notation as in the proof of Theorem 1:

CPC vs. CPC

$$\int_0^{\varphi_i v_i^a} F_Z(z)^{N-1} dz \geq \frac{\varphi_i}{\theta} \int_0^{\theta v_i^a} F_Z(z)^{N-1} dz \ \ \forall i \in I$$

CPA vs. CPA

$$\int_0^{v_i^a} F_V(v)^{N-1} dv \geq \frac{\theta}{\varphi_i} \int_0^{\frac{\varphi_i}{\theta} v_i^a} F_V(v)^{N-1} dv \ \ \forall i \in I$$

CPA vs. CPC

We now consider bounds for the two possible hybrid BNE. Without loss of generality, suppose bidder 1 selects CPA while 2 selects CPC. If this bidding is to be a hybrid BNE, 1 must prefer choosing CPA given that 2 chooses CPC. Likewise, 2 must prefer CPC given that 1 chooses CPA. Formally:

$$\int_0^{\varphi_i v_i^a} F_Z(z)^{N-1} dz \leq \frac{\varphi_i}{\theta} \int_0^{\theta v_i^a} F_Z(z)^{N-1} dz$$

$$\int_0^{v_i^a} F_V(v)^{N-1} dv \leq \frac{\theta}{\varphi_i} \int_0^{\frac{\varphi_i}{\theta} v_i^a} F_V(v)^{N-1} dv$$

We now return to each of the cases, substituting for F_z and G_z using the distributions derived above:

CPC vs. CPC

Using the uniform distributions in assumptions 2-3:

$$\int_0^{\varphi_i v_i^a} \frac{z}{\lambda}\left(ln\frac{\lambda}{z}+1\right)dz \geq \frac{\varphi_i}{\theta}\int_0^{\theta v_i^a} \frac{z}{\lambda}\left(ln\frac{\lambda}{z}+1\right)dz \;\; \forall i \in I \tag{1}$$

$$\frac{(\varphi_i v_i^a)^2}{4\lambda}\left(2ln\frac{\lambda}{\varphi_i v_i^a}+3\right) \geq \frac{\varphi_i}{\theta}\frac{(\theta_i v_i^a)^2}{4\lambda}\left(2ln\frac{\lambda}{\theta v_i^a}+3\right) \;\forall i \in I$$

$$\varphi_i(2ln\lambda - 2ln\varphi_i v_i^a + 3) \geq \theta(2ln\lambda - 2ln\theta v_i^a + 3) \;\forall i \in I$$

Let $H(a) = a(2ln\lambda - 2lnav_i^a + 3)$. Then,

$$H'(a) = 2ln\lambda - 2lnz - 2lnv_i^a + 1 > 0.$$

So $H(a)$ is a strictly increasing function in the open interval $(0,1)$. Thus, $\varphi_i \geq \theta$ implies (1). If $\varphi_i \geq \theta \;\forall i$, then CPC vs. CPC is BNE.

CPA vs. CPA

$$\int_0^{v_i^a} \frac{1}{\lambda}vdv \geq \frac{\theta}{\varphi_i}\int_0^{\frac{\varphi_i}{\theta}v_i^a} \frac{1}{\lambda}vdv \;\;\forall i \in I \tag{2}$$

We now solve directly for the necessary condition:

$$\frac{1}{2\lambda}(v_i^a)^2 \geq \frac{\theta}{\varphi_i}\cdot\frac{1}{2\lambda}\left(\frac{\varphi_i}{\theta}v_i^a\right)^2$$

$$1 \geq \frac{\theta}{\varphi_i}\left(\frac{\varphi_i}{\theta}\right)^2$$

$$\theta \geq \varphi_i \;\forall i \in I$$

If $\varphi_i \leq \theta \;\forall i$, then CPA vs. CPA is BNE.

CPA vs. CPC

$$\frac{(\varphi_a v_i^a)^2}{4\lambda}\left(2ln\frac{\lambda}{\varphi_a v_i^a}+3\right) \leq \frac{\varphi_a}{\theta}\frac{(\theta_i v_i^a)^2}{4\lambda}\left(2ln\frac{\lambda}{\theta v_i^a}+3\right) \;\forall i \in I \tag{3}$$

$$\frac{1}{2\lambda}(v_i^a)^2 \leq \frac{1}{2\lambda}\left(\frac{\varphi_c}{\theta}v_i^a\right)^2 \tag{4}$$

Bidder 1 chooses CPA and hence is bound by (3): $\varphi_a \leq \theta$.

Bidder 2 chooses CPC and hence is bound by (4): $\theta \leq \varphi_c$.

Combining the two inequalities yields:

$$\varphi_a \leq \theta \leq \varphi_c \tag{5}$$

in a hybrid BNE in which bidder 1 chooses CPA and bidder 2 chooses CPC.

The preceding analysis took θ as given. If the ad platform's estimate of θ is correct in expectation, it must be the case that

$$
\begin{aligned}
\theta &= E_{k_i}E_{\varphi_i}\left[\varphi_i \middle| \{k_1 \times k_2 \times \ldots \times k_N, k_i \in \{c,a\}\}\right] \\
&= \sum_{i=0}^{N}\left(\frac{1}{F_\varphi(\theta)}\frac{i}{N}\int_0^\theta \varphi f_\varphi(\varphi)d\varphi + \frac{1}{1-F_\varphi(\theta)}\frac{N-i}{N}\int_\theta^1 \varphi f_\varphi(\varphi)d\varphi\right)\binom{N}{i}F_\varphi(\theta)^i\left(1-F_\varphi(\theta)\right)^{N-i} \\
&= \sum_{i=0}^{2}\left(\frac{1}{\theta}\cdot\frac{i}{2}\int_0^\theta \varphi d\varphi + \frac{1}{1-\theta}\cdot\frac{2-i}{2}\int_\theta^1 \varphi d\varphi\right)\binom{2}{i}\theta^i(1-\theta)^{2-i} \\
&= \left(\frac{1}{1-\theta}\int_\theta^1 \varphi d\varphi\right)(1-\theta)^2 + \frac{1}{2}\left(\frac{1}{\theta}\int_0^\theta \varphi d\varphi + \frac{1}{1-\theta}\int_\theta^1 \varphi d\varphi\right)2\theta(1-\theta) + \left(\frac{1}{\theta}\int_0^\theta \varphi d\varphi\right)\theta^2 \\
&= \left(\frac{1+\theta}{2}\right)(1-\theta)^2 + \left(\frac{1+2\theta}{2}\right)\theta(1-\theta) + \frac{\theta^3}{2}
\end{aligned}
$$

We solve for θ:

$$\tilde{\theta} = 0.5 = E[\varphi]$$

Expected Revenue to Ad Platform

Suppose an ad platform expands its CPC pricing metric to offer CPA payments also. From the perspective of the ad platform, the desirability of such an offer turns in large part on the revenue implications. Will offering CPA yield higher or lower revenue? In this section, we seek to show that given sufficiently many bidders, an ad platform achieves more revenue through a hybrid auction than through an all-CPC auction (Theorem 3), and more revenue through an all-CPC auction than through an all-CPA auction (Theorem 4).

Theorem 3

Let assumptions 1 through 7 be true. In a hybrid second-price auctions, $\tilde{\theta} = E[\varphi]$ yields weakly higher expected revenue to the ad platform than all-CPC or all-CPA auctions.

Proof: See appendix.

If $\theta = 0$, the ad platform is effectively operating in an all-CPC auction. Conversely, if $\theta = 1$, then the ad platform is effectively operating an all-CPA auction. Thus if $\tilde{\theta} = E[\varphi]$ maximizes expected revenue to the ad platform in a hybrid auction, then $\tilde{\theta} = E[\varphi]$ weakly dominates both all-CPC and all-CPA auctions for the ad platform.

Suppose an ad platform were constrained to offer either all-CPA or all-CPC, but not to allow hybrid bidding. As between all-CPC and all-CPA, which choice would offer higher revenue? Theorem 4 offers an answer:

Theorem 4

Let assumptions 1 through 7 be true. If N is sufficiently large, expected revenue to the ad platform is higher in an all-CPC auction than in an all-CPA auction.

Proof: See appendix.

Theorem 4 is limited to auctions with sufficiently many bidders. See appendix for a note and simulation confirming that results can flip when an auction has fewer bidders.

Comparing outcomes between all-CPC and all-CPA auctions implies comparing:

$$E\left[\varphi v_{(N-1)}^{a} | \text{all-CPA}\right] \lesseqgtr E\left[z_{(N-1)} | \text{all-CPC}\right]$$

$$E[\varphi] \int_{0}^{\infty} vN(N-1)F_v(v)^{N-2}\big(1-F_v(v)\big)f_v(v)dv \lesseqgtr \int_{0}^{\infty} zN(N-1)F_z(z)^{N-2}\big(1-F_z(z)\big)f_z(z)dz$$

where $z = \varphi v^a$. Theorem 4 offers a general result, namely that all-CPC auctions have higher revenue.

Implications

An ad platform seeking to add CPA measurement metrics to its existing CPC offering might naturally appraise a CPA bid based on the average conversion rate of existing CPC advertisers. Our analysis indicates that this would be a mistake: The advertisers that choose CPA are likely to differ systematically from advertisers that prefer to remain CPC. (Lemma 1)

Furthermore, our analysis reveals a counterintuitive result as to which advertisers are most likely to choose CPA measurement if it is available. One might expect that CPA measurement would be particularly attractive to the advertisers that most intensely measure and focus on their conversion rates – e.g. those that are selling products online, with easily-measured conversions. But if such advertisers' efforts produce higher conversion rates, they may instead choose to *avoid* CPA. In particular, if an ad platform fails to reward high-conversion advertisers for their high conversion rates, such advertisers will choose CPC – leaving the ad platform with only low-conversion advertisers choosing CPA. Our analysis thus confirms the crucial importance of an ad platform's conditioning on all available information.

Despite the possible revenue losses from offering optional CPA rather than only CPC, we see good reasons why an ad platform might nonetheless want to allow CPA payments. For one, bidders uncertain of ad platform quality may see CPA as a low-risk entrée to a new platform. Offering CPA thus attracts bidders a platform might otherwise be unable to recruit. Moreover, even with bidder participation fixed, if an ad platform correctly sets its estimated conversion rate, $\tilde{\theta}$, Theorem 3 indicates that the ad platform's revenue under hybrid bidding will be at least as large as its revenue when permitting only CPC bids.

In future analysis, with data from an ad platform that offers both CPC and CPA measurement metrics, we hope to calibrate our model of hybrid bidding and estimate the effects of offering CPA payments.

Appendix

Example: N=3, Conversion Rates and Valuations are Arbitrary (continued from page 4)

Bidder 2:

Suppose bidders 1 and 3 use CPC and CPC. Bidder 2 loses the auction by choosing C for any θ, whereas he wins the auction for $\theta \geq 0.34$ by choosing A. Thus he is weakly better-off by choosing A.

Suppose bidders 1 and 3 use CPC and CPA. If $\theta < 0.25$, bidder 2 loses the auction by choosing A but wins by choosing C. If $0.33 > \theta > 0.25$, bidder 2 wins both auctions, receiving payoff $\alpha(1.2 - 1)$ in C and $\alpha \cdot 0.3\left(4 - \frac{1}{\theta}\right)$ in A. Then bidder 2 is strictly better-off by choosing C since $\theta < 1$. If $0.4 > \theta \geq 0.33$, bidder 2 wins both auctions, receiving payoff $\alpha(1.2 - 3\theta)$ in C and $\alpha \cdot 0.3(4 - 3)$ in A. Then he is strictly better-off by choosing A since $\theta > 0.3$. Otherwise, bidder 2 loses the auction by choosing C and wins by choosing A. Thus we see that bidder 2 is strictly better-off by choosing C if $0.33 > \theta$ and strictly better-off by choosing A otherwise.

Suppose bidders 1 and 3 use CPA and CPC. Bidder 2 loses both auctions regardless of her choice of pricing metric. Thus, he is indifferent.

Suppose bidders 1 and 3 use CPA and CPA. Bidder 2 loses the auction by choosing A for any θ, whereas he wins the auction for $\theta \leq 0.24$ by choosing C. Thus he is weakly better-off by choosing C.

Bidder 3:

Suppose bidders 1 and 2 use CPC and CPC. If $\theta < 0.4$, bidder 3 loses the auction by choosing A, whereas he wins the auction by choosing C. If $\theta \geq 0.4$, bidder 3 wins both auctions receives payoff of $\alpha(1.35 - 1.2)$ in C and $\alpha \cdot 0.45\left(3 - \frac{1.2}{\theta}\right)$ in A. Thus, he is weakly better-off by choosing A. Then bidder 3 is strictly better-off by choosing C for $0.45 > \theta \geq 0.4$ and weakly better-off by choosing A for $\theta \geq 0.45$.

Suppose bidders 1 and 2 use CPC and CPA. Bidder 3 loses the auction by choosing A for any θ, whereas he wins the auction for $\theta \geq 0.34$ by choosing C. Thus he is weakly better-off by choosing C.

Suppose bidders 1 and 2 use CPA and CPC. Bidder 3 loses the auction by choosing A for any θ, whereas he wins the auction for $\theta \leq 0.27$ by choosing C. Thus he is weakly better-off by choosing C.

Suppose bidders 1 and 2 use CPA and CPA. Bidder 3 loses the auction by choosing A for any θ, whereas he wins the auction for $\theta < 0.27$ by choosing C. Thus he is weakly better-off by choosing C.

Lemma 1

Let assumptions 1 through 7 be true. Then it is a weakly dominant strategy for each bidder to reveal his true valuation of click or conversion, and bidding is truthful. That is, each advertiser bids $b_i^{k*} = v_i^k$ in all-CPC, all-CPA and hybrid equilibria.

Proof: Suppose all bidders choose CPC or all bidders choose CPA. Then this auction reduces to an ordinary Vickrey single-unit second-price auction, with valuations of $\varphi_i v_i^a$ and v_i^a in the all-CPC and all-CPA cases, respectively. So, for each bidder i, $b_i^{c*} = \varphi_i v_i^a$ and $b_i^{a*} = v_i^a$ are weakly dominant strategies, respectively.

Now consider the case when at least one bidder chooses CPA and at least one bidder chooses CPC. Without loss of generality, suppose bidder i bids in CPC and all others bid optimally. The ad platform deflates all CPA bids using $r\left(b_i^{k_i}\right)$, then compares the deflated CPA bids with CPC bids to find the maximand. Since bidder i bids in CPC, his bid is invariant to the ad platform's deflation. Thus the auction outcome matches an all-CPC auction, and i's optimal bid is $b_i^{c*} = \varphi_i v_i^a$.

Suppose bidder i bids in CPA and all others bid optimally. Then bidder i's bid is deflated to an estimated CPA bid following the ad platform's deflation. If i wins the auction, i pays $b_{(N-1)} = p(b_{(N-1)}, a)$ where $(N-1)$ denotes the bidder whose assessed value is the second-highest among all bidders' assessed values. Suppose bidder i bids $b_i^a > v_i^a$. If the second-highest bidder is bidding (in either CPC or CPA) such that $r(b_i^a) > b_{(N-1)} > \theta v_i^a$, then i's payment $p(b_{(N-1)}, a) = \frac{b_{(N-1)}}{\theta} > v_i^a$, so i would be better off by reducing his bid. Alternatively, suppose bidder i bids $b_i^a < v_i^a$. If the bidder immediately above i is the highest bidder such that $r(b_i^a) < b_{(N)} < \theta v_i^a$, then i could increase his bid to win, and receive a positive pay-off. Finally, suppose i bids $b_i^a = v_i^a$. If i is the highest bidder and wins $r(b_i^a) = \theta v_i^a > b_{(N-1)}$, then i receives a positive payoff of $v_i^a - \frac{b_{(N-1)}}{\theta}$ per conversion. If i is the second-highest bidder such that $r(b_i^a) = \theta v_i^a < b_{(N)}$, i would be worse off by increasing his bid to win because, if he won, he would receive a negative pay-off of $v_i^a - \frac{b_{(N)}}{\theta}$ per conversion. Thus, $b_i^{a*} = v_i^a$ is a weakly dominant strategy for each CPA bidder.

<u>Theorem 1</u>

Let assumptions 1 through 7 be true. Let A denote set of indices of bidders bidding in CPA and C denote set of indices of bidders bidding in CPC.

1) *There exists exactly one all-CPC Bayesian Nash Equilibrium if and only if $\theta \le \varphi_i\ \forall i$.*
2) *There exists exactly one all-CPA Bayesian Nash Equilibrium if and only if $\theta \ge \varphi_i\ \forall i$.*
3) *There exists exactly one hybrid Bayesian Nash Equilibrium if and only if $\varphi_a \le \theta \le \varphi_c\ \forall a \in A, c \in C$.*

Proof: Consider the case in which all bidders choose CPC or all bidders choose CPA.

The model's assumptions imply:

$$E\left[\pi_i^k\right] = E\left[s_i^k\left(v_i^k - b_{(N-1)}^k\right)\middle|\text{i wins}\right]P(\text{i wins}) + 0P(\text{i loses})$$
$$= s_i^k\left(v_i^k - b_{(N-1)}^k | b_i^k = \max_{j\in I}\{b_j^k\}]\right)P\left(b_i^k = \max_{j\in I}\{b_j^k\}\right)$$

In equilibrium, player i must be weakly better off by playing symmetrically:

$$E[\pi_i^{c*}|\text{-i play CPC}] \ge E[\pi_i^{a*}|\text{-i play CPC}]$$
$$E[\pi_i^{a*}|\text{-i play CPA}] \ge E[\pi_i^{c*}|\text{-i play CPA}]$$

Using the results in Lemma 1, we can rewrite the inequalities as follow:

$$\alpha\left(\varphi_i v_i^a - E\left[b_{(N-1)}^c \middle| \varphi_i v_i^a = \max_{j\in I}\{\varphi_j v_j^a\}\right]\right) P\left(\varphi_i v_i^a = \max_{j\in I}\{\varphi_j v_j^a\}\right)$$
$$\geq \alpha\varphi_i\left(v_i^a - E\left[p\left(b_{(N-1)}, a\right)\middle|\theta v_i^a = \max_{j\in I}\left\{r\left(b_j^{k_j}\right)\right\}\right]\right) P\left(\theta v_i^a = \max_{j\in I}\left\{r\left(b_j^{k_j}\right)\right\}\right)$$

$$\alpha\varphi_i\left(v_i^a - E\left[v_{(N-1)}^a \middle| v_i^a = \max_{j\in I}\{v_j^a\}\right]\right) P\left(v_i^a = \max_{j\in I}\{v_j^a\}\right)$$
$$\geq \alpha\left(\varphi_i v_i^a - E\left[p\left(b_{(N-1)}, c\right)\middle|\varphi_i v_i^a = \max_{j\in I}\left\{r\left(b_j^{k_j}\right)\right\}\right]\right) P\left(\varphi_i v_i^a = \max_{j\in I}\left\{r\left(b_j^{k_j}\right)\right\}\right)$$

Now let $z = \varphi_i v_i^a$ then $z \in [0, \infty)$ with continuous PDF f_Z and continuous CDF F_Z. After further simplifications:

$$\left(\varphi_i v_i^a - E\left[z_{(N-1)} | \varphi_i v_i^a \geq z_j \forall j\right]\right) P\left(\varphi_i v_i^a \geq z_j \forall j\right) \geq \left(\varphi_i v_i^a - \frac{\varphi_i}{\theta} E\left[z_{(N-1)} | \theta v_i^a \geq z_j \forall j\right]\right) P\left(\theta v_i^a \geq z_j \forall j\right) \quad (6)$$

$$\varphi_i\left(v_i^a - E\left[v_{(N-1)}^a | v_i^a \geq v_j^a \forall j\right]\right) P\left(v_i^a \geq v_j^a \forall j\right) \geq \left(\varphi_i v_i^a - \theta E\left[v_{(N-1)}^a \middle| \frac{\varphi_i}{\theta} v_i^a \geq v_j^a \forall j\right]\right) P\left(\frac{\varphi_i}{\theta} v_i^a \geq v_j^a \forall j\right) \quad (7)$$

Note that $P\left(\varphi_i v_i^a \geq z_j \forall j\right) = P\left(\varphi_i v_i^a \geq z_1, \varphi_i v_i^a \geq z_2, \ldots, \varphi_i v_i^a \geq z_{N-1}\right) = F_z\left(\varphi_i v_i^a\right)^{N-1}$ and $f_{Z_{(N-1)}}\big(z | \varphi_i v_i^a \geq$ *zj∀j=N−1FzφiviaN−1FzzN−2fzz* with *FZN*−1*z|φivia≥zj∀j=FzzN−1FzφiviaN−1* because *zi* is i.i.d.

Define

$$G_z(y) \coloneqq E\left[z_{(N-1)} \middle| y \geq z_j \forall j\right] P\left(y \geq z_j \forall j\right) = \int_0^y z(N-1) F_Z(z)^{N-2} f_Z(z)\, dz$$

and for the CPA case,

$$G_v(y) \coloneqq E\left[v_{(N-1)}^a \middle| y \geq v_j^a \forall j\right] P\left(y \geq v_j^a \forall j\right) = \int_0^y v(N-1) F_V(v)^{N-2} f_V(v)\, dv$$

Using integration by parts, we get

$$G_X(y) = y F_X(y)^{N-1} - \int_0^y F_X(x)^{N-1} dx\,, X \in \{V, Z\}$$

Rewriting (6):

$$\varphi_i v_i^a\left[F_z\left(\varphi_i v_i^a\right)^{N-1} - F_z\left(\theta v_i^a\right)^{N-1}\right]$$
$$\geq \varphi_i v_i^a F_Z\left(\varphi_i v_i^a\right)^{N-1} - \int_0^{\varphi_i v_i^a} F_Z(z)^{N-1} dz - \frac{\varphi_i}{\theta}\left[\theta v_i^a F_Z\left(\theta v_i^a\right)^{N-1} - \int_0^{\theta v_i^a} F_Z(z)^{N-1} dz\right]$$
$$= \varphi_i v_i^a\left[F_Z\left(\varphi_i v_i^a\right)^{N-1} - F_Z\left(\theta v_i^a\right)^{N-1}\right] - \int_0^{\varphi_i v_i^a} F_Z(z)^{N-1} dz + \frac{\varphi_i}{\theta}\int_0^{\theta v_i^a} F_Z(z)^{N-1} dz$$

which implies

$$\int_0^{\varphi_i v_i^a} F_Z(z)^{N-1} dz \geq \frac{\varphi_i}{\theta}\int_0^{\theta v_i^a} F_Z(z)^{N-1} dz \quad (8)$$

Define $H(\theta) := \frac{\varphi_i}{\theta}\int_0^{\theta v_i^a} F_Z(z)^{N-1}dz$. Then:

$$H'(\theta) = -\frac{\varphi_i}{\theta^2}\int_0^{\theta v_i^a} F_Z(z)^{N-1}dz + \frac{\varphi_i}{\theta}\cdot v_i^a \cdot F_Z(\theta v_i^a)^{N-1}$$

By the Mean Value Theorem,

$$\frac{1}{\varepsilon}\int_0^{\varepsilon} F_Z(z)^{N-1}dz = F_Z(\zeta)^{N-1}, \zeta \in (0, \varepsilon)$$

By monotonicity of F_Z,

$$-\frac{\varphi_i}{\theta^2}\cdot \theta v_i^a \cdot F_Z(\zeta)^{N-1} + \frac{\varphi_i v_i^a}{\theta} F_Z(\theta v_i^a)^{N-1} = -\frac{\varphi_i v_i^a}{\theta} F_Z(\zeta)^{N-1} + \frac{\varphi_i v_i^a}{\theta} F_Z(\theta v_i^a)^{N-1} \geq 0$$

This means $H(\theta)$ is a non-decreasing function. Hence, (8) is true if and only if:

$$\theta \leq \varphi_i, \forall i$$

Applying similar reasoning for (7):

$$\int_0^{v_i^a} F_V(v)^{N-1}dv \geq \frac{\theta}{\varphi_i}\int_0^{\frac{\varphi_i}{\theta}v_i^a} F_V(v)^{N-1}dv \tag{9}$$

which is true if and only if:

$$\theta \geq \varphi_i, \forall i$$

This proves parts 1 and 2 of the Theorem.

We now turn to part 3 of the Theorem. We must bound the following inequalities:

$E[\pi_c^{c*}|$a plays CPA$] \geq E[\pi_c^{a*}|$a plays CPA$]$ and $E[\pi_a^{a*}|$c plays CPC$] \geq E[\pi_a^{c*}|$c plays CPC$], \forall a \in A\ \forall c \in C$

where A and C represent the sets of indices of bidders bidding in CPA and CPC, respectively.

Because at least one bidder chooses a bidding metric that differs from others' choices, the ad platform deflates CPA bids according to the estimated conversion rate θ. This allows the ad platform to compare all bids in CPC.

If advertisers c are bidding CPC, we claim the advertisers a prefer to bid CPA. To prefer CPA, these advertisers must face exactly the opposite of the condition derived in (8). Conversely, if advertisers A are bidding CPA, advertisers C must face the opposite of the condition in (9). Combining the results from the proof of parts 1 and 2 of the Theorem, reversing the inequalities in each instance:

$$\int_0^{\varphi_i v_i^a} F_Z(z)^{N-1}dz \leq \frac{\varphi_i}{\theta}\int_0^{\theta v_i^a} F_Z(z)^{N-1}dz \tag{10}$$

$$\int_0^{v_i^a} F_V(v)^{N-1} dv \leq \frac{\theta}{\varphi_i} \int_0^{\frac{\varphi_i}{\theta} v_i^a} F_V(v)^{N-1} dv \tag{11}$$

Applying the derivation from (8) and (9), we must have:

$$\varphi_a \leq \theta \leq \varphi_c$$

This proves part 3 of the Theorem.

Theorem 2

Let assumptions 1 through 7 be true. Then there exists some estimated conversion probability $\tilde{\theta}$ that is correct in expectation. That is, if the ad platform anticipates that CPA ad clicks will convert with probability $\tilde{\theta}$, the ad platform turns out to be right in expectation. Formally, there exists some $0 < \tilde{\theta} < 1$ such that $\tilde{\theta} = E_{k_i} E_{\varphi_i}\left[\varphi_i \middle| \{k_1 \times k_2 \times \ldots \times k_N, k_i \in \{c, a\}\}\right]$. Moreover, $\tilde{\theta} = E[\varphi]$ and $\tilde{\theta}$ maximizes the expected revenue to ad platform in hybrid second-price auctions.

Proof: Using Theorem 1, we know the conditions in which bidders will prefer to bid CPC or CPA. If the ad platform's estimated conversion probability θ is to be right on average, θ must equal the average conversion rate in each type of equilibrium, weighted by the probability of such equilibrium. Substituting, using the conditions from Theorem 1:

$$\begin{aligned}\theta &= \sum_{i=0}^{N} \left(\frac{i}{N} E[\varphi_a | \theta \geq \varphi_a] + \frac{N-i}{N} E[\varphi_c | \theta \leq \varphi_c] \right) \binom{N}{i} F_\varphi(\theta)^i \left(1 - F_\varphi(\theta)\right)^{N-i} \\ &= \sum_{i=0}^{N} \left(\frac{1}{F_\varphi(\theta)} \frac{i}{N} \int_0^{\theta} \varphi f_\varphi(\varphi) d\varphi + \frac{1}{1 - F_\varphi(\theta)} \frac{N-i}{N} \int_\theta^1 \varphi f_\varphi(\varphi) d\varphi \right) \binom{N}{i} F_\varphi(\theta)^i \left(1 - F_\varphi(\theta)\right)^{N-i}\end{aligned} \tag{12}$$

We now seek to show that (12) is equivalent to $E[\varphi]$.

$$\begin{aligned}&\frac{1}{F_\varphi(\theta)} \frac{i}{N} \int_0^{\theta} \varphi f_\varphi(\varphi) d\varphi + \frac{1}{1 - F_\varphi(\theta)} \frac{N-i}{N} \int_\theta^1 \varphi f_\varphi(\varphi) d\varphi \\ &= \frac{1}{N F_\varphi(\theta)[1 - F_\varphi(\theta)]} \left([1 - F_\varphi(\theta)] i \int_0^{\theta} \varphi f_\varphi(\varphi) d\varphi + F_\varphi(\theta)(N-i) \int_\theta^1 \varphi f_\varphi(\varphi) d\varphi \right) \\ &= \frac{1}{N F_\varphi(\theta)[1 - F_\varphi(\theta)]} \left(i \left(\int_0^{\theta} \varphi f_\varphi(\varphi) d\varphi - F_\varphi(\theta) E[\varphi] \right) + N F_\varphi(\theta) \int_\theta^1 \varphi f_\varphi(\varphi) d\varphi \right)\end{aligned}$$

Define A $:= \int_0^{\theta} \varphi f_\varphi(\varphi) d\varphi - F_\varphi(\theta) E[\varphi]$ and $B := N F_\varphi(\theta) \int_\theta^1 \varphi f_\varphi(\varphi) d\varphi$

Then substituting each component back into (12) yields:

$$= \frac{1}{N F_\varphi(\theta)[1 - F_\varphi(\theta)]} \left[A \sum_{i=0}^{N} i \binom{N}{i} F_\varphi(\theta)^i \left(1 - F_\varphi(\theta)\right)^{N-i} + B \sum_{i=0}^{N} \binom{N}{i} F_\varphi(\theta)^i \left(1 - F_\varphi(\theta)\right)^{N-i} \right]$$

Note that $\sum_{i=0}^{N}\binom{N}{i}F_{\varphi}(\theta)^{i}\left(1-F_{\varphi}(\theta)\right)^{N-i}$ is simply a binomial series. Also we can rewrite

$$\sum_{i=0}^{N} i\binom{N}{i}F_{\varphi}(\theta)^{i}\left(1-F_{\varphi}(\theta)\right)^{N-i}=\sum_{i=1}^{N} i\binom{N}{i}F_{\varphi}(\theta)^{i}\left(1-F_{\varphi}(\theta)\right)^{N-i}$$

Let $k=i-1$, then:

$$\sum_{k=0}^{N-1}\frac{N!}{k!\,(N-1-k)!}F_{\varphi}(\theta)^{k+1}\left(1-F_{\varphi}(\theta)\right)^{N-1-k}$$
$$=NF_{\varphi}(\theta)\sum_{k=0}^{N-1}\binom{N-1}{k}F_{\varphi}(\theta)^{k}\left(1-F_{\varphi}(\theta)\right)^{N-1-k}$$
$$=NF_{\varphi}(\theta)$$

(12) becomes:

$$=\frac{1}{NF_{\varphi}(\theta)[1-F_{\varphi}(\theta)]}\left[\left(\int_{0}^{\theta}\varphi f_{\varphi}(\varphi)d\varphi-F_{\varphi}(\theta)E[\varphi]\right)NF_{\varphi}(\theta)+NF_{\varphi}(\theta)\int_{\theta}^{1}\varphi f_{\varphi}(\varphi)d\varphi\right]$$
$$=\frac{1}{1-F_{\varphi}(\theta)}\left[\left(\int_{0}^{\theta}\varphi f_{\varphi}(\varphi)d\varphi-F_{\varphi}(\theta)E[\varphi]\right)+\int_{\theta}^{1}\varphi f_{\varphi}(\varphi)d\varphi\right]$$
$$=\frac{E[\varphi]}{1-F_{\varphi}(\theta)}\left(1-F_{\varphi}(\theta)\right)$$
$$=E[\varphi]$$

Thus we have: $\theta=E[\varphi]$.

Theorem 3

Let assumptions 1 through 7 be true. In a hybrid second-price auctions, $\tilde{\theta}=E[\varphi]$ yields weakly higher expected revenue to the ad platform than all-CPC or all-CPA auctions.

Proof: Analyzing the ad platform revenues associated with $\tilde{\theta}=E[\varphi]$ is challenging because bids are not i.i.d. In particular, bids are non-identically distributed because bidders are choosing their payment metrics endogenously, and bidders who choose CPC differ from those who choose CPA. Hence ordinary results of i.i.d. order statistics do not apply. Moreover, depending on the highest bidder's choice of payment metric, ad platform revenues may be either the second highest bid (if the highest bidder chooses CPC) or the second highest bid multiplied by the ratio of the highest bidder's conversion rate and $\tilde{\theta}$ (if the highest bidder chooses CPA).

The proof of Theorem 3 proceeds by partitioning the set of bidders, bounding the expectation of the largest element of each subset, and using these subset bounds to characterize the highest bidder in the whole set.

Suppose we draw N vectors of bidder information, $\Gamma=\{(\varphi_i,v_i),\forall i\in I\}$. Let $z_i=\varphi_i v_i$, and let the ad platform's conversion estimate be $\tilde{\theta}$.

We partition the set of bidders into those who prefer to pay CPC and those who prefer to pay CPA. By Theorem 1, $\varphi_{c_i} \geq \tilde{\theta} \geq \varphi_{a_i}$ where $a_i \in A, c_i \in C$ and $|A \cup C| = N$. Let $B = B_c \cup B_a$ where $B_c = \{\varphi_{c_i} v_{c_i} | c_i \in C\}$ and $B_a = \{\tilde{\theta} v_{a_i} | a_i \in A\}$. Then we have:

$$\max B = \max(B_c \cup B_a) \subseteq \max B_c \cup \max B_a$$
$$\max B = \max(B_c \cup B_a) \geq \max B_c, \max B_a$$

Without loss of generality, assume that $|\max B| = 1$ since $|\max B| > 1$ only implies that there are repeated elements.

Using Theorem 2, we have $\theta = E[\varphi]$.

$$\begin{aligned} E[\max B_c] &= E\left[z_{(|C|)} = \varphi_k v_k \middle| \varphi_k \geq \theta\right] \\ &\geq E\left[z_{(|C|)} = \varphi_k v_k\right] \end{aligned}$$

$$\begin{aligned} E[\max B_a] &= E\left[\theta v_{(|A|)} = \theta v_j \middle| \varphi_j \leq \theta\right] \\ &= \theta E\left[v_j\right] \\ &= E[\varphi] E\left[v_{(|A|)}\right] \end{aligned}$$

Combining these results:

$$E[\max B] \geq \max\left\{E\left[z_{(|C|)}\right], E[\varphi] E\left[v_{(|A|)}\right]\right\}, \qquad \forall |C| \in [0, N]$$

Choosing $|A| = N$ and $|C| = N$:

$$E[\max B] \geq \max\left\{E\left[z_{(N)}\right], E\left[\varphi v_{(N)}\right]\right\} \tag{13}$$

Recall Theorem 5.2.2 from *Order Statistics* by David and Nagaraja (p. 97). Restating that result for the reader's convenience:

> Let $\{X_1, \ldots, X_N\}$ be independent and non-identically distributed set of random variables and $\{Y_1, \ldots, Y_N\}$ be independent and identically distributed set of random variables.
>
> a) If $E\left[X_{(N)}\right] \geq E\left[Y_{(N)}\right]$ then $E\left[X_{(r)}\right] \geq E\left[Y_{(r)}\right]$ for each $r \in [1, N]$, provided each expectation exists.
> b) If $E\left[X_{(1)}\right] \leq E\left[Y_{(1)}\right]$ then $E\left[X_{(r)}\right] \leq E\left[Y_{(r)}\right]$ for each $r \in [1, N]$, provided each expectation exists.
> c) If $E\left[X_{(N)}\right] \geq E\left[Y_{(N)}\right]$ and $E\left[X_{(1)}\right] \leq E\left[Y_{(1)}\right]$ then sets $\{X_1, \ldots, X_N\}$ and $\{Y_1, \ldots, Y_N\}$ are identically distributed.

We now wish to apply David and Nagaraja's Theorem 5.2.2. The X's in 5.2 are independent but non-identical – like bids in a hybrid auction. Conversely, the X's are i.i.d., as for bids from an all-CPC or all-CPA auction. (13) confirms that the hybrid auction's N^{th} order statistic is larger than the respective order statistic in an all-CPC and all-CPA auctions. This satisfies the hypotheses of David and Nagaraja's Theorem 5.2.2.(a), yielding:

$$E\left[B_{(N-1)}\right] \geq \max\left\{E\left[\varphi v_{(N-1)} | \text{all-CPA}\right], E\left[z_{(N-1)} | \text{all-CPC}\right]\right\}$$

where $B_{(N-1)}$ denotes the second-largest element of the set B.

Theorem 4

Let assumptions 1 through 7 be true. If N is sufficiently large, expected revenue to the ad platform is higher in an all-CPC auction than in an all-CPA auction.

Proof: We can show that the expected revenue to the ad platform is higher in all-CPC for a sufficiently large number of bidders. First we show that the expectation of the highest-order statistic is asymptotically equivalent to the expectation of the second-highest-order statistic. Then we show that the all-CPC auction yields higher revenue for sufficiently large N using highest-order statistics in both auctions.

Suppose $x_i \sim iid f_x, 1 \leq i \leq N$ then:

$$E\left[x_{(N)}\right] = \lim_{a\to\infty} \int_0^a N x F_x(x)^{N-1} f_x(x) dx$$

$$= \lim_{a\to\infty} a F_x(a)^N - \int_0^a F_x(x)^N dx$$

$$E\left[x_{(N-1)}\right] = \lim_{a\to\infty} \int_0^a N(N-1) x F_x(x)^{N-2}\left(1 - F_x(x)\right) f_x(x) dx$$

$$= \lim_{a\to\infty} a[N F_x(a)^{N-1} - (N-1) F_x(a)^N] - \int_0^a N F_x(x)^{N-1} - (N-1) F_x(x)^N dx$$

$$E\left[x_{(N)}\right] - E\left[x_{(N-1)}\right] = \lim_{a\to\infty} N \int_0^a F_x(x)^{N-1} - F_x(x)^N dx$$

$$\lim_{N\to\infty} E\left[x_{(N)}\right] - E\left[x_{(N-1)}\right] = \lim_{a\to\infty} \left[\lim_{N\to\infty} N \int_0^a F_x(x)^{N-1} - F_x(x)^N dx\right]$$

$$= \lim_{a\to\infty} \left[\int_0^a \lim_{N\to\infty} N F_x(x)^{N-1} - N F_x(x)^N dx\right]$$

$$= \lim_{a\to\infty} \left[\int_0^a 0 dx\right] = 0$$

Suppose we draw N vectors of bidder information, $\gamma_i = (\varphi_i, v_i)$ where $z_i = \varphi_i v_i$ then:

$$\begin{aligned} E\left[z_{(N-1)} | \text{all-CPC}\right] &\approx E\left[z_{(N)}\right] \\ &\geq E[z_i], \forall i \\ &= E[\varphi_i v_i] \\ &= E[\varphi_i] E[v_i] \\ &\geq E[\varphi] E[v_{(N)}] \approx E\left[\varphi v_{(N-1)}^a | \text{all-CPA}\right] \end{aligned}$$

Thus expected revenue to ad platform is higher in all-CPC auction as N becomes sufficiently large.

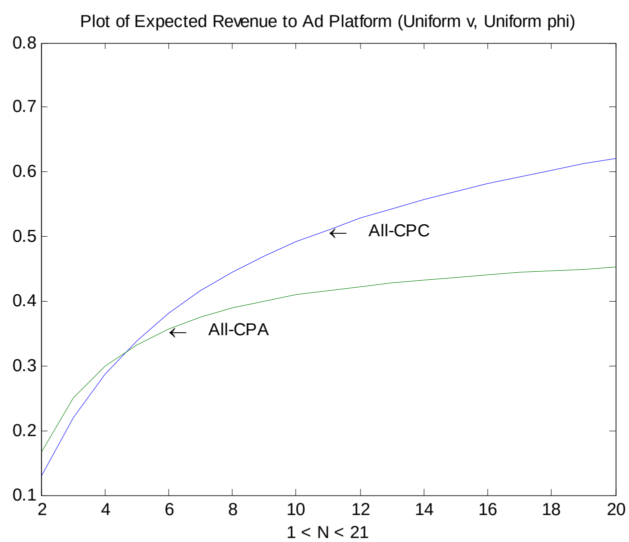
Theorem 4 – Outcomes with Fewer Bidders

Consider valuations and conversion rates drawn from uniform distributions:

1. $v_i^a \sim Unif(0,\lambda), \lambda > 1$
2. $\varphi_i \sim Unif(0,1)$

$$E\left[\varphi v_{(N-1)}^a \middle| \text{all} - \text{CPA}\right] = \frac{1}{2}\int_0^\lambda \frac{N(N-1)}{\lambda} v \cdot \left(\frac{v}{\lambda}\right)^{N-2}\left(1-\frac{v}{\lambda}\right) dv = \frac{1}{2}\left(N-1-\frac{N(N-1)}{N+1}\right)\lambda$$
$$= \frac{1}{2}\left(\frac{N-1}{N+1}\right)\lambda \xrightarrow{N\ large} \frac{1}{2}\lambda$$
$$E\left[z_{(N-1)} \middle| \text{all} - \text{CPC}\right] = N(N-1)\int_0^\lambda zF_z(z)^{N-2}\left(1-F_z(z)\right)f_z(z)dz$$
$$= \lambda[NF_z(\lambda)^{N-1} - (N-1)F_z(\lambda)^N] - \int_0^\lambda NF_z(z)^{N-1} - (N-1)F_z(z)^N dz$$
$$= \lambda\left[1 - \frac{N!}{N^N}\sum_{k=0}^{N-1}\frac{N^k}{k!} + (N-1)\frac{N!}{(N+1)^{N+1}}\sum_{k=0}^{N}\frac{(N+1)^k}{k!}\right] \xrightarrow{N\ large} \lambda$$

Plotting the comparison as λ varies:

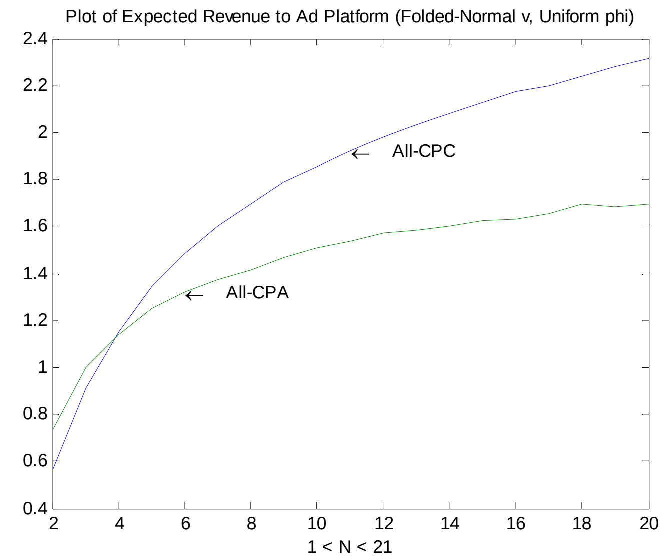


Notice that for $N < 5$, all-CPA has higher expected ad platform revenue than all-CPC. For $N > 5$, all-CPC has higher expected ad platform revenue.

Alternatively, consider valuations drawn from a folded normal distribution. In particular, consider the following specifications:

1. $v_i^a \sim FoldedNormal(2,1)$
2. $\varphi_i \sim Unif(0,1)$

It is intractable to compare ad platform revenues under all-CPC auctions to all-CPA auctions. We therefore turn to simulation. Running 10,000 draws, the following plot presents expected revenue under all-CPC and all-CPA auction rules.



Notice that for $N \leq 3$, all-CPA has higher expected ad platform revenue than all-CPC. For $N > 3$, all-CPC has higher expected ad platform revenue.