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A Noncooperative Support for Equal Division in Estate Division Problems

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A Noncooperative Support for Equal Division in Estate Division Problems*

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Abstract

We consider estate division problems, a generalization of bankruptcy problems. We show that in a direct revelation claim game, if the underlying division rule satisfies *efficiency*, *equal treatment of equals*, and *weak order preservation*, then *all* (pure strategy) Nash equilibria induce equal division. Next, we consider division rules satisfying *efficiency*, *equal treatment of equals*, and *claims monotonicity*. For claim games with at most three agents, again *all* Nash equilibria induce equal division. Surprisingly, this result does not extend to claim games with more than three agents. However, if *nonbossiness* is added, then equal division is restored.
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1 Introduction

We consider estate division problems, a generalization of bankruptcy problems, in which a positive estate has to be divided among a set of agents. Clearly, if the agents' claims add up to less than the estate, no conflict occurs and each agent can receive his claimed amount. However, if the sum of the agents' claims exceeds the estate, then bankruptcy occurs. The class of bankruptcy problems has been extensively studied using various approaches such as the normative (axiomatic) or the game theoretical approach (cooperative or noncooperative). For extensive surveys of the literature, we refer to Moulin (2002) and Thomson (2003).

In bankruptcy problems the agents' claims are normally considered as fixed inputs to the problem. However, in many real life situations it is impossible or difficult to check the validity of claims, e.g., if the profit of a joint project should be split among the project participants, but inputs are not perfectly observable or difficult to compare. Other examples are claims based on moral property rights, entitlements (see Gächter and Riedl, 2005) or subjective needs (see Pulido, Sanchez-Soriano, and Llorca, 2002). If the authority in charge of the estate lacks the ability to verify claims or verification is too costly, agents are likely to behave strategically to ensure larger shares of the estate for themselves.

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We model this type of situation with a simple noncooperative game. Given the estate to divide and based on a (division) rule, agents simply submit claims which are restricted to not exceed a common upper bound. We analyze the (pure strategy) Nash equilibria of the resulting *claim game*. We do not fix any specific rule, but only require the rule to satisfy basic and appealing properties.

First, we require the rule to satisfy *efficiency*, *equal treatment of equals*, and *weak order preservation*.[1] Then, all agents claiming the largest possible amount is a Nash equilibrium and *all* Nash equilibria lead to equal division (Theorem 1). Two corresponding results are obtained using *order preservation* or *others oriented claims monotonicity* (Corollaries 1 and 2). Second, we replace *weak order preservation* with *claims monotonicity*.[2] Again, all agents claiming the largest possible amount is a Nash equilibrium. However, in difference to the previous results, we show that equal division is guaranteed for *all* Nash equilibria only for claim games with at most three agents (Theorem 2). This result does not extend to claim games with more than three agents (Example 1). Nevertheless, if *nonbossiness* is added (i.e., if we require *efficiency*, *equal treatment of equals*, *claims monotonicity*, and *non-bossiness*), then equal division in *all* Nash equilibria is restored (Theorem 3).[3]

All our results point towards the same intuitive message: if it is impossible or difficult to test the legitimacy of claims, the conflict will escalate to the highest possible level at which claims are no longer informative. As a result, equal division is the "non-discriminating" outcome in Nash equilibrium. In other words, equal division is not only a normatively appealing division method, but it is also the result of a natural noncooperative game.[4] These findings might explain why in many instances equal division is applied right away even without asking agents' claims. For instance, pre-1975 U.S. Admiralty law divides liabilities equally among parties if they are both found negligent (see Feldman and Jeonghyun, 2005). British Shipping Law, until the act of 1911, applied equal division of costs in case of a collision between two ships, however much the degree of their faults or negligence may differ. This practice has originated from a medieval rule, which was originally intended to be applied only in cases where negligence cannot be perfectly proven (see Porges and Thomas, 1963).

A number of articles also consider strategic aspects (see Thomson, 2003, Section 7). The articles closely related to ours are Chun (1989), Thomson (1990), Moreno-Ternero (2002), Herrero (2003), and Bochet and Sakai (2008) in that the games they consider do not focus on a specific rule, but a class of rules that is determined by basic properties. Chun (1989) considers a noncooperative game where agents propose rules and a sequential revision procedure then converges to equal division. Moreno-Ternero (2002) constructs a noncooperative game, the equilibrium of which converges to the proportional rule. A noncooperative game similar (in a sense dual) to the one in Chun (1989) is constructed by Herrero (2003) who shows convergence to the constrained equal losses rule. Thomson (1990) and Bochet and Sakai (2008) consider the problem of allocating an estate when agents have single-peaked preferences and study a direct revelation game where agents report either their preferences (Thomson, 1990) or their peaks (Bochet and Sakai, 2008). While Thomson (1990) shows for various rules (e.g., the symmetrically proportional rule, the equal distance rule, and the

[1] *Efficiency*: the estate is allocated if the sum of claims is larger than (or equal to) the estate. *Equal treatment of equals*: any two agents with identical claims receive the same awards. *Weak order preservation*: if an agent has a higher claim than another agent, then he does not receive less than that agent.

[2] *Claims monotonicity*: an agent does not receive less after an increase in his claim.

[3] *Nonbossiness*: no agent can change other agents' awards by changing his claim unless his award changes as well.

[4] A game-theoretical interpretation of our result is that if a rule satisfies certain natural and appealing properties (see our results above), it can be used to implement equal division.

equal sacrifice rule) that uniform division is the only Nash equilibrium outcome, Bochet and Sakai (2008, Theorem 2) prove this result for rules satisfying certain properties.[5]

The paper is organized as follows. In Section 2, we introduce estate division problems and properties of rules. Furthermore, we establish logical relations between properties and define three well-known rules (the proportional, the constrained equal awards, and the constrained equal losses rule). In Section 3, we introduce claim games and establish various equal division Nash equilibria results (Theorems 1, 2, and 3, Corollaries 1 and 2, and Example 1), including a discussion of the independence of assumptions needed to establish our results. We conclude in Section 4.

2 Estate Division Problems and Properties of Rules

An amount $E \in \mathbb{R}_{++}$ has to be divided among a set of agents $N = \{1, \dots, n\}$. For every $i \in N$ let $c_i \in \mathbb{R}_+$ denote agent i's claim, and let $c \equiv (c_i)_{i \in N} \in \mathbb{R}^N_+$ be the *claims vector*. An *estate division problem*, or *problem* for short, is a pair $(c; E) \in \mathbb{R}^N_+ \times \mathbb{R}_{++}$. For all $c \in C$ and all $S \subseteq N$, $S \neq \emptyset$, let $c_S = \sum_{i \in S} c_i$. Estate division problem $(c; E)$ is a *bankruptcy* (or claims) *problem* if $c_N \geq E$. We denote by $\mathcal{E}^N$ the *class of all estate division problems.*

A *(division) rule* is a function $R : \mathcal{E}^N \to \mathbb{R}^N$ that associates with each problem $(c; E) \in \mathcal{E}^N$ an *awards vector* $x \in \mathbb{R}^N_+$ such that $\sum x_i \leq E$ and $x \leq c$.[6] An awards vector x for $(c; E)$ is *efficient* if it assigns the largest possible amount of E taking claims as upper bounds, i.e., if $c_N \geq E$, then $\sum x_i = E$ and if $c_N \leq E$, then $x = c$. Note that we do not require that E has to be completely allocated among the agents if no bankruptcy occurs.[7] We now introduce some properties of rules, which are extensions of standard properties for bankruptcy rules to our more general estate division model (see Thomson, 2003, for a comprehensive survey on the axiomatic and game-theoretic analysis of bankruptcy problems).

Efficiency: A rule R satisfies *efficiency* if for all $(c; E) \in \mathcal{E}^N$, $R(c; E)$ is efficient.

The following property requires that the awards to agents whose claims are equal should be equal.

Equal Treatment of Equals: A rule R satisfies *equal treatment of equals* if for all $(c; E) \in \mathcal{E}^N$ and all $i, j \in N$ such that $c_i = c_j$, $R_i(c; E) = R_j(c; E)$.

By the next requirement the rule should respect the ordering of claims as follows: if agent i's claim is at least as large as agent j's claim, he should receive at least as much as agent j and the differences, calculated agent by agent, between claims and awards, should be ordered as the claims are (see Aumann and Maschler, 1985).

Order Preservation: A rule R satisfies *order preservation* if for all $(c; E) \in \mathcal{E}^N$ and all $i, j \in N$ such that $c_i \geq c_j$, $R_i(c; E) \geq R_j(c; E)$ and $c_i - R_i(c; E) \geq c_j - R_j(c; E)$.

Lemma 1. *Order preservation implies equal treatment of equals.*

[5] The properties they consider – *efficiency*, *strict own peak monotonicity*, *others peak monotonicity*, *peak order preservation*, and *own peak continuity* – are similar in spirit to the ones we consider for estate division problems. However, Bochet and Sakai (2008) require more properties to obtain their result due to the difference between their model and our estate division model.

[6] Note that $x \leq c$ if and only if for all $i \in N$, $x_i \leq c_i$.

[7] In Appendix B we describe what happens if we require that the estate E is always completely allocated among the agents. Then, *efficiency* is already incorporated in the definition of a rule and results essentially do not change.

Proof. Let rule R satisfy *order preservation*. Let $(c;E) \in \mathcal{E}^N$ and $i,j \in N$ such that $c_i = c_j$. Hence, by *order preservation*, $[c_i \geq c_j$ implies $R_i(c;E) \geq R_j(c;E)]$ and $[c_j \geq c_i$ implies $R_j(c;E) \geq R_i(c;E)]$. Thus, $R_i(c;E) = R_j(c;E)$ and rule R satisfies *equal treatment of equals*. □

For bankruptcy problems, *order preservation* reflects a duality concerning awards and losses by requiring order preservation of awards *and* order preservation of losses. A weaker order preservation property results if only order preservation of awards is required: if agent i's claim is larger than agent j's claim, i should receive at least as much as agent j does.

Weak Order Preservation: A rule R satisfies *weak order preservation* if for all $(c;E) \in \mathcal{E}^N$ and all $i,j \in N$ such that $c_i > c_j$, $R_i(c;E) \geq R_j(c;E)$.

The following monotonicity property requires that if an agent's claim increases, he should receive at least as much as he did initially. We use the following notation. Let $c \in \mathbb{R}^N_+$, $i \in N$, and $c'_i \in \mathbb{R}_+$. Then, $(c'_i, c_{-i}) \in \mathbb{R}^N_+$ denotes the claims vector obtained from c by replacing c_i with c'_i.

Claims Monotonicity: A rule R satisfies *claims monotonicity* if for all $(c;E) \in \mathcal{E}^N$ and all $i \in N$ such that $c_i < c'_i$, $R_i(c;E) \leq R_i(c'_i, c_{-i};E)$.

Focusing on the other agents, we can formulate another monotonicity property: if an agent's claim increases, then all other agents should receive at most as much as they did initially (Thomson, 2003, mentions this property after introducing *claims monotonicity* without giving it a name).

Others Oriented Claims Monotonicity: A rule R satisfies *others oriented claims monotonicity* if for all $(c;E) \in \mathcal{E}^N$ and all $i \in N$ such that $c_i < c'_i$, $R_j(c;E) \geq R_j(c'_i, c_{-i};E)$ for all $j \neq i$.

Note that *others oriented claims monotonicity* together with *efficiency* implies *claims monotonicity*.[8] For two-agent problems the inverse conclusion is also true.

Lemma 2.

(a) *Efficiency and others oriented claims monotonicity imply claims monotonicity.*
(b) *If $|N| = 2$, then efficiency and claims monotonicity imply others oriented claims monotonicity.*

Proof.
(a) Let rule R satisfy *efficiency* and *others oriented claims monotonicity*. Let $(c;E) \in \mathcal{E}^N$, $i \in N$, and c'_i such that $c'_i > c_i$. By *efficiency*, $\sum R_k(c;E) \leq \sum R_k(c'_i, c_{-i};E)$, and by *others oriented claims monotonicity*, $\sum_{j \neq i} R_j(c;E) \geq \sum R_{j \neq i}(c'_i, c_{-i};E)$. Hence, $R_i(c;E) = \sum R_k(c;E) - \sum_{j\neq i} R_j(c;E) \leq \sum R_k(c'_i, c_{-i};E) - \sum R_{j\neq i}(c'_i, c_{-i};E) = R_i(c'_i, c_{-i};E)$. Thus R satisfies *claims monotonicity*.
(b) Let rule R satisfy *efficiency* and *claims monotonicity*. Let $(c;E) \in \mathcal{E}^N$ and assume without loss of generality that $c_1 < c'_1$. By *claims monotonicity*, $R_1(c;E) \leq R_1(c'_1, c_2;E)$. Hence, (i) $E - R_1(c;E) \geq E - R_1(c'_1, c_2;E)$. By *efficiency*, (ii) $R_2(c;E) = \min\{c_2, E - R_1(c;E)\}$ and $R_2(c'_1, c_2;E) = \min\{c_2, E - R_1(c'_1, c_2;E)\}$. Thus, (i) and (ii) imply $R_2(c;E) \geq R_2(c'_1, c_2;E)$ and R satisfies *others oriented claims monotonicity*. □

Using Lemma 2 (b), we can easily establish a "weak inverse" of Lemma 1 for two agent problems if we add *efficiency* and *claims monotonicity*.

[8]For bankruptcy problems, this result is also mentioned in Thomson (2003, p. 269).

Lemma 3. *If $|N| = 2$, then efficiency, equal treatment of equals, and claims monotonicity imply weak order preservation.*

Proof. Let rule R satisfy *efficiency*, *equal treatment of equals*, and *claims monotonicity*. Hence, by Lemma 2 (b), R satisfies *others oriented claims monotonicity*. Let $N = \{1, 2\}$, $(c; E) \in \mathcal{E}^N$, and assume without loss of generality that $c_1 \leq c_2$.

Let $c_2' = c_1$. By *efficiency* and *equal treatment of equals*, $R_1(c_1, c_2'; E) = R_2(c_1, c_2'; E) = \min\{c_1, \frac{E}{2}\}$. Since $c_2' \leq c_2$, by *claims monotonicity*, (i) $\min\{c_1, \frac{E}{2}\} = R_2(c_1, c_2'; E) \leq R_2(c; E)$ and by *others oriented claims monotonicity*, (ii) $\min\{c_1, \frac{E}{2}\} = R_1(c_1, c_2'; E) \geq R_1(c; E)$. Thus, (i) and (ii) imply $R_2(c; E) \geq R_1(c; E)$ and R satisfies *weak order preservation*. □

In Section 3 we implicitly show that both Lemma 2 (b) and Lemma 3 cannot be extended to any number of agents.[9]

Finally, using Lemma 2 (a), we show that *efficiency*, *equal treatment of equals*, and *others oriented claims monotonicity* imply *weak order preservation*.

Lemma 4. *Efficiency, equal treatment of equals and others oriented claims monotonicity imply weak order preservation.*

Proof. Let R satisfy *efficiency*, *equal treatment of equals* and *others oriented claims monotonicity*. Then, by Lemma 2 (a), R satisfies *claims monotonicity*. Let $(c; E) \in \mathcal{E}^N$ and $i, j \in N$ such that $c_i > c_j$. Let $c_j' = c_i$. By *equal treatment of equals*, (i) $R_j(c_j', c_{-j}; E) = R_i(c_j', c_{-j}; E)$. Note that $c_j < c_j'$. Hence, by *claims monotonicity*, (ii) $R_j(c; E) \leq R_j(c_j', c_{-j}; E)$ and by *others oriented claims monotonicity*, (iii) $R_i(c; E) \geq R_i(c_j', c_{-j}; E)$. Thus, (i), (ii), and (iii) imply $R_i(c; E) \geq R_j(c; E)$ and R satisfies *weak order preservation*. □

Most well-known bankruptcy rules satisfy all the properties mentioned above; e.g., the constrained equal awards, the constrained equal losses, and the proportional rule. We introduce efficient extensions of these well-known bankruptcy rules to (estate division) rules.

The constrained equal awards rule allocates the estate as equally as possible taking claims as upper bounds.

Constrained Equal Awards Rule: For all $(c; E) \in \mathcal{E}^N$,

(i) if $c_N \leq E$, then $CEA(c; E) = c$ and
(ii) if $c_N \geq E$, then for all $j \in N$, $CEA_j(c; E) = \min\{c_j, \lambda_{cea}\}$,
where λ_{cea} is such that $\sum \min\{c_i, \lambda_{cea}\} = E$.

The constrained equal losses rule allocates the shortage of the estate in an equal way, keeping shares bounded below by zero.

Constrained Equal Losses Rule: For all $(c; E) \in \mathcal{E}^N$,

(i) if $c_N \leq E$, then $CEL(c; E) = c$ and
(ii) if $c_N \geq E$, then for all $j \in N$, $CEL_j(c; E) = \max\{0, c_j - \lambda_{cel}\}$,
where λ_{cel} is such that $\sum \max\{0, c_i - \lambda_{cel}\} = E$.

[9]The rule described in Example 1 satisfies *efficiency*, *equal treatment of equals*, *claims monotonicity*, but neither *others oriented claims monotonicity* nor *weak order preservation*.

The proportional rule allocates the estate proportionally with respect to claims.

Proportional Rule: For all $(c; E) \in \mathcal{E}^N$,

(i) if $c_N \leq E$, then $P(c; E) = c$ and
(ii) if $c_N \geq E$, then $P(c; E) = \lambda_p c$, where $\lambda_p = \frac{E}{c_N}$.

3 Claim Games, Nash Equilibria, and Equal Division

Given an estate $E \in R^N_{++}$, assume that each agent can choose a claim from a strategy set $C_i \subseteq \mathbb{R}_+$. Let $C = C_1 \times \ldots \times C_n$. Then, for each rule R and estate E we define the *claim game* $\Gamma(R, E)$ by assigning to each reported claims vector $c \in C$ the awards vector $R(c; E)$. A claims vector $\hat{c} \in C$ is a *Nash equilibrium (in pure strategies) of claim game* $\Gamma(R, E)$ if for all $i \in N$ and all $c'_i \in C_i$, $R_i(\hat{c}; E) \geq R_i(c'_i, \hat{c}_{-i}; E)$. We call $R(\hat{c}; E)$ the *Nash equilibrium outcome.* We require that for all agents $i \in N$ the strategy set C_i is compact. Hence, for all $i \in N$, $\bar{c}_i \equiv \max C_i$ and $\bar{c} \equiv (\bar{c}_i)_{i\in N}$ are well-defined. For example, we could assume $C_i = [0, E]$ for all $i \in N$.[10]

Since all agents' preferences are strictly monotonic over the amount of the estate that they receive, any Nash equilibrium of a claim game that is based on an *efficient* rule has to distribute the whole estate if that is possible given upper bounds on reported claims, $\bar{c}$. This implies that at any Nash equilibrium $\hat{c}$ in which agents do not claim their maximal possible amounts ($\hat{c} \neq \bar{c}$), the sum of reported claims already adds up to at least the estate ($\hat{c}_N \geq E$). We show this formally.

Lemma 5. *If R is efficient, then for any Nash equilibrium $\hat{c}$ of the claim game $\Gamma(R, E)$, $\hat{c} \neq \bar{c}$ implies $\hat{c}_N \geq E$.*

Proof. Let R be *efficient* and assume that $\hat{c} \neq \bar{c}$ is a Nash equilibrium of the claim game $\Gamma(R, E)$ such that $\hat{c}_N < E$. Let $\alpha \equiv E - \hat{c}_N > 0$ and define for some $j \in N$ such that $\hat{c} \neq \bar{c}_j$, $c'_j \equiv \min\{\bar{c}_j, \hat{c}_j + \alpha\} > \hat{c}_j$. Then, by *efficiency*, $R(\hat{c}; E) = \hat{c}$ and $R(c'_j, \hat{c}_{-j}; E) = (c'_j, \hat{c}_{-j})$. Hence, $R_j(\hat{c}; E) = \hat{c}_j < c'_j = R_j(c'_i, \hat{c}_{-i}; E)$; contradicting that $\hat{c}$ is a Nash equilibrium of $\Gamma(R, E)$. □

We denote by $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^N_{++}$ the *one-vector.*

Equal Division: Given an estate $E \in R^N_{++}$, $\frac{E}{n}\mathbf{1} \in \mathbb{R}^N_{++}$ denotes the corresponding *equal division vector.*

Next, we present our first main result: we show that for claim games where agents have equal maximal strategies and the underlying rule satisfies *efficiency*, *equal treatment of equals*, and *weak order preservation*, (a) claiming the maximal amount is always a Nash equilibrium and (b) all Nash equilibria induce equal division.

Theorem 1. *Assume that for some $k \in \mathbb{R}_{++}$, $\bar{c} = k\mathbf{1}$. Let R satisfy efficiency, equal treatment of equals, and weak order preservation. Then,*

(a) *$\hat{c} = \bar{c}$ is a Nash equilibrium of the claim game $\Gamma(R, E)$ and*
(b) *all Nash equilibria of the claim game $\Gamma(R, E)$ have the equal division vector $\min\{k, \frac{E}{n}\}\mathbf{1}$ as equilibrium outcome.*

[10] Alternatively, instead of requiring that all strategy sets are compact we could require that for all $i \in N$, $C_i = \mathbb{R}_+$ and rule R satisfies claims truncation invariance, i.e., for all $(c; E) \in \mathcal{E}^N$, $R(c; E) = R(\min\{c_1, E\}, \ldots, \min\{c_n, E\}; E)$.

Proof.

(a) We prove that $\hat{c} = \bar{c} = k\mathbf{1}$ is a Nash equilibrium of the claim game $\Gamma(R, E)$. By *efficiency* and *equal treatment of equals*, $R(\hat{c}, E) = \min\{k, \frac{E}{n}\}\mathbf{1}$. If $R(\hat{c}, E) = k\mathbf{1}$, then each agent already gets the largest possible amount and $\hat{c}$ is a Nash equilibrium.

Thus, assume that $R(\hat{c}, E) = \frac{E}{n}\mathbf{1} < k\mathbf{1}$. Let $i \in N$ and $c'_i \neq \hat{c}_i$. Thus, for all $j \neq i$, $c'_i < k = \hat{c}_j$. Hence, by *weak order preservation*, for all $j \neq i$, $R_j(c'_i, \hat{c}_{-i}; E) \geq R_i(c'_i, \hat{c}_{-i}; E)$. Suppose that $R_i(c'_i, \hat{c}_{-i}; E) > \frac{E}{n}$. Then, for all $l \in N$, $R_l(c'_i, \hat{c}_{-i}; E) > \frac{E}{n}$ and $\sum R_l(c'_i, \hat{c}_{-i};, E) > E$; a contradiction. Thus, $R_i(c'_i, \hat{c}_{-i}; E) \leq \frac{E}{n} = R_i(\hat{c}; E)$ and $\hat{c}$ is a Nash equilibrium of the claim game $\Gamma(R, E)$.

(b) Suppose that $\hat{c}$ is a Nash equilibrium of the claim game $\Gamma(R, E)$ and $R(\hat{c}; E) \neq \min\{k, \frac{E}{n}\}\mathbf{1}$. Then, for some $i \in N$, $R_i(\hat{c}; E) < \min\{k, \frac{E}{n}\}$. Let $c'_i = k$ (possibly $c'_i = \hat{c}_i$). Since $\hat{c}$ is a Nash equilibrium, $R_i(c'_i, \hat{c}_{-i}; E) \leq R_i(\hat{c}; E) < \min\{k, \frac{E}{n}\}$. In particular, (i) $R_i(c'_i, \hat{c}_{-i}; E) < \frac{E}{n}$.

Since by (a), $\bar{c}$ is a Nash equilibrium such that $R(\bar{c}; E) = \min\{k, \frac{E}{n}\}\mathbf{1}$, we know that $\hat{c} \neq \bar{c}$. Thus, by Lemma 5, $\hat{c}_N \geq E$. Recall that $c'_i = k \geq \hat{c}_i$. Hence, $c'_i + \sum_{l \neq i} \hat{c}_l \geq E$ and by *efficiency*, (ii) $\sum R_l(c'_i, \hat{c}_{-i}; E) = E$. For all $j \neq i$ such that $\hat{c}_j < k = c'_i$, by *weak order preservation* and (i), $R_j(c'_i, \hat{c}_{-i}; E) \leq R_i(c'_i, \hat{c}_{-i}; E) < \frac{E}{n}$. For all $j \neq i$ such that $\hat{c}_j = k = c'_i$, by *equal treatment of equals* and (i), $R_j(c'_i, \hat{c}_{-i}; E) = R_i(c'_i, \hat{c}_{-i}; E) < \frac{E}{n}$. Hence, $\sum R_l(c'_i, \hat{c}_{-i}; E) < E$; a contradiction to (ii). □

Remark 1. ***Independence of Assumptions in Theorem 1***

(i) Suppose that there exist $i, j \in N$ such that $\bar{c}_i \neq \bar{c}_j$. Then, an "unequal" Nash equilibrium outcome is possible for a rule satisfying *efficiency*, *equal treatment of equals*, and *weak order preservation*; e.g., for the proportional rule, for all $(c; E) \in \mathcal{E}^N$, $\bar{c}$ is a Nash equilibrium, but for all i, j such that $\bar{c}_i \neq \bar{c}_j$, $P_i(\bar{c}; E) \neq P_j(\bar{c}; E)$.

(ii) The following rule R' satisfies *equal treatment of equals*, *weak order preservation*, but not *efficiency*. If $c \neq \bar{c}$, then $R'(c; E) = P(c; E)$ and $R'(\bar{c}; E) = 0\mathbf{1}$. Clearly, $\bar{c}$ is not a Nash equilibrium of the claim game $\Gamma(E, R')$ and the equal division vector $\min\{k, \frac{E}{n}\}\mathbf{1}$ is never an equilibrium outcome.

(iii) A serial dictatorship rule that first serves agents with the highest claims lexicographically (i.e., if several agents have the highest claim, then first serve the agent with the lowest index and so on) satisfies *efficiency* and *weak order preservation*, but not *equal treatment of equals*. There are Nash equilibria, e.g., $\hat{c} = k\mathbf{1}$ when $nk > E$, at which agent 1 receives more than agent n.

(iv) The following rule R'' satisfies *efficiency* and *equal treatment of equals*, but not *weak order preservation*. Rule R'' first assigns the estate E proportionally (and *efficiently*) among all agents who have a claim different from that of agent 1. Then, if some part of the estate is left, R'' allocates it equally (and *efficiently*) among the remaining agents. For $\bar{c} \geq E\mathbf{1}$, $\bar{c}$ is not a Nash equilibrium of the claim game $\Gamma(E, R'')$ and the equal division vector $\min\{k, \frac{E}{n}\}\mathbf{1}$ is not an equilibrium outcome. ⋄

Theorem 1 and Lemma 1 imply the following corollary.

Corollary 1. *Assume that for some $k \in \mathbb{R}_{++}$, $\bar{c} = k\mathbf{1}$. Let R satisfy efficiency and order preservation. Then,*

- **(a)** *$\hat{c} = \bar{c}$ is a Nash equilibrium of the claim game $\Gamma(R, E)$ and*
- **(b)** *all Nash equilibria of the claim game $\Gamma(R, E)$ have the equal division vector $\min\{k, \frac{E}{n}\}\mathbf{1}$ as equilibrium outcome.*

Theorem 1 and Lemma 4 imply the following corollary.

Corollary 2. *Assume that for some $k \in \mathbb{R}_{++}$, $\bar{c} = k\mathbf{1}$. Let R satisfy efficiency, equal treatment of equals, and others oriented claims monotonicity. Then,*

- **(a)** *$\hat{c} = \bar{c}$ is a Nash equilibrium of the claim game $\Gamma(R, E)$ and*
- **(b)** *all Nash equilibria of the claim game $\Gamma(R, E)$ have the equal division vector $\min\{k, \frac{E}{n}\}\mathbf{1}$ as equilibrium outcome.*

Next, we present our second main result: we show that for claim games where agents have equal maximal strategies and the underlying rule satisfies *efficiency*, *equal treatment of equals*, and *claims monotonicity*, (a) claiming the maximal amount is always a Nash equilibrium and (b) for $n \leq 3$, all Nash equilibria induce equal division.

Theorem 2. *Assume that for some $k \in \mathbb{R}_{++}$, $\bar{c} = k\mathbf{1}$. Let R satisfy efficiency, equal treatment of equals, and claims monotonicity. Then,*

- **(a)** *$\hat{c} = \bar{c}$ is a Nash equilibrium of the claim game $\Gamma(R, E)$ and*
- **(b)** *for $n \leq 3$, all Nash equilibria of the claim game $\Gamma(R, E)$ have the equal division vector $\min\{k, \frac{E}{n}\}\mathbf{1}$ as equilibrium outcome.*

Proof.
(a) By *claims monotonicity*, for each agent i it is a weakly dominant strategy to claim $\bar{c}_i$. Hence, $\bar{c}$ is a Nash equilibrium of $\Gamma(R, E)$.

(b) For $n = 1$, the proof is obvious and therefore omitted.

Let $n = 2$. By Lemma 3, *efficiency*, *equal treatment of equals*, and *claims monotonicity* imply *weak order preservation*. By Theorem 1 (b), *efficiency*, *equal treatment of equals*, and *weak order preservation* imply the result.

Let $n = 3$. Suppose that $\hat{c}$ is a Nash equilibrium of the claim game $\Gamma(R, E)$ and $R(\hat{c}; E) \neq \min\{k, \frac{E}{3}\}\mathbf{1}$. Without loss of generality, we assume that $\hat{c}_1 \leq \hat{c}_2 \leq \hat{c}_3$.

Case 1: $i \in \{1, 2\} \equiv \{i, j\}$ and $R_i(\hat{c}; E) < \min\{k, \frac{E}{3}\}$.
Let $c'_i = \hat{c}_3$ (possibly $c'_i = \hat{c}_1$). Since $\hat{c}$ is a Nash equilibrium, $R_i(c'_i, \hat{c}_{-i}; E) \leq R_i(\hat{c}; E) < \min\{k, \frac{E}{3}\}$. In particular, (i) $R_i(c'_i, \hat{c}_{-i}; E) < \frac{E}{3}$.

Since by (a), $\bar{c}$ is a Nash equilibrium such that $R(\bar{c}; E) = \min\{k, \frac{E}{n}\}\mathbf{1}$, we know that $\hat{c} \neq \bar{c}$. Thus, by Lemma 5, $\hat{c}_N \geq E$. Hence, $c'_i + \sum_{l \neq i} \hat{c}_l \geq E$ and by *efficiency*, (ii) $\sum R_l(c'_i, \hat{c}_{-i}; E) = E$. By *equal treatment of equals* and (i), $R_3(c'_i, \hat{c}_{-i}; E) = R_i(c'_i, \hat{c}_{-i}; E) < \frac{E}{3}$. Hence, (ii) implies (iii) $R_j(c'_i, \hat{c}_{-i}; E) > \frac{E}{3}$.

Recall that $j \in \{1, 2\}$ and therefore, $\hat{c}_j \leq \hat{c}_3$. Let $c'_j = \hat{c}_3$ and consider $(c'_i, c'_j, \hat{c}_3; E) = (\hat{c}_3\mathbf{1}; E)$. By *equal treatment of equals*, (i) and (iii) imply $\hat{c}_j < \hat{c}_3$. Hence, $c'_i + c'_j + \hat{c}_3 > E$ and by *efficiency*, (iv) $\sum R_l(c'_i, c'_j, \hat{c}_3; E) = E$. By *claims monotonicity*, $R_j(c'_i, c'_j, \hat{c}_3; E) \geq R_j(c'_i, \hat{c}_{-i}; E) > \frac{E}{3}$ and by *equal treatment of equals*, $R_j(c'_i, c'_j, \hat{c}_3; E) = R_i(c'_i, c'_j, \hat{c}_3; E) = R_3(c'_i, c'_j, \hat{c}_3; E) > \frac{E}{3}$. Hence, $\sum R_l(c'_i, c'_j, \hat{c}_3; E) > E$; a contradiction to (iv).

Case 2: $R_3(\hat{c}; E) < \min\{k, \frac{E}{3}\}$.
First, $R_3(\hat{c}; E) < \min\{k, \frac{E}{3}\}$ implies (v) $R_3(\hat{c}; E) < \frac{E}{3}$. Furthermore, if $\hat{c}_2 = \hat{c}_3$, then by *equal treatment of equals*, $R_2(\hat{c}; E) = R_3(\hat{c}; E) < \min\{k, \frac{E}{3}\}$, and we are done by Case 1. Hence, assume that $\hat{c}_2 < \hat{c}_3$. Let $c'_3 = \hat{c}_2$ and consider $(c'_3, \hat{c}_{-3}; E)$.

Since by (a), $\bar{c}$ is a Nash equilibrium such that $R(\bar{c}; E) = \min\{k, \frac{E}{n}\}\mathbf{1}$, we know that $\hat{c} \neq \bar{c}$. Thus, by Lemma 5, $\hat{c}_N \geq E$. Hence, by *efficiency*, (vi) $\sum R_l(\hat{c}; E) = E$. Then, (v) and (vi) imply $R_1(\hat{c}; E) > \frac{E}{3}$ or $R_2(\hat{c}; E) > \frac{E}{3}$. Note that $R_1(\hat{c}; E) \leq \hat{c}_1 \leq \hat{c}_2$ and $R_2(\hat{c}; E) \leq \hat{c}_2$. Thus, $c'_3 > R_3(\hat{c}; E)$ and $\hat{c}_1 + \hat{c}_2 + c'_3 \geq \sum R_l(\hat{c}; E) = E$, where the last equality follows from (vi). By *efficiency*, (vii) $\sum R_l(c'_3, \hat{c}_{-3}; E) = E$. By *claims monotonicity*, (viii) $R_3(c'_3, \hat{c}_{-3}; E) \leq R_3(\hat{c}; E) < \frac{E}{3}$. By *equal treatment of equals*, $R_2(c'_3, \hat{c}_{-3}; E) = R_3(c'_3, \hat{c}_{-3}; E) < \frac{E}{3}$. Hence, (vii) implies (ix) $R_1(c'_3, \hat{c}_{-3}; E) > \frac{E}{3}$.

Let $c'_1 = \hat{c}_2$ and consider $(c'_1, c'_3, \hat{c}_2; E) = (\hat{c}_2\mathbf{1}; E)$. By *equal treatment of equals*, (viii) and (ix) imply $\hat{c}_1 < c'_3 = \hat{c}_2$. Hence, $c'_1 + c'_3 + \hat{c}_2 > E$ and by *efficiency*, (x) $\sum R_l(c'_1, c'_3, \hat{c}_2; E) = E$. By *claims monotonicity*, $R_1(c'_1, c'_3, \hat{c}_2; E) \geq R_1(c'_3, \hat{c}_{-3}; E) > \frac{E}{3}$ and by *equal treatment of equals*, $R_1(c'_1, c'_3, \hat{c}_2; E) = R_2(c'_1, c'_3, \hat{c}_2; E) = R_3(c'_1, c'_3, \hat{c}_2; E) > \frac{E}{3}$. Hence, $\sum R_l(c'_1, c'_3, \hat{c}_2; E) > E$; a contradiction to (x). □

Remark 2. *Independence of Assumptions in Theorem 2*
Note that *claims monotonicity* alone implies Theorem 2 (a). Hence, we show independence only for Theorem 2 (b).

(i) The proof that the assumption $\bar{c} = k\mathbf{1}$ is needed is the same as in Remark 1 (i).

(ii) The following rule $\tilde{R}$ satisfies *claims monotonicity* and *equal treatment of equals*, but not *efficiency*. If at c exactly one agent i claims $c_i = \bar{c}_i$, then he receives $\tilde{R}_i(c; E) = \min\{k, \frac{E}{n}\}$ and for all $j \neq i$, $\tilde{R}_i(c; E) = 0$. Furthermore, $\tilde{R}(\bar{c}; E) = \min\{k, \frac{E}{n}\}\mathbf{1}$ and for all other claim vectors c, $\tilde{R}(c; E) = 0\mathbf{1}$. Then, Nash equilibria which do not induce equal division exist, e.g., for $N = \{1, 2, 3\}$ and $E = k = 1$, $\tilde{c} = (1, 0, 0)$ resulting in the equilibrium outcome $\tilde{R}(\tilde{c}; 1) = (1/3, 0, 0)$.

(iii) To prove that *equal treatment of equals* is needed one can use the serial dictatorship rule as described in Remark 1 (iii) – it satisfies *efficiency* and *claims monotonicity*, but not *equal treatment of equals*.

(iv) To prove that *claims monotonicity* is needed one can use rule R'' as described in Remark 1 (iv) – it satisfies *efficiency* and *equal treatment of equals*, but not *claims monotonicity*. ◇

In the following example, we show that when $n > 3$, *efficiency*, *equal treatment of equals*, and *claims monotonicity* are not sufficient to guarantee equal division in all Nash equilibria of the claim game $\Gamma(R, E)$.

Example 1. Let $N = \{1, 2, 3, 4\}$, $E = 1$, and for all $i \in N$, $C_i = [0, 1]$. Before we define rule R of claim game $\Gamma(R, 1)$, we introduce some notation.

Let $H = 1/2$ and $L = 1/3$ be two points, which we will use to partition the set of claim profiles. For all profiles $c \in C$, let $LH(c) = \{i \in N : L \leq c_i \leq H\}$, $L(c) = \{i \in N : c_i < L\}$, and $H(c) = \{i \in N : c_i > H\}$. For all $j = 0, 1, \ldots, 4$, denote by C^j the set of claim profiles in which j agents claim between L and H and $n - j$ agents claim more than H. That is, $C^j = \{c \in C : |LH(c)| = j \text{ and } |H(c)| = n - j\}$. Let $P = C \setminus \cup_{j=0}^{4} C^j$. Note that for all claim profiles $c \in P$, there exists some agent $i \in N$ for which $c_i < L$. Furthermore, note that the collection of sets C^j $(j = 1, \ldots, 4)$ and P partition the set of claim profiles C.

For all $c \in P$, define $B(c)$ to be the maximal set of agents such that

(i) $c_{B(c)} \leq 1$, i.e., the sum of claims of agents in $B(c)$ does not exceed the estate, and
(ii) for all $i, l \in N$, if $c_i \geq c_l$ and $l \in B(c)$, then $i \in B(c)$, i.e., if agent l is a member of $B(c)$, then all agents with claims larger than or equal to c_l are also members of $B(c)$.

Let $D(c) = \{i \in N \setminus B(c) : \text{for all } l \in N \setminus B(c),\ c_i \geq c_l\}$, i.e., if $B(c) \neq N$, then $D(c) \neq \emptyset$ contains the set of agents that have the highest claim among the agents in $N \setminus B(c)$. Note that $D(c) = \emptyset$ if and only if $c_N \leq 1$. Finally, let $A(c) = B(c) \cup D(c)$.

Roughly speaking, rule R works as follows. For claim profiles in $\cup_{j=0}^{4} C^j$, we specify awards to agents according to their claims being larger than H or not. For claim profiles in P, rule R does the following: it first ranks agents from highest claim to lowest claim. Then, to all agents in the set $B(c)$, R gives their full claim, and allocates the residual amount equally to agents in $D(c)$. All other agents receive 0.

$$R_i(c,1) = \begin{cases} 1/4, & & & c \in C^4 \cup C^0, \\ 1/6, & c_i \leq H & \text{and} & c \in C^3, \\ 1/2, & c_i > H & \text{and} & c \in C^3, \\ 1/3, & c_i \leq H & \text{and} & c \in C^2, \\ 1/6, & c_i > H & \text{and} & c \in C^2, \\ 0, & c_i \leq H & \text{and} & c \in C^1, \\ 1/3, & c_i > H & \text{and} & c \in C^1, \\ c_i, & c \in P & \text{and} & i \in B(c), \\ \frac{1-c_{B(c)}}{|D(c)|}, & c \in P & \text{and} & i \in D(c), \\ 0, & c \in P & \text{and} & i \notin A(c). \end{cases}$$

We prove in Appendix A (Claim 1) that rule R satisfies *efficiency*, *equal treatment of equals*, and *claims monotonicity*. Next, we show that the profile of claims $c = (1, 1/3, 1/3, 1/3)$ is an equilibrium in $\Gamma(R, 1)$ and since $R(c; 1) = (1/2, 1/6, 1/6, 1/6)$, we have a violation of equal division in equilibrium.

Note that $c \in C^3$. Then, a unilateral deviation by agent 1 can only result in a claim profile that belongs to one of the sets C^3, C^4, or P, which induces the amounts (for agent 1) 1/2, 1/4, or 0, respectively. Since at c agent 1 obtains 1/2, no unilateral deviation from c is beneficial for agent 1. Next, a unilateral deviation by agent $k \in \{2, 3, 4\}$ can only result in a claim profile that belongs to one of the sets C^2, C^3, or P, which induces the amounts (for agent k) 1/6, 1/6, or 0, respectively. Since at c agent k obtains 1/6, no unilateral deviation from c is beneficial for agent k. ◇

Note that the rule described in Example 1 violates *weak order preservation* and *others oriented claims monotonicity*.

Finally, we show that equal division is restored in the equilibrium result of Theorem 2 for more than three agents by adding a non-manipulation property: *nonbossiness* (Satterthwaite and Sonnenschein, 1981) requires that no agent can change other agents' awards by changing his claim without changing his own award.

Nonbossiness: A rule R satisfies *nonbossiness* if for all $(c; E) \in \mathcal{E}^N$ and all $i \in N$ such that $R_i(c; E) = R_i(c'_i, c_{-i}; E)$, $R_j(c; E) = R_j(c'_i, c_{-i}; E)$ for all $j \neq i$.

Theorem 3. *Assume that for some $k \in \mathbb{R}_{++}$, $\bar{c} = k\mathbf{1}$. Let R satisfy efficiency, equal treatment of equals, claims monotonicity, and nonbossiness. Then,*

(a) *$\hat{c} = \bar{c}$ is a Nash equilibrium of the claim game $\Gamma(R, E)$ and*

(b) *all Nash equilibria of the claim game $\Gamma(R, E)$ have the equal division vector $\min\{k, \frac{E}{n}\}\mathbf{1}$ as equilibrium outcome.*

Proof.

(a) This part follows immediately from *claims monotonicity*.

(b) We first prove that *equal treatment of equals*, *claims monotonicity*, and *nonbossiness* imply equal division for all Nash equilibria. Suppose that $\hat{c}$ is a Nash equilibrium of the claim game $\Gamma(R, E)$ and for some $i, j \in N$, (i) $R_i(\hat{c}; E) \neq R_j(\hat{c}; E)$. Hence, by *equal treatment of equals* $\hat{c}_i \neq \hat{c}_j$. Without loss of generality assume that $\hat{c}_i < \hat{c}_j$. Let $c'_i = \hat{c}_j$ and consider $(c'_i, \hat{c}_{-i}; E)$. Since $\hat{c}$ is a Nash equilibrium, $R_i(c'_i, \hat{c}_{-i}; E) \leq R_i(\hat{c}; E)$. By *claims monotonicity*, $R_i(c'_i, \hat{c}_{-i}; E) \geq R_i(\hat{c}; E)$. Hence, (ii) $R_i(c'_i, \hat{c}_{-i}; E) = R_i(\hat{c}; E)$. Thus, by *nonbossiness*, (iii) $R_j(c'_i, \hat{c}_{-i}; E) = R_j(\hat{c}; E)$. Then, (i), (ii), and (iii) imply $R_i(c'_i, \hat{c}_{-i}; E) \neq R_j(c'_i, \hat{c}_{-i}; E)$; a contradiction to *equal treatment of equals*. Therefore, for all $i, j \in N$, $R_i(\hat{c}; E) = R_j(\hat{c}; E)$, which proves an equal division vector is induced by all Nash equilibria. By *efficiency*, this equal division vector equals $\min\{k, \frac{E}{n}\}\mathbf{1}$. □

From the proof of Theorem 3 it becomes clear that even without *efficiency* Nash equilibria outcomes respect equal division. However, without *efficiency*, some part of the estate might be wasted.

Note that the rule described in Example 1 violates *nonbossiness*.

Remark 3. ***Independence of Assumptions in Theorem 3***
Note that *claims monotonicity* alone implies Theorem 2 (a). Hence, we show independence only for Theorem 2 (b).

(i) The proof that the assumption $\bar{c} = k\mathbf{1}$ is needed is the same as in Remark 1 (i).

(ii) As explained after the proof of Theorem 3, *efficiency* is only needed to obtain the *efficient* equal division vector as equilibrium outcome. The following inefficient proportional rule satisfies *equal treatment of equals*, *claims monotonicity*, and *nonbossiness*, but not *efficiency*: for all $(c; E) \in \mathcal{E}^N$, $P'(c; E) = P(c; E/2)$.

(iii) To prove that *equal treatment of equals* is needed one can use the serial dictatorship rule as described in Remark 1 (iii) – it satisfies *efficiency*, *claims monotonicity*, and *nonbossiness* but not *equal treatment of equals*

(iv) To prove that *claims monotonicity* is needed one can use rule R'' as described in Remark 1 (iv) – it satisfies *efficiency*, *equal treatment of equals*, and *nonbossiness*, but not *claims monotonicity*.

(v) To prove that *nonbossiness* is needed one can use the same rule as in Example 1 – it satisfies *equal treatment of equals*, and *claims monotonicity*, but not *nonbossiness*. ◇

Remark 4. ***Nash Equilibria in*** $\Gamma(P, E)$***,*** $\Gamma(CEA, E)$ ***and*** $\Gamma(CEL, E)$
The proportional rule, the constrained equal awards rule, and the constrained equal losses rule satisfy all properties introduced in this article. Hence, for these rules, claiming the largest possible amount is always an equal division Nash equilibrium. For the proportional rule and the constrained equal losses rule, this is the unique Nash equilibrium of the associated claim game. However, if agents are allowed to claim more than an equal share of the estate, the constrained equal awards rule admits multiple (in fact infinitely many) equal division Nash equilibria. This difference stems from the fact that under the proportional rule and the constrained equal losses rule, claiming the whole estate is a strictly dominant strategy for all agents whereas under the constrained equal awards rule, it is a weakly dominant strategy.

4 Concluding Remarks

We analyze situations where an estate should be distributed among a set of agents, but claims to the estate are impossible or difficult to verify. We model a simple and intuitive claim game where, given the estate and a rule satisfying some basic properties, agents simply announce their claims. Our results show that first of all, claiming the largest possible amount is *always* a Nash equilibrium. Of course, this is an intuitive and not very surprising result. However, in addition, we show that even though we do not focus on any specific rule to be used in our claim game, equal division is the unique Nash equilibrium outcome. Since most well-known rules satisfy all the properties we require (e.g., the proportional, the constrained equal awards, and the constrained equal losses rule), our results can be interpreted as a noncooperative support for equal division in estate division. Finally, future research on this topic might analyze situations in which partial verification is possible and agents spend resources to support their claims (e.g., hiring a lawyer in a court case).

Appendix A

Claim 1. Rule R as defined in Example 1 satisfies *equal treatment of equals*, *efficiency*, and *claims monotonicity*.

Proof. Equal treatment of equals follows immediately from the definition of rule R.
Efficiency: Note that for all $c \in \cup_{j=0}^{4} C^j$, $c_N \geq 1$ and $\sum R_l(c;1) = 1$. Assume that $c \in P$. If $c_N \leq 1$, then $R(c;1) = c$. Finally, if $c_N > 1$, then $\sum R_l(c;1) = \sum_{i \in B(c)} R_i(c;1) + \sum_{i \in D(c)} R_i(c;1) + \sum_{i \in N \setminus A(c)} R_i(c;1) = c_{B(c)} + \sum_{i \in D(c)} \frac{1 - c_{B(c)}}{|D(c)|} + 0 = 1$.
Claims Monotonicity: Let $i \in N$, $c = (c_i, c_{-i})$, and $c' = (c'_i, c_{-i})$ such that $c_i < c'_i$. We show that $R_i(c;1) \leq R_i(c'_i;1)$ for the following (exhaustive) cases.
Case 1: $c, c' \in P$.
If $i \notin A(c)$, then $R_i(c;1) = 0 \leq R_i(c';1)$. If $i \in A(c)$, then $i \in A(c')$, i.e., $i \in B(c')$ or $i \in D(c')$. If $i \in B(c')$, then $R_i(c;1) \leq c_i \leq c'_i = R_i(c';1)$ and we are done. Assume that $i \in D(c')$. Since $c_i < c'_i$, for all $j \in N \setminus B(c)$, $c_j < c'_i$. Hence, $A(c') \setminus \{i\} \subseteq B(c)$. Therefore, for all $j \in B(c')$, $R_j(c';1) = R_j(c;1)$, and for all $j \in D(c') \setminus \{i\}$, $R_j(c';1) \leq R_j(c;1)$. Thus, we showed that for all $j \in A(c') \setminus \{i\}$, $R_j(c';1) \leq R_j(c;1)$. Since for all $j \in N \setminus A(c')$, $R_j(c';1) = 0$ and R is *efficient* it follows that $R_i(c;1) \leq R_i(c';1)$.
Case 2: $c \in P$ and for some $j \in \{0,1,2,3,4\}$, $c' \in C^j$.
Note that $L(c) = \{i\}$ and $LH(c) \cup H(c) = N \setminus \{i\}$. Since $L \geq 1/3$, $c_{N \setminus \{i\}} \geq 1$ and $\sum_{l \neq i} R_l(c;1) = 1$. Thus, $R_i(c;1) = 1 - \sum_{l \neq i} R_l(c;1) = 0 \leq R_i(c';1)$.
Case 3: for some $j \in \{0,1,2,3,4\}$, $c \in C^j$ and $c' \in C^j$.
Note that either $[i \in LH(c)$ and $i \in LH(c')]$ or $[i \in H(c)$ and $i \in H(c')]$. Thus, $R_i(c;1) = R_i(c';1)$.
Case 4: for some $j \in \{1,2,3,4\}$, $c \in C^j$ and $c' \in C^{j-1}$.
Note that $i \in LH(c)$ and $i \in H(c')$. Then, by the definition of R, for $j = 1,4$, $R_i(c;1) < R_i(c';e)$ and for $j = 2,3$, $R_i(c;1) = R_i(c';1)$. □

Appendix B

In this appendix we describe what happens if we require that the estate E is always completely allocated among the agents. Formally, a *(full division) rule* is a function $R : \mathcal{E}^N \to \mathbb{R}^N$ that associates with each problem $(c; E) \in \mathcal{E}^N$ an *awards vector* $x \in \mathbb{R}^N_+$ such that $\sum x_i = E$.

First, all our property results from Section 2 (i.e., Lemmas $1 - 4$) can be stated similarly without mentioning *efficiency* and in the proofs the fact that the whole estate is always allocated is used instead of *efficiency*.

Second, we consider our results in Section 3. Note that Lemma 5 does not hold anymore, i.e., it is not always the case that in every Nash equilibrium $\hat{c}$ of the claim game $\Gamma(R, E)$, $\hat{c}_N \geq E$; e.g., for the constant rule that always assigns E/n to each agent, every claim vector is a Nash equilibrium. Although this lemma is used in some of our proofs, it is used only in order to show that in equilibrium the entire estate is allocated. Hence, again the fact that the whole estate is always allocated is used instead of Lemma 5. Furthermore, all results that state that the division vector in a Nash equilibrium is $\min\{k, \frac{E}{n}\}\mathbf{1}$ are changed to have the $\frac{E}{n}\mathbf{1}$ division vector. To summarize, Theorems $1 - 3$ and Corollaries 1 and 2 hold with minimal changes in the statements and proofs. Finally, the only adjustment of Example 1 needed to fit the model described here is to change rule R in Example 1 to rule $\tilde{R}$ as follows: for every $c \notin P$ let $\tilde{R}(c; E) = R(c; E)$ and for every $c \in P$ let $\tilde{R}(c; E) = \frac{E}{4}$.

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