

THE 8-UNIVERSALITY CRITERION IS UNIQUE

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ABSTRACT. Using the methods developed for the proof that the 2-universality criterion is unique, we partially characterize criteria for the n -universality of positive-definite integer-matrix quadratic forms. We then obtain the uniqueness of Oh's 8-universality criterion as an application of our characterization results.

1. INTRODUCTION

A degree-two homogenous polynomial in n independent variables is called a *quadratic form* (or just *form*) of rank n . For a rank- n quadratic form $Q(x_1, \dots, x_n) = \sum_{i,j} a_{ij}x_i x_j$ (where $a_{ij} = a_{ji}$), the matrix given by $L = (a_{ij})$ is the *Gram Matrix* of a \mathbb{Z} -lattice L equipped with a symmetric bilinear form $B(\cdot, \cdot)$ such that $B(L, L) \subseteq \mathbb{Z}$. Then, $Q(\mathbf{x}) = \mathbf{x}^T L \mathbf{x} = B(L\mathbf{x}, \mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$.

A rank- n quadratic form Q is said to *represent* an integer k if there exists an $\mathbf{x} \in \mathbb{Z}^n$ such that $Q(\mathbf{x}) = k$. More generally, a \mathbb{Z} -lattice L *represents* another \mathbb{Z} -lattice ℓ if there exists a \mathbb{Z} -linear, bilinear form-preserving injection $\ell \rightarrow L$. A quadratic form is called *universal* if it represents all positive integers. Analogously, a lattice is called *n -universal* if it represents all positive-definite integer-matrix rank- n quadratic forms. Connecting these two notions of universality, we observe that a rank- n quadratic form Q is universal if and only if it is 1-universal, as for an integer k ,

$$k = Q(x_1, \dots, x_n) \iff Q(x_1x, \dots, x_nx) = kx^2.$$

In 1993, Conway and Schneeberger announced their celebrated *Fifteen Theorem*, giving a criterion characterizing the universal positive-definite integer-matrix quadratic forms. Specifically, they showed that any positive-definite integer-matrix form which represents the set of nine critical numbers

$$\mathcal{S}_1 = \{1, 2, 3, 5, 6, 7, 10, 14, 15\}$$

is universal (see [C2, Bh]). Kim, Kim, and Oh [KKO2] presented an analogous criterion for 2-universality, showing that a positive-definite integer-matrix lattice is 2-universal if and only if it represents the set of forms

$$\mathcal{S}_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \right\}.$$

Oh [Oh] gave a similar criterion for 8-universality, which we state in Theorem 3 of Section 4.

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A set \mathcal{S} of rank- n lattices having the property that a lattice L is n -universal if and only if L represents every lattice in \mathcal{S} is called an n -*criterion set*. Thus, for example, the set \mathcal{S}_2 obtained by Kim, Kim, and Oh [KKO2] is a 2-criterion set and the set \mathcal{S}_1 found by Conway [C2] naturally gives the 1-criterion set

$$\{x^2, 2x^2, 3x^2, 5x^2, 6x^2, 7x^2, 10x^2, 14x^2, 15x^2\}.$$

The set \mathcal{S}_1 of the Fifteen Theorem is known to be unique (see [KKO1]), in the sense that if \mathcal{S}'_1 is a set of integers such that a quadratic form is universal if and only if it represents the full set \mathcal{S}'_1 , then $\mathcal{S}_1 \subseteq \mathcal{S}'_1$. The author [Ko] recently obtained an analogous uniqueness result for the 2-criterion set \mathcal{S}_2 .

Kim, Kim, and Oh [KKO1] have proven that n -criterion sets exist for all positive integers n . However, the problems of finding and determining the uniquenesses of criterion sets have both proven to be difficult (see [KKO1]). Here, we advance both problems: We obtain the first characterization results for arbitrary n -criterion sets, from which we obtain the uniqueness of Oh's 8-universality criterion as a corollary.

2. NOTATIONS AND TERMINOLOGY

We use the lattice-theoretic language of quadratic form theory. A complete introduction to this approach may be found in [O'M].

For a \mathbb{Z} -lattice (or hereafter, just *lattice*) L with basis $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$, we write $L \cong \mathbb{Z}\mathbf{x}_1 + \dots + \mathbb{Z}\mathbf{x}_n$. If L is of the form $L = L_1 \oplus L_2$ for sublattices L_1, L_2 of L and $B(L_1, L_2) = 0$ then we write $L \cong L_1 \perp L_2$ and say that L_1 and L_2 are *orthogonal*.

For a sublattice ℓ of $L_1 \perp L_2$ which can be expressed in the form

$$\ell \cong \mathbb{Z}(\mathbf{x}_{1,1} + \mathbf{x}_{2,1}) + \dots + \mathbb{Z}(\mathbf{x}_{1,n} + \mathbf{x}_{2,n})$$

with $\mathbf{x}_{i,j} \in L_i$, we denote $\ell(L_i) := \mathbb{Z}\mathbf{x}_{i,1} + \dots + \mathbb{Z}\mathbf{x}_{i,n}$. We naturally extend this notation to lattices ℓ represented by $L_1 \perp L_2$. We then say that a lattice is *additively indecomposable* if either $\ell(L_1) \cong 0$ or $\ell(L_2) \cong 0$ whenever $L_1 \perp L_2$ represents ℓ . Otherwise, we say that ℓ is *additively decomposable*.

Finally, we use the lattice notation of Conway [C1]. In particular, I_n is the rank- n lattice of the form $\langle 1, \dots, 1 \rangle$ and E_8 is the unique even unimodular lattice of rank 8.

3. CHARACTERIZATION RESULTS FOR n -CRITERION SETS

In this section, we prove two results which partially characterize the contents of arbitrary n -criterion sets.

Proposition 1. *Any n -criterion set must include the lattice I_n .*

Proof. If \mathcal{T} is a finite, nonempty set of rank- n lattices not containing I_n , then every lattice $T \in \mathcal{T}$ may be written in the form $T \cong I_k \perp T'$, where $0 \leq k < n$, the sublattice T' is of rank $n - k$, and the first minimum of T' is larger than 1. Indeed, any I_k -sublattice of T is unimodular and therefore splits T ; the condition on T' follows from Minkowski reduction.

We may therefore write \mathcal{T} in the form

$$\mathcal{T} = \bigcup_{k=0}^{n-1} \{I_k \perp T_{k,i}\}_{i=0}^{i_k},$$

where $0 < |\mathcal{T}| = \sum_{k=0}^{n-1} i_k$ and each $T_{k,i}$ is a rank- $(n-k)$ lattice with first minimum greater than 1. Then, the lattice

$$I_{n-1} \perp \left(\left(\perp_{i=0}^{i_1} T_{0,i} \right) \perp \cdots \perp \left(\perp_{i=0}^{i_{n-1}} T_{n-1,i} \right) \right)$$

represents all of \mathcal{T} but does not represent I_n . It follows that \mathcal{T} is not an n -criterion set, so any n -criterion set must contain I_n . \square

Proposition 2. *Let \mathcal{E} be the set of additively indecomposable lattices of rank n . If $|\mathcal{E}| > 0$, then any n -criterion set must include at least one lattice $E \in \mathcal{E}$.*

Proof. If $\mathcal{T} = \{T_i\}_{i=1}^k$ is a finite, nonempty set of rank- n lattices with $\mathcal{T} \cap \mathcal{E} = \emptyset$, then every lattice $T_i \in \mathcal{T}$ is additively decomposable. It follows that the lattice

$$T_1 \perp \cdots \perp T_k$$

represents all of \mathcal{T} but does not represent any lattice in \mathcal{E} , since $T_1 \perp \cdots \perp T_k$ has no rank- n additively indecomposable sublattices. Thus, \mathcal{T} is not an n -criterion set. It then follows that any n -criterion set must contain some lattice $E \in \mathcal{E}$. \square

Remark. It is clear that direct analogues of these two propositions hold in the more general setting of \mathcal{S} -universal lattices discussed in [KKO1]. In particular, suppose that \mathcal{S} is an infinite set of lattices. Then, if $n = \max\{k : I_k \in \mathcal{S}\} > 0$, any finite set $\mathcal{S}_5 \subset \mathcal{S}$ with the property that a lattice L represents every lattice $\ell \in \mathcal{S}$ if and only if L represents every $\ell \in \mathcal{S}_5$ must contain I_n . Similarly, such a set \mathcal{S}_5 must contain an additively indecomposable lattice if \mathcal{S} does.

4. UNIQUENESS OF THE 8-CRITERION SET

Oh obtained the following 8-criterion set in [Oh, remark on Theorem 3.1]:

Theorem 3 (Oh). *The set $\mathcal{S}_8 = \{I_8, E_8\}$ is an 8-criterion set.*

The set \mathcal{S}_8 is clearly a *minimal* 8-criterion set, as for each $\ell \in \mathcal{S}_8$ there is a lattice which represents $\mathcal{S}_8 \setminus \ell$ but does not represent ℓ . (The single lattice in $\mathcal{S}_8 \setminus \ell$ suffices.) Meanwhile, our characterization results imply the following corollary which strengthens Theorem 3:

Corollary 4. *Every 8-criterion set must contain \mathcal{S}_8 as a subset.*

Proof. Since E_8 is the unique additively indecomposable lattice of rank 8, the result follows directly from Propositions 1 and 2. \square

Corollary 4, when combined with Theorem 3, shows that \mathcal{S}_8 is the unique minimal 8-criterion set.

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