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A Perceptions Framework for Categorizing Inventory Policies in Single-stage Inventory Systems

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In this paper we propose a perceptions framework for categorizing a range of inventory decision making that can be employed in a single-stage supply chain. We take the existence of a wide range of inventory decision making processes, as given and strive not to model the reasons that the range persists but seek a way to categorize them via their effects on inventory levels, orders placed given the demand faced by the inventory system. Using a perspective that we consider natural and thus appealing, the categorization involves the use of conceptual perceptions of demand to underpin the link across three features of the inventory system: inventory levels, orders placed and actual demand faced. The perceptions framework is based on forecasting with Auto-Regressive Integrated Moving Average (ARIMA) time series models. The context in which we develop this perceptions framework is of a single stage stochastic inventory system with periodic review, constant leadtimes, infinite supply, full backlogging, linear holding and penalty costs and no ordering costs. Forecasting ARIMA time series requires tracking forecast errors (interpolating) and using these forecast errors and past demand realizations to predict future demand (extrapolating). So called optimal inventory policies are categorized here by perceptions of demand that align with reality. Naturally then, deviations from optimal inventory policies are characterized by allowing the perception about demand implied by the interpolations or extrapolations to be primarily different from the actual demand process. Extrapolations and interpolations being separate activities, can in addition, imply differing perceptions from each other and this can further categorize inventory decision making.

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1 Introduction

In this paper we propose a perceptions framework for categorizing a range of inventory decision making in a single-stage supply chain. The perceptions framework is based on forecasting with Auto-Regressive Integrated Moving Average (ARIMA) time series models. In this paper, we take the existence of a wide range of inventory decision making processes as given and strive not to model the reasons that the range persists but seek a way to categorize them via their effects on inventory levels, orders placed given the demand faced by the inventory system. Using a perspective that we consider natural and thus appealing, the categorization involves the use of conceptual perceptions of demand to underpin the link across three features of the inventory system: inventory levels, orders placed and actual demand faced. With limited emphasis on the cost consequences of the inventory policies, (sake in the distinction of optimal and suboptimal,) our categorization is less an economics-based one, but rather process-based, focusing instead on the stochastic processes that characterize the inventory levels, orders and demand levels at the inventory system and how they can be considered inter-related. We also examine the implications of our inventory decision making categories with respect to the variance and uncertainty of these stochastic processes, characteristics which we expect would have significant influence on the costs for an inventory system and for its supply chain.

Our use of the term perception is motivated by its use in organizational behavior theory on organizational decision making, March and Simon (1993), Cyert and March (1992). In such theory, perception of reality in terms of environment, decision alternatives, goals or consequences of actions are considered significant drivers of decision making. These perceptions can be misaligned with reality (assuming an objective assessment of this reality) since they are considered the outcome of psychological, sociological and operational processes with their attendant strengths and weaknesses.

Why do we need such a framework? In the operations literature, the assumption of the completely rational agent has dominated the research surrounding inventory policies. However deviations from the optimal policy are prevalent in practice and analytical models to understand the effects of inventory dynamics on practice may require means of modeling them. Modeling deviations from optimal policy can also augment our understanding of inventory systems and supply chains, in a manner akin to sensitivity analysis in mathematical programming, as the dynamics surrounding optimal management of these systems may not completely describe these systems. For example, although the optimal EOQ is determined via strict optimization of standard setup and holding costs, the inventory cost function is well-known to be fairly "flat" around this point.

Inventory decision making, in addition to being influenced by the above mentioned perceptions of reality, will also be the outcome of a combination of psychological, sociological, organizational and operational processes, Oliva and Watson (2004, 2007). We refer to a particular combination of perceptions and decision making processes as an inventory decision making *setting*. In order to categorize these settings, we make the *strong* assumption that such settings can be grouped based on the inventory levels, orders and demand persistently exhibited within the system, despite these settings being otherwise considered different from each other. For example, when the operations literature has assumed a rational agent for particular inventory decisions say in response to i.i.d. demand, we would infer a reference to all settings where the particular combination of perceptions and decision making processes results in processes for inventory levels and orders along with demand that resemble rational behavior. In this paper, the commonality used for grouping settings is represented by stochastic processes, serving as conceptual perceptions of demand. These perceptions form both a conceptual and a satisfying analytic link between inventory levels and orders placed and the actual demand faced by the inventory. The details of this analytic link and thus the particular operationalization of our perceptions framework make use of the ARIMA time series models, Box et. al. (1994).

From Box et. al., forecasting ARIMA time series requires tracking forecast errors (interpolating) and using these forecast errors and past demand realizations to predict future demand (extrapolating). Both interpolation and extrapolation are actions that can be considered to imply a belief about the demand process or a perception, as they are based on a particular time series demand model when performed by a completely rational agent. So called optimal inventory policies are categorized here by perceptions of demand that align with reality. Naturally then, categorizing deviation from optimal inventory policies is possible if we allow that the time series used for interpolating or extrapolating, that is, what we interpret as a belief about the demand process or perception, may not be the same as the actual demand process. Further deviations can even be categorized by allowing that the perception implied by interpolations do not match those implied by extrapolations.

In this paper, we establish these conceptual perceptions as analytic links connecting inventory levels or orders and the actual demand. In so doing, we also provide a prediction for the effects of a range of inventory decision making settings on inventory levels and orders. For example, we can suggest the effects of inventory decision making that ignore any non-stationary properties of demand. In an effort to further our appreciation of the framework, we interpret simple forecasting techniques used for inventory policies such as the moving average and exponential smoothing using our framework. Interestingly we show that in order to categorize the use of the moving average, we need slightly different perceptions in interpolation versus that of extrapolation justifying the need for the framework's flexibility. In an effort to improve our understanding of the effects of suboptimal policies on the process characteristics of inventory levels and orders, we also examine separately, the effects of misaligned perceptions along the autoregressive and then the moving average dimensions of the ARIMA modeled demand process and focus on the implications for variance and uncertainty of the inventory and order processes. Given that these are the two critical dimensions of the ARIMA processes which model our perceptions, understanding how each dimension separately affects inventory levels and orders should help our understanding of their effects in tandem.

The use of ARIMA processes as the foundation for this perceptions framework is motivated by the following two important observations: First as cited in Gaur et. al. real-life demand patterns often follow higher-order autoregressive processes due to the presence of seasonality and business cycles. For example, the monthly demand for a seasonal item can be an AR(12) process. More general ARMA processes are found to fit demand for long lifecycle goods such as fuel, food products, machine tools, etc., as observed in Chopra and Meindl (2001) and Nahmias (1993). Second, the results in this paper along with recent research show that ARMA demand processes occur naturally in multistage supply chains. For example, Zhang (2004) studied the time-series characterization of the order process for a decision maker serving ARMA demand and using a periodic-review order-up-to policy. They show an "ARMA-in-ARMA-out" principle for perceptions aligned with actual demand, i.e., if demand follows an ARMA(p,q) process, then the order process is an ARMA $(p, \max[p, q-l])$ process, where l is the replenishment lead time. Gilbert (2005) extended these results to ARIMA process but still for aligned perceptions. We find a similar "ARIMA-in-ARIMA-out" principle for orders given a wider range of inventory policies. Such a result increases our expectations for the prevalence of these processes in real world supply chains.

It needs to be pointed out that in our framework we do not use perception in its more conventional sense. We are not making any claims about the actual perception of any manager in this paper and how decisions are made by these managers. However, there are real world scenarios which resemble details of our perception framework more closely than others. One scenario involves fitting ARIMA stochastic models to historical demand data, Gooijer (1985), Koreisha and Yoshimoto (1991). Different methods for fitting these stochastic processes including the Box-Jenkins approach, the corner method and extended sample autocorrelations, have been shown to have varying performance in correctly identifying these processes. Poor identification then could result in a situation where the demand process used for inventory management is misaligned with reality. A second scenario involves subjective forecasting. Lawrence and O'Connor (1992) in a forecasting study investigate the extent to which some of the widely documented judgemental biases and heuristics apply to time series forecasting. The time series used were modeled from stationary ARMA processes. The authors found that subjects' forecasts could be modeled as an AR(1) process that was not always aligned with actual.

Supply chain issues which we expect could benefit from our framework include the bullwhip effect and its mitigation Lee, et. al. (1997), value of information sharing in a supply chain, Lee, et. al. (2000), Cachon and Fisher (2000), Raghunathan (2001), Gaur et. al. (2005) and the general coordination of a supply chain whether via incentives Cachon (2000) or more structural approaches such as leadtime reduction, Cachon and Fisher, or network rationalization. For example, in section 4.2, we make some comments on the value of sharing information in a supply chain. In Watson and

Zheng (2005), misaligned perceptions implying deviations from optimal inventory policies, were also used to test the robustness of decentralization schemes for serial supply chains, that is, how well the schemes mitigated the resulting increase in total systems costs.

Section 2 presents the general demand model and inventory system while Section 3 introduces the perceptions framework within the context of optimal inventory management. Section 4 continues the introduction but within the context of suboptimal inventory policies focusing, for sake of simplicity, on true demand modeled as stationary ARMA stochastic processes. Section 5 examines misaligned perceptions along the autoregressive and moving average dimensions separately. In Section 6 we also examine policies based on simple forecasting techniques. Section 7 completes the presentation of the perceptions framework within the context of suboptimal inventory policies with a treatment of demand modeled as non-stationary ARIMA processes.

2 The Demand Model and Inventory System

In this paper, we use the general class of Auto-Regressive Integrated Moving Average (ARIMA) time-series models to model both demand and perceptions of demand. In this section we describe the methodology for forecasting with these time series models once they have been specified. Since perceptions are also modeled as ARIMA time series models, the description is applicable for understanding the details of the mechanics of the perceptions framework.

We follow the notation of Box et. al. (1994). Stationary demand will be represented by an autoregressive moving average model, ARMA(p, q) with p the number of autoregressive terms and q the number of moving average terms:

$$\begin{aligned} Z_t &= \mu + \phi_1 (Z_{t-1} - \mu) + \phi_2 (Z_{t-2} - \mu) + ... + \phi_p (Z_{t-p} - \mu) + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - ... - \theta_q a_{t-q} \\ &= \mu + \sum_{i=1}^{p} \phi_i (Z_{t-i} - \mu) + a_t + \sum_{i=1}^{q} -\theta_i a_{t-i}, \end{aligned} \tag{1}$$

where Z_t is the demand at time, μ is the level or mean of demand and a_t is a noise series of independent identically distributed random variables with mean 0 and variance σ_a^2.

The ARMA models can be represented more concisely and are more easily manipulated mathematically by using a backshift operator. Let B denote the time-series backshift operator such that

$$BZ_t = Z_{t-1} \text{ and } B^n Z_t = Z_{t-n}.$$

The above stationary series (1) can be written using the backshift operator as follows:

$$\phi(B)(Z_t - \mu) = \theta(B) a_t,$$

where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p$ and is referred to as the *autoregressive operator* with order p, and $\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - ... - \theta_q B^q$ which is referred to as the *moving average operator* with order q.

Non-stationary demand can be modeled by assuming that some differencing results in a stationary ARMA series. ∇ will be used to denote the difference operation such that $\nabla Z_t = (1 - B) Z_t$ and $\nabla^d Z_t = (1 - B)^d Z_t$ where $\nabla^d = (1 - B)^d$ is the polynomial resulting from raising $1 - B$ to the dth power. Assuming the dth difference of a non-stationary demand series is stationary, the series can be represented by an autoregressive integrated moving average or ARIMA(p, d, q) series:

$$\phi(B) \nabla^d Z_t = \theta(B) a_t \tag{2}$$

or

$$\varphi(B) Z_t = \theta(B) a_t,$$

where $\varphi(B) = \phi(B) \nabla^d$ is a polynomial of order $p + d$ and is referred to as the *generalized autoregressive operator*.

2.1 ARIMA(p, d, q) Forms and Forecasting

There are three different "explicit" forms for the general model (2). The current value of Z_t can be expressed:

1. In terms of the realizations of previous $Z_{t-j} : j > 0$ and current and previous realizations a_{t-j}, $j = 0, 1, ...$ as implied by (2) :

$$(Z_t - \mu) = (1 - \varphi(B))(Z_{t-1} - \mu) + \theta(B) a_t.$$

2. In terms of the current and previous realizations a_{t-j}, $j = 0, 1, ...$ This representation is seen by rewriting (2) as follows

$$(Z_t - \mu) = \psi(B) a_t, \tag{3}$$

   where $\psi(B) = 1 + \sum_{j=1}^{\infty} \psi_j B^j = \theta(B) / \varphi(B)$. In particular:

$$\psi_j = \varphi_{p+d} \psi_{j-p-d} + ... + \varphi_1 \psi_{j-1} - \theta_j : j > 0$$

where $\psi_0 = 1$, $\psi_j = 0$ for $j < 0$, and $\theta_j = 0$ for $j > q$. For the ARMA time series, the stationary property is guaranteed if the series $\psi(B)$ is infinitely summable, that is, if $\sum_{j=1}^{\infty} |\psi_j| < \infty$. For an ARIMA time series Z_t, although $\nabla^d Z_t$ is stationary, Z_t is not. Therefore the infinite sum (3) above, is not summable and so does not converge in any sense. Alternatively we can rewrite (18) in terms of the dth difference which does have a converging sum of random shocks. As a notational device, for ARIMA time series we will refer to this infinite random shock form to represent the infinite random shock form of the dth difference.

3. In terms of the realizations of previous Z_{t-j} and current a_t. This representation is seen by rewriting (2) as follows

$$\pi(B)(Z_t - \mu) = a_t,$$

where $\pi(B) = 1 - \sum_{j=1}^{\infty} \pi_j B^j = \varphi(B)/\theta(B)$. This implies the following

$$(Z_t - \mu) = \sum_{j=1}^{\infty} \pi_j (Z_{t-j} - \mu) + a_t.$$

In particular:

$$\pi_j = \theta_q \pi_{j-q} + ... + \theta_1 \pi_{j-1} + \phi_j : j > 0$$

where $\pi_0 = -1$, $\pi_j = 0$ for $j < 0$ and $\phi_j = 0$ for $j > p$. An invertibility property for (2) implies that this is a legitimate form. The property is guaranteed by restricting the series $\pi(B)$ to be infinitely summable, that is, $\sum_{j=1}^{\infty} |\pi_j| < \infty$. This property can hold for stationary ARMA as well as non-stationary ARIMA time series.

These three forms can be used for forecasting Z_t once the model of demand process has been estimated or specified and are equivalent, i.e., the forecast is the same independent of which forecasting form is used. We show that their effect on inventory systems are not always equivalent for perceptions that are not aligned with reality.

2.2 The Inventory System

The inventory context in which we develop the perceptions framework is the following. We assume an inventory system managed by a retailer with periodic review and constant leadtime L, infinite supply, full backlogging, linear holding and penalty costs, no ordering costs and infinite history. Demand is assumed to follow an ARIMA process (2) with $d \geq 0$ satisfying the stationary property but not necessarily the invertibility property. The sequence of operations in a period are that 1) there is receipt of inventory previously ordered 2) demand arrives within a period and is met from on-hand inventory, and then 3) orders are placed. Let I_t be the inventory at the retailer at the end of period t. Consistent with the assumptions on the costs of the inventory system, we will assume an order up-to policy for the retailer with target s_t for period t. Where necessary we allow costless returns which implies that orders from the retailer in period t are given by

$$O_t = Z_t + s_{t+1} - s_t. \tag{4}$$

The primary purpose of this paper is to examine the use of perceptions as a framework for categorizing inventory policies, therefore we will for simplicity and brevity assume that the inventory system has zero leadtime, that is, $L = 0$. The results for arbitrary leadtime have similar themes.

3 The Perceptions Framework Applied to Optimal Inventory Policies

In this section we introduce the perceptions framework within the context of optimal inventory management.

3.1 Aligned Perceptions: Optimal Inventory policies.

Consider the general demand process for the retailer (2) and the optimal inventory policy for minimizing expected single period costs which, in this setting, is the myopic order up-to policy. For this optimal policy, at the end of period $t-1$, the retailer calculates the forecast $\hat{Z}_{t-1}$ which minimizes the forecast error for period t and combines this forecast with a constant safety stock T. From Box et. al. as mentioned earlier this forecast can be generated using any of the three explicit forms of the demand process. The first two forms require, however, the retailer to calculate at least the most recent random shock a_{t-1}. This is calculated as the difference between the actual realization and the one period ahead forecast, that is, $a_{t-1} = Z_{t-1} - \hat{Z}_{t-2}$ for an ARMA time series or $a_{t-1} = \nabla^d Z_{t-1} - \nabla^d \hat{Z}_{t-2}$ for an ARIMA time series. (Here $\nabla^d \hat{Z}_{t-2}$ serves as convenient notation for the forecasted dth difference.) When our retailer calculates the difference between the actual realization and the one period ahead forecast, she interpolates the realizations of the unseen random shocks. When the retailer makes her forecast $\hat{Z}_{t-1}$ using any of the explicit forms, she is extrapolating, that is, using previous realizations and interpolations to predict future realizations. Both interpolation and extrapolation as actions can be considered to imply a belief about the demand process as they are based on a particular time series model. We attribute this time series implied by either forecasting operation of interpolation or extrapolation as a perception about the demand process. The resulting optimal forecasts and inventory policies can then be considered optimal partly because the perception of the retailer, that is, the belief of the demand process, is aligned (or matches) with the actual demand process. The terms introduced here, *perception*, *interpolation* and *extrapolation* will be used in the next section for categorizing a wider range of inventory policies beyond the optimal policy for an inventory system.

The theorems below characterize the stochastic process for the inventory levels at the retailer and her orders assuming the myopic policy described above and comes from Gilbert. These results will be compared with those obtained when we examine the perceptions framework within a context of misaligned perceptions of demand.

Theorem 3.1 *An aligned perception is associated with a time series of inventory I_t which is ARIMA(0,0,1) with the form $I_t - T = a_t^I$ where $a_t^I = -a_t$.*

Theorem 3.2 *An aligned perception is associated with a time series of orders O_t which is $ARIMA\big(p, d, q^O\big)$ with the form*

$$\varphi(B)(O_t - \mu) = \theta^O(B)\, a_t^O$$

where $a^O = K^O a_t$, $\theta^O(B) = \frac{\theta(B) - \varphi(B)}{BK^O} + \frac{\varphi(B)}{K^O}$, $q^O = \max(p + d, q - 1)$ *and* $K^O = 1 + \varphi_1 - \theta_1$.

Proposition 3.3 *For stationary ARMA demand process, the variance of the time series $(O_t - \mu)$ is greater than that of $(Z_t - \mu)$ if and only if $\phi_1 - \theta_1 > 0$.*

3.2 Discussion of Optimal Policy

Theorem 3.1 suggests that inventory would rarely follow a process more complex than a simple i.i.d. process despite the complexity of the demand process and further, that the mean of this inventory process is equal to the intended safety stock T. Theorem 3.2 shows that the optimal orders follow an ARIMA process similar to that of demand except for differences in the variance of the random shock and in the moving average operator of the process. The random shock for the order process has variance $\big(K^O \sigma_a\big)^2 = (1 + \varphi_1 - \theta_1)^2 \sigma_a^2$. In Gilbert and Watson and Zheng (2008) the coefficient $|K^O|$ is interpreted as an alternative measure of the bullwhip effect complementing the more conventional measure of the ratio of variance of orders to demand. Watson and Zheng refer to the coefficient $|K^O|$ as the bullwhip effect *modulo full information* and interpret it as the best case increase (or decrease) in uncertainty for a potential supplier given full distributional knowledge about demand and knowledge about the inventory policy of the retailer. The greater the coefficient $|K^O|$, the greater the expected uncertainty for the supplier despite her extensive information. Proposition 3.3 shows that for aligned perceptions, this bullwhip effect modulo full information is consistent with the traditional measure of the bullwhip effect.

The results for aligned perceptions here provide added argument for the need for a framework on deviations from the optimal policy. Theorem 3.1 provides very strict expectations for the inventories of a retailer, (that it follows a ARIMA (0,0,1) process,) which we would be surprised, if we found completely characterized a majority of inventory systems.

3.3 Formalizing Framework Concepts: Perceptions, Interpolations and Extrapolations

We assimilate the concepts of perception, interpolation and extrapolation introduced in the previous subsections within a context of forecasting with ARIMA time series models in order to provide our framework. *Perception* we will define as the implied characterization of the demand process facing the inventory system. Conceptually, perceptions can be implied from interpolations and extrapolations. In this paper, *interpolation* is the action of inferring past but hidden realizations, in this case, the unseen random shock from demand realizations. *Extrapolation* is the action of using historical realizations and/or interpolations to predict future demand realizations. In order to impute a characterization of demand, that is, a perception, from an action requires an explicit mapping of demand processes to actions. Our mapping corresponds to that generated by a completely rational agent making forecasts for a given ARIMA demand specification. In essence, a perception is implied from an interpolation or extrapolation by assuming the action to be performed by a completely rational agent and then inferring the demand process which drives the action. By our definition of perception, it will not be necessary for the perception imputed from interpolations to be the same with that imputed from extrapolations (as implicitly assumed in the optimal policy above) even though the implicit assumption is that the actions (and perceptions) categorize the same inventory system. More importantly neither of these perceptions need be aligned with reality. Using the above introduced definitions, we can examine a range of inventory policy responses by our retailer and study the effects on her orders and inventory levels.

4 Perceptions Framework for Suboptimal policies: Misaligned Stationary Perceptions

In this section we move beyond optimal policies and begin to examine inventory decision making that can be termed suboptimal. In our framework, such policies are categorized by perceptions which do not align with reality and thus we will term them *misaligned perceptions*. In this section we will concentrate on stationary demand processes and stationary perceptions, that is, ARMA (ARIMA with $d = 0$) processes as in (1), while in Section 7 we focus on non-stationary ARIMA ($d > 0$) demand processes. Our categorization will be based on two perceptions, one for interpolations and one for extrapolations. Therefore we can consider two variations in the extent of the misaligned perceptions. The first (section 4.2) considers the same misaligned perceptions in both extrapolations and interpolations and the second considers different perceptions for extrapolations and interpolations. Although the first variation concerns perceptions that could be described as internally *consistent* since that used for extrapolation matches that used for interpolations, the second variation follows from recognizing that these perceptions do not have to be so. In section 4.3 we consider one example where the perception used for interpolation is different from that used for extrapolation. In particular we consider misaligned perceptions in extrapolations but aligned perceptions in interpolations. Furthermore, in section 6, we interpret the use of the moving average and exponential smoothing using our perceptions framework. The example of the moving average policy considered in section 6.1 will serve as another example where the perception used for interpolation will be different from the perception used for extrapolation. Our main goal in this section (and paper) is to establish these perceptions as the fundamental component of an analytic link between the demand process and the process for inventory levels and orders. In so doing, we also provide a prediction for the effects of "suboptimal" inventory decision making on inventory levels and orders.

4.1 Misaligned Stationary Perceptions : ARMA$(\tilde{p}, \tilde{q})$ processes

In this subsection we discuss the general stationary perceptions that can be attributed to interpolations and extrapolations which we will allow to differ from the actual demand process. All stationary and invertible ARMA processes can serve as misaligned perceptions, thus follow an ARMA$(\tilde{p}, \tilde{q})$ process:

$$\tilde{\phi}(B)(Z_t - \tilde{\mu}) = \tilde{\theta}(B)\tilde{a}_t \tag{5}$$

where $\tilde{\phi}(B) = 1 - \tilde{\phi}_1 B - \tilde{\phi}_2 B^2 - ... - \tilde{\phi}_{\tilde{p}} B^{\tilde{p}}$, $\tilde{\theta}(B) = 1 - \tilde{\theta}_1 B - \tilde{\theta}_2 B^2 - ... - \tilde{\theta}_{\tilde{q}} B^{\tilde{q}}$, and $\tilde{a}_t$ is a random noise series with mean 0 and variance $\tilde{\sigma}_a$. The stationary and invertible properties from section 2.1 are assumed to hold and thus all three explicit forms of forecasting will be legitimate. In (5) it is possible for $\tilde{p} = p$, $\tilde{q} = q$, $\tilde{\mu} = \mu$, $\tilde{\phi}_j = \phi_j$, $\tilde{\theta}_j = \theta_j$ for some j or $\tilde{\sigma}_a = \sigma_a$.

We will assume that the forecasts for extrapolations are denoted by $\hat{Z}_t^e[k]$ which specifically denotes the forecast of demand at time $t + k$ made at time t after demand is observed. Since we concentrate on zero leadtimes, we only deal with single period ahead forecasts, $\hat{Z}_t^e[1]$, which we will denote $\hat{Z}_t^e$. Recall from section 2.1 that there are three different explicit forms for the general ARIMA models and that these forms imply three different approaches to calculating $\hat{Z}_t^e$. We will denote these three forms as $\hat{Z}_t^e(1)$, $\hat{Z}_t^e(2)$ and $\hat{Z}_t^e(3)$ which coincide sequentially with the three explicit forms from Section 2.1 and define them explicitly as follows:

$$\hat{Z}_t^e(1) = \tilde{\mu} + \frac{1-\tilde{\phi}(B)}{B}(Z_t - \tilde{\mu}) + \frac{\tilde{\theta}(B)-1}{B}\tilde{a}_t, \tag{6}$$

$$\hat{Z}_t^e(2) = \tilde{\mu} + \frac{\tilde{\psi}(B)-1}{B}\tilde{a}_t, \tag{7}$$

$$\hat{Z}_t^e(3) = \tilde{\mu} + \frac{1-\tilde{\pi}(B)}{B}(Z_t - \tilde{\mu}), \tag{8}$$

where $\tilde{\psi}(B) = \tilde{\theta}(B)/\tilde{\phi}(B)$ and $\tilde{\pi}(B) = \tilde{\phi}(B)/\tilde{\theta}(B)$. The above formulas are generated by taking the expectation at time t of demand in period $t+1$, Z_{t+1}, using the three explicit forms listed in Section 2.1 for the process (5). Note that at time t, $E\tilde{a}_{t+1} = 0$. Furthermore the realizations $\tilde{a}_k$, $k \leq t$, are determined from interpolations. Note finally that $\hat{Z}_t^e(3)$ does not use any interpolations.

For interpolations, recall that it is the difference between the actual realization and the one period ahead forecast that is used to generate the realization for the demand shock $\tilde{a}_t$ that are used for extrapolations in (6) and (7) above. We let $\hat{Z}_t^i$ denote this one period ahead forecast based on the perceptions used for the interpolation. As with extrapolation, we can denote three forms for $\hat{Z}_t^i$ as $\hat{Z}_t^i(1)$, $\hat{Z}_t^i(2)$ and $\hat{Z}_t^i(3)$ which again coincide sequentially with the three explicit forms listed in Section 2.1. Specifically $\tilde{a}_t = (Z_t - \tilde{\mu}) - \left(\hat{Z}_{t-1}^i(J) - \tilde{\mu}\right)$ where the forecast used in the interpolation could be of any of the three forms $J = 1, 2, 3$. For $J = 3$, is it is easy to show that we can then write $\tilde{a}_t = \tilde{\pi}(B)(Z_t - \tilde{\mu})$ where $\tilde{\pi}(B) = \tilde{\phi}(B)/\tilde{\theta}(B)$. Note that since the perception (5) is invertible, $\tilde{\pi}(B)$ is legitimate. Note that the other two forms $J = 1, 2$ also make use of demand realizations determined from prior interpolations. For these forms it only takes a little more work to show that we still get $\tilde{a}_t = \tilde{\pi}(B)(Z_t - \tilde{\mu})$. We write this as a proposition:

Proposition 4.1 *For misaligned interpolations, either form of the single period ahead forecast* $\hat{Z}_t^i(J)$, $J = 1, 2, 3$, *results in realization of demand shocks which can be formulated as* $\tilde{a}_t = \tilde{\pi}(B)(Z_t - \tilde{\mu})$ *where* $\tilde{\pi}(B) = \tilde{\phi}(B)/\tilde{\theta}(B)$.

4.2 Consistent Misaligned Perceptions in Interpolation and Extrapolation

In this subsection by characterizing the stochastic processes for the inventory levels and orders given the same misaligned perception in both interpolations and extrapolations, we establish this perception as the basis for the link between the process characteristics of the system. After the results provided below we describe the more interesting relationships between the categorizing perception and the process results.

Assume without loss of generality that this misaligned perception is defined as the ARMA$(\tilde{p}, \tilde{q})$ process (5). We have the following theorems:

Theorem 4.2 *A misaligned perception attributed to both interpolation and extrapolation is associated with a time series of inventory $\tilde{I}_t$ which is ARMA $\left(p^I, q^I\right)$ is with the form*

$$\phi^I(B)\left(\tilde{I}_t - T^I\right) = \theta^I(B)\, a_t^I$$

and:

$a_t^I = K^I a_t$	$p^I = \tilde{q} + p$	$\theta^I(B) = \frac{-\tilde{\phi}(B)\theta(B)}{K^I}$
$K^I = -1$	$q^I = \tilde{p} + q$	
$T^I = T + \left(\frac{1-B-\tilde{\pi}(B)}{B}\right)(\mu - \tilde{\mu})$	$\phi^I(B) = \phi(B)\,\tilde{\theta}(B)$	

This time series is stationary and independent of the explicit forecasting form used for either interpolation or extrapolation. The time series is also invertible if the demand process is invertible.

Theorem 4.3 *A misaligned perception attributed to both interpolation and extrapolation is associated with a time series of orders $\tilde{O}_t$ which is ARMA$\left(p^O, q^O\right)$ with the form*

$$\phi^O(B)\left(\tilde{O}_t - \mu\right) = \theta^O(B)\, a_t^O$$

and:

$a_t^O = K^O a_t$	$p^O = p + \tilde{q}$	$\theta^O(B) = \left(\frac{\tilde{\theta}(B)-\tilde{\phi}(B)(1-B)}{BK^O}\right)\theta(B)$
$K^O = 1 + \tilde{\phi}_1 - \tilde{\theta}_1$	$q^O = \max(\tilde{p} + q, q + \tilde{q} - 1)$	
	$\phi^O(B) = \tilde{\theta}(B)\,\phi(B)$	

This time series is stationary and independent of the explicit form used for either interpolation or extrapolation but not necessarily invertible.

Perceptions Perspective on the Inventory Level Process: Theorem 4.2 above shows that the inventory level process for a misaligned perception in both interpolations and extrapolations is not i.i.d. (versus results for aligned perception in Theorem 3.1) but can follow a wide range of ARMA processes. The inventory level process is driven by both the perception and the demand process. In particular, the autoregressive and moving average operators from both the perception and the demand process determine the operators of the inventory level process. Furthermore, since $T^I = T + \left(\frac{1-B-\tilde{\pi}(B)}{B}\right)(\mu - \tilde{\mu})$, a perception which has mean $\tilde{\mu} \neq \mu$ results in essentially a different safety factor as the mean of the inventory level process. Here notice that except for the mean of actual demand, this safety factor is completely determined by the perception and is independent of the stochastic process for actual demand. As expected this safety factor T^I is usually lower than T if $\mu \geq \tilde{\mu}$ since $\left(\frac{1-B-\tilde{\pi}(B)}{B}\right)$ tends to be negative (since -1 is the constant in the polynomial $\left(\frac{1-B-\tilde{\pi}(B)}{B}\right)$,) however this relation does not have to hold.

Perceptions Perspective on the Order Process: As with the inventory level process, the order process is determined by the perception and the demand process. This is an important observation given its relationship to supply chain research on sharing demand for improving supply chain coordination, e.g., Lee, So and Tang. Researchers have argued that sharing such demand would help supply chain coordination in three ways: 1) historical demand information would help upstream stages forecast end-customer demand more accurately, which would then improve the forecasts of future orders from downstream stages, 2) knowing real-time demand realizations would also improve accuracy of forecasts of future orders, Lee, So and Tang, and 3) end-customer demand can prove a more robust guide for orders placed by upstream partners than their incoming orders, Watson and Zheng (2005). Our perceptions perspective here speak to the first two arguments. To the first argument, our results suggest that for improving forecasts of orders, an understanding of the demand process needs to be complemented by an understanding of the inventory policy of the downstream stage creating the orders. Historically, these orders have been assumed to come from aligned perceptions with the demand process, but if in reality these perceptions are not aligned, then forecasting demand is not sufficient for forecasting incoming orders. In particular, we see in Theorem 4.3 that the bullwhip effect modulo information for orders $|K^O|$, that is the uncertainty inherent in orders, is related only to the parameters of the misaligned perception and not to that of the actual demand process. With respect to the second argument, the results suggest interestingly that sharing current demand realizations does not improve accuracy of order forecasts since the order process is also ARMA and thus completely characterized by previous orders and interpolated errors in forecasting, see Raghunathan (2001) for similar finding with aligned perceptions.

4.3 Inconsistent Misaligned Perceptions in Extrapolation and Interpolation

In this subsection we continue our examination of inventory decision making that can be termed suboptimal, focusing on those which are categorized by differing perceptions in extrapolations versus interpolations. We will consider a simple example of this variation where the perceptions from extrapolations are not aligned with the actual demand process (1), while the perceptions from interpolations are so aligned. This variation is considered to be one example but not the only such example where the perception used for interpolation is different from that used for extrapolation. Our results show that allowing perceptions to differ between interpolations and extrapolations, allows us to categorize a larger range of inventory decision making settings than when the perceptions are not allowed to differ. The example of the moving average policy considered in section 6.1 will serve as an example where the perception used for interpolation will be different from the perception used for extrapolation.

First consider interpolations. Since the perceptions here is of a belief that matches with reality, we assume that the one period ahead forecasts used here, $\hat{Z}_t^i$, are equal to those of the completely rational agent, that is, $\hat{Z}_t^i = \hat{Z}_t$. This gives generated realizations for the random shocks, which will be used in extrapolation, that match with reality, i.e., $\tilde{a}_t = a_t$.

Now consider extrapolations. Assume w.l.o.g. that the misaligned perception attributed to our extrapolation follows the ARMA$(\tilde{p}, \tilde{q})$ process (5). For clarity, we write below the three explicit forms of the one period ahead forecasts:

$$\hat{Z}_t^e(1) = \tilde{\mu} + \frac{1 - \tilde{\phi}(B)}{B}(Z_t - \tilde{\mu}) + \frac{\tilde{\theta}(B) - 1}{B} a_t; \tag{9}$$

$$\hat{Z}_t^e(2) = \tilde{\mu} + \frac{\tilde{\psi}(B) - 1}{B} a_t; \tag{10}$$

$$\hat{Z}_t^e(3) = \tilde{\mu} + \frac{1 - \tilde{\pi}(B)}{B}(Z_t - \tilde{\mu}); \tag{11}$$

Note that in the first two explicit forms of $\hat{Z}_t^e$ we use the random shock terms a_t instead of $\tilde{a}_t$ to reflect interpolations with aligned perceptions.

The following are theorems characterizing the inventory levels and orders from our retailer for the first and second explicit forecasting form. Note that since the third explicit forecasting form is the same here as when extrapolations and interpolations are both misaligned, the results in theorems 4.2 and 4.3 apply in that case. The first difference to note between the results here and in section 4.2 is that here for inconsistent misaligned perceptions, the form of forecasting affects the results for inventory and orders, whereas for consistent misaligned perceptions in section 4.2, it does not.

Theorem 4.4 *A misaligned perception in extrapolation only and where extrapolations use the first explicit form of forecasts, is associated with a time series of inventory $\tilde{I}_t$ which is ARMA (p, q^I) with the form*

$$\phi(B)(\tilde{I}_t - T^I) = \theta^I(B) a_t^I$$

and:

$a_t^I = K^I a_t$	$q^I = \max\{\tilde{p} + q - 1, p + q, p + \tilde{q}\}$
$K^I = -(1 + \phi_1 - \tilde{\phi}_1)$	$\theta^I(B) = \frac{1}{K^I}\left(\frac{\phi(B) - \tilde{\phi}(B)}{B}\theta(B) + \phi(B)\left(\tilde{\theta}(B) - 1 - \theta(B)\right)\right)$
$T^I = T + \left(\frac{1 - B - \tilde{\phi}(B)}{B}\right)(\mu - \tilde{\mu})$	

and associated with a time series of orders $\tilde{O}_t$ which is ARMA$\left(p, q^O\right)$ with the form

$$\phi(B)\left(\tilde{O}_t-\mu\right)=\theta^O(B)\, a_t^O$$

and:

$a_t^O = K^O a_t$	$q^O = \max\{\tilde{p}+q, p+\tilde{q}\}$
$K^O = 1+\tilde{\phi}_1-\tilde{\theta}_1$	$\theta^O(B)=\frac{1}{K^O}\left(\frac{\theta(B)}{B}-\left(\phi(B)\left(\tilde{\theta}(B)-1\right)-\theta(B)\tilde{\phi}(B)\right)\frac{(1-B)}{B}\right)$

Both time series $\tilde{I}_t$ and $\tilde{O}_t$ are stationary but not necessarily invertible.

Theorem 4.5 *A misaligned perception in extrapolations only and where extrapolations use the second explicit form of forecasts, is associated with a time series of inventory $\tilde{I}_t$ which is ARMA $\left(p^I, q^I\right)$ with the form*

$$\phi^I(B)\,(\tilde{I}_t-T^I)=\theta^I(B)\, a_t^I$$

and:

$a_t^I = K^I a_t$	$q^I = \max\{\tilde{p}+q, \tilde{q}+p, \tilde{p}+p\}$
$K^I = -1$	$\theta^I(B)=\frac{1}{K^I}\left(\tilde{\theta}(B)\phi(B)-\theta(B)\tilde{\phi}(B)-\tilde{\phi}(B)\phi(B)\right)$
$T^I = T-(\mu-\tilde{\mu})$	
$p^I = \tilde{p}+p$	$\phi^I(B)=\tilde{\phi}(B)\phi(B)$

and associated with a time series of orders $\tilde{O}_t$ which is ARMA$\left(p^O, q^O\right)$ with the form

$$\phi^O(B)\left(\tilde{O}_t-\mu\right)=\theta^O(B)\, a_t^O$$

and:

$a_t^O = K^O a_t$	$q^O = \max\{\tilde{p}+q, \tilde{q}+p, \tilde{p}+p\}$
$K^O = 1+\tilde{\phi}_1-\tilde{\theta}_1$	$\theta^O(B)=\frac{1}{K^O}\theta(B)\tilde{\phi}(B)+\frac{\tilde{\theta}(B)-\tilde{\phi}(B)}{B}\phi(B)(1-B)$
$p^O = \tilde{p}+p$	$\phi^O(B)=\tilde{\phi}(B)\phi(B)$

Both time series $\tilde{I}_t$ and $\tilde{O}_t$ are stationary but not necessarily invertible.

Theorems 4.4 and 4.5 above show that the resulting processes for inventory and orders depend on which form of forecasting is used. For example, inventory and orders under the first form of forecasting in Theorem 4.4 have the same autoregressive operator as the demand process while under the second form, in Theorem 4.5, the autoregressive operator for inventory and orders can be quite different from actual demand, i.e., $\phi(B)\neq\phi^I(B)$ and $\phi(B)\neq\phi^O(B)$. Comparing Theorems 4.4 and 4.5 for inconsistent misaligned perceptions with Theorems 4.2 and 4.3 for consistent misaligned perceptions further shows how the inventory and order processes can differ when the perceptions are not consistent. For example, the variational coefficient K^I in the inventory model in Theorem 4.4 is rarely equal to -1 for inconsistent misaligned perceptions while it is always equal to -1 in Theorem 4.2. Furthermore inconsistent misaligned perceptions can result in inventory level processes that are not invertible while consistent misaligned perceptions always result in invertible inventory level processes if the demand process is invertible. This implies that allowing the perceptions for interpolations and extrapolations to differ from each other can help categorize a greater range of inventory decision making settings than when the perceptions are confined to be the same.

5 Investigating the Effects of Consistent Misaligned Perceptions along Auto-Regressive and Moving Average Dimensions

In an effort to improve our understanding of the effects of suboptimal policies on the process characteristics of inventory levels and orders, and promote the utility of our perceptions framework, we examine perceptions where the misalignment is either along the autoregressive dimension or moving average dimension but not both. Given that these are the two critical dimensions of the ARIMA processes which model our perceptions, understanding how each dimension separately affects inventory levels and orders should help our understanding of their effects in tandem. We will refer to these misalignments as autoregressive based misalignments and moving average based misalignments respectively and restrict our attention to consistent misaligned perceptions. We will examine the implications of both types of misalignment with respect to the variance and uncertainty of the inventory level and order processes.

5.1 Auto-regressive based Misalignments

In this subsection we consider misaligned perceptions with the misalignment along the autoregressive dimension. Given that demand follows the ARMA process (1), an autoregressive based misaligned perception is the following ARMA$(\tilde{p}, q)$ process

$$\tilde{\phi}(B)(Z_t - \mu) = \theta(B)\tilde{a}_t \tag{12}$$

where $\tilde{\phi}(B) = 1 - \tilde{\phi}_1 B - \tilde{\phi}_2 B^2 - ... - \tilde{\phi}_{\tilde{p}} B^{\tilde{p}}$. Note that we assume that the mean of perception is the same as that of actual demand. The following proposition will be used to discuss this type of misalignment.

Proposition 5.1 *An autoregressive based misaligned perception in both the interpolation and extrapolation is associated with a time series of inventory I_t which is ARMA (p, q^I) is with the form*

$$\phi(B)(I_t - T) = \theta^I(B)\, a_t^I$$

and:

$a_t^I = K^I a_t$	$K^I = -1$	$q^I = \tilde{p}$	$\theta^I(B) = \left(-\frac{\tilde{\phi}(B)}{K^I}\right)$

and associated with a time series of orders O_t which is ARMA$\left(p, q^O\right)$ with the form

$$\phi(B)(O_t - \mu) = \theta^O(B)\, a_t^O$$

and:

$a_t^O = K^O a_t$	$q^O = \max(\tilde{p}, q-1)$	$K^O = 1 + \tilde{\phi}_1 - \theta_1$	$\theta^O(B) = \frac{\theta(B)}{BK^O} - \frac{\tilde{\phi}(B)(1-B)}{BK^O}$

These inventory and order time series are independent of the explicit form used for either interpolations or extrapolations.

Perception Perspective on the Inventory Level Process: Comparing results for aligned perceptions from Theorem 3.1 with results for autoregressive based misalignments from Proposition 5.1, we see that an autoregressive based misalignment introduces an autoregressive operator into the inventory level process which is the same as that of the demand process. Recall that under aligned perceptions, the inventory level process was i.i.d. A moving average operator is also introduced into the inventory level process. As expected, in Proposition 5.2, we show that aligned perceptions result in the lowest variance for the inventory process.

Proposition 5.2 *If $\phi(B) \neq \tilde{\phi}(B)$ then $\left(\tilde{I}_t - T\right)$ has a larger variance than $(I_t - T)$.*

Monotonic results for changes in variance of inventory with changes in parameters of the misaligned perception depend on the parameters of actual demand. In Proposition 5.3 below when the coefficients of the autoregressive operator for actual demand are all positive and either all greater than or less than those of the misaligned perception, then variance will increase with the divergence of the misaligned perception from that of actual demand. Such monotonic properties do not always hold however for example, if some of the autoregressive coefficients are negative.

Proposition 5.3 *Assume $\tilde{p} \leq p$ and $\phi_i \geq 0$, for all i, if $\phi_i - \tilde{\phi}_i \geq 0$ for all i or $\phi_i - \tilde{\phi}_i \leq 0$ for all i, then the variance of $\tilde{I}_t$ is increasing in $\left|\phi_i - \tilde{\phi}_i\right|$ for all i.*

Perception Perspective on the Order Process: Comparing results for aligned perceptions from Theorem 3.2 with results for autoregressive based misalignment from Proposition 5.1, we see an autoregressive based misalignment does not change the autoregressive operator of the order process from what would have resulted from an aligned perception. Furthermore, the bullwhip effect modulo full information, that is the uncertainty inherent in orders, is determined by the mis-estimate of the first coefficient of the autoregressive operator $\tilde{\phi}_1$. When $\tilde{\phi}_1 \geq \phi_1$, this implies that the bullwhip effect modulo full information is greater than for aligned perceptions, else lower. With respect to the variance of orders, Proposition 5.4 provides monotonic results for changes in the coefficients of the autoregressive operator of the perception. We find that given positive coefficients in the autoregressive operator of demand, the variance of orders increases as the coefficients of the autoregressive operator of the perception increase. The assumption in the proposition that $\tilde{\psi}_i$ for $i \geq 1$ is a decreasing sequence is not too onerous due to the assumption of the stationarity property but it does mean that $\tilde{\psi}_i \geq 0$ for all i.

Proposition 5.4 *Given $\tilde{O}_t - \mu = \tilde{\psi}(B) a_t^o$, assume that the sequence $\tilde{\psi}_i$ for $i \geq 1$ is a decreasing sequence. If $\phi_i \geq 0$ for all i, then for all i, $Var\left(\tilde{O}_t - \mu\right)$ increases as $\tilde{\phi}_i$ increases.*

5.2 Moving Average based Misalignments

In this subsection we consider misaligned perceptions with the misalignment along the moving average dimension. Given that demand follows the ARMA process (1), a moving average based misaligned perception is the following ARMA$(p, \tilde{q})$ process

$$\phi(B)(Z_t - \mu) = \tilde{\theta}(B)\tilde{a}_t. \tag{13}$$

where $\tilde{\theta}(B) = 1 - \tilde{\theta}_1 B - \tilde{\theta}_2 B^2 - ... - \tilde{\theta}_{\tilde{q}} B^{\tilde{q}}$. Note that we again assume that the mean of perception is the same as that of actual demand.

Proposition 5.5 *A moving average based misalignment in both the interpolation and extrapolation is associated with a time series of inventory I_t which is ARMA $\left(p^I, q\right)$ is with the form*

$$\phi^I(B)(I_t - T) = \theta(B) a_t^I$$

and:

$a_t^I = K^I a_t$	$p^I = \tilde{q}$	$\phi^I(B) = \frac{1}{K^I}\left(-\tilde{\theta}(B)\right)$
$K^I = -1$		

and associated with a time series of orders O_t which is ARMA$\left(p^O, q^O\right)$ with the form

$$\phi^O(B)(O_t - \mu) = \theta^O(B) a_t^O$$

and:

$a_t^O = K^O a_t$	$p^O = \tilde{q} + p$	$\phi^O(B) = \phi(B)\tilde{\theta}(B)$
$K^O = 1 + \phi_1 - \tilde{\theta}_1$	$q^O = \max(p + q, q + \tilde{q} - 1)$	$\theta^O(B) = \left(\frac{\tilde{\theta}(B)}{BK^O} - \frac{\phi(B)(1-B)}{BK^O}\right)\theta(B)$

This order time series is independent of the explicit form used for either interpolations or extrapolations.

Perceptions Perspective on the Inventory Level Process: Comparing the results for aligned perceptions in Theorem 3.1 with the results for moving average based misalignments in Proposition 5.5, we see that a moving average based misalignment introduces a moving average operator to the inventory process which is the same as that of the demand process. Recall again that under aligned perceptions, the inventory level process was i.i.d. An autoregressive operator is also introduced into the inventory process since $p^I = \tilde{q}$. As expected, from Proposition 5.6 below, misaligned perceptions increase the variance of the inventory level process versus aligned perceptions.

Proposition 5.6 *If* $\theta(B) \neq \tilde{\theta}(B)$ *then* $\left(\tilde{I}_t - T\right)$ *has a larger variance than* $(I_t - T)$.

As with autoregressive based misalignments, monotonic results for changes in variance of inventory with changes in parameters of the misaligned perception depend on the parameters of actual demand. In Proposition 5.7 below, when the coefficients of the moving average operator of the perception are all positive and either all greater than or less than those of the demand, then variance will increase with the divergence of the misaligned perception from that of actual demand. Such monotonic properties do not always hold however for example, if some of all of the moving average coefficients in the misaligned perception are negative.

Proposition 5.7 *Assume* $q \leq \tilde{q}$ *and* $\tilde{\theta}_i \geq 0$, *for all* i, *if* $\tilde{\theta}_i - \theta_i \geq 0$ *for all* i *or* $\tilde{\theta}_i - \theta_i \leq 0$ *for all* i, *then the variance of* $\tilde{I}_t$ *is increasing in* $\left|\tilde{\theta}_i - \theta_i\right|$ *for all* i.

Perception Perspective on the Order Process: Comparing the results for aligned perceptions in Theorem 3.2 with the results for moving average based misalignments in Proposition 5.5, we see that a moving average based misalignment has an effect on both the autoregressive and on the moving average dimensions of the order process compared to aligned perceptions. The bullwhip effect modulo full information is determined by the mis-estimate of the first moving average coefficient $\tilde{\theta}_1$. If $\tilde{\theta}_1 \geq \theta_1$ then the uncertainty surrounding orders is less than that for aligned perceptions, else greater. With respect to the variance of orders we are unable to prove monotonicity results for changes in the parameters of the misaligned perception. Rather, Proposition 5.8 provides monotonic results for changes in the coefficients of the moving average operator of the actual demand holding the moving average operator of the misaligned perception constant. (This is the reverse of our findings for the autoregressive based alignments above, justifying somewhat the need for separating the analysis along the two dimensions as we do here.) We find that given positive coefficients in the moving average operator of perception, the variance of orders increases as the coefficients of the moving average operator of the demand increase. Again the assumption that $\tilde{\psi}_i$ for $i \geq 1$ is a decreasing sequence is not too onerous due to the assumption of the stationarity property but it does mean that $\tilde{\psi}_i \geq 0$ for all i.

Proposition 5.8 *Given $\tilde{O}_t - \mu = \tilde{\psi}(B) a_t^o$, assume that the sequence $\tilde{\psi}_i$ for $i \geq 1$ is a decreasing sequence. If $\tilde{\theta}_i \geq 0$ for all i, then for all i, $\tilde{O}_t - \mu$ increases as θ_i increases.*

6 Perceptions Perspective on Moving Average Forecasting and Exponential Smoothing

In this section we will see how simple forecasting approaches such as the moving average and exponential smoothing can be examined using the perceptions framework. The main goal will be to describe the perceptions that could be attributed to the use of these forecasting approaches. For completeness, the predictions on the stochastic processes for inventory levels and orders are also provided.

6.1 Moving Average Forecasting

The moving average involves the use of the simple average of the previous n demand realizations, $\bar{Z}_t^n$ as an estimate of the mean of the demand process. Here

$$\bar{Z}_t^n = \frac{\sum_{i=1}^n Z_{t-i+1}}{n}.$$

Forecast errors for the moving average are calculated as $\bar{Z}_{t-i}^n - Z_{t-i+1}$ for all i and the simple average of the previous n forecast errors, $\bar{\sigma}_t^n$ is also used as an estimate of the standard deviation of the forecast error and used to determine a proxy for the safety stock of the inventory policy. Here

$$\bar{\sigma}_t^n = \frac{\sum_{i=1}^n \left(\bar{Z}_{t-i}^n - Z_{t-i+1}\right)}{n}. \tag{14}$$

The order up-to target for the inventory system is $\bar{Z}_t^n + \bar{\sigma}_t^n k$ with $\bar{\sigma}_t^n k$ as the implied safety factor. Here the estimate of the mean of the demand process and the safety factor comprise the order-up-to target, implying that our safety stock $T = 0$. This results in the following claim for inconsistent misaligned perceptions which categorize the use of the moving average for setting inventory policies. The claim follows since it directly mimics the use of the moving average just described and lemmas 6.1 and 6.2 show the perceptions to be stationary and invertible.

Claim 1 *The perceptions implied by the use of an order up-to target* $\bar{Z}_t^n + \bar{\sigma}_t^n k$ *is of the following* $ARMA(n, 0)$ *process for interpolations:*

$$\tilde{\phi}(B)(Z_t - \mu) = \tilde{a}_t$$

where $\tilde{\phi}(B) = 1 - \sum_{i=1}^{n} \frac{1}{n} B^i$ *and the following* $ARMA(n, n)$ *process for extrapolations:*

$$\hat{\phi}(B)(Z_t - \mu) = \hat{\theta}(B)\tilde{a}_t$$

where $\hat{\theta}(B) = 1 - \sum_{i=1}^{n} \frac{-k}{n} B^i$*, and* $\hat{\phi}(B) = \tilde{\phi}(B)$.

Lemma 6.1 *The time series*

$$\tilde{\phi}(B)(Z_t - \mu) = \tilde{a}_t$$

where $\tilde{\phi}(B) = 1 - \sum_{i=1}^{n} \frac{1}{n} B^i$ *is stationary and invertible.*

Lemma 6.2 *The time series*

$$\hat{\phi}(B)(Z_t - \mu) = \hat{\theta}(B)\tilde{a}_t$$

where $\hat{\theta}(B) = 1 - \sum_{i=1}^{n} \frac{-k}{n} B^i$*, and* $\hat{\phi}(B) = \tilde{\phi}(B)$ *is stationary and invertible.*

Note here that the mean of the demand process assumed in the perception is the same as actual since the moving average is used intuitively to estimate the actual level rather than assume a preconceived one. The explicit form of forecasting used for either interpolation or extrapolation here is the first explicit form, that is, $J = 1$. The perception implied by interpolations matches with how forecast errors are calculated for the moving average, thus note that it is an ARMA $(n, 0)$ process. The perception implied by extrapolation matches with how inventory targets are set, thus note that the moving average dimension of this perception calculates the simple average of the previous n forecast errors and then combines it with the safety factor k. For completeness we provide the results for inventory levels and orders that are associated with the perceptions in Claim 1, given that demand follows the ARMA process (1).

Proposition 6.3 *The use of the moving average of the n previous demand realizations for an order up-to* $\bar{Z}_t^n + \bar{\sigma}_t^n k$ *inventory policy is associated with a time series of inventory* I_t *which is ARMA* (p, q^I) *with the form*

$$\phi(B) I_t = \theta^I(B) a_t^I$$

and:

$a_t^I = K^I a_t$	$q^I = 2n + q$
$K^I = -1$	$\theta^I(B) = \tilde{\phi}(B)\,\theta(B)\left(\hat{\theta}(B) - 2\right)$

and associated with a time series of orders O_t which is ARMA$\left(p, q^O\right)$ with the form

$$\phi(B)(O_t - \mu) = \theta^O(B)\, a_t^O$$

and:

$a_t^O = K^O a_t$	$q^O = 2n + q$
$K^O = 1 + \frac{1+k}{n}$	$\theta^O(B) = \left(\frac{1}{BK^O} - \frac{2\tilde{\phi}(B)(1-B)}{BK^O} + \frac{\tilde{\phi}(B)\hat{\theta}(B)(1-B)}{BK^O}\right)\theta(B)$

6.2 Exponential Smoothing

Exponential smoothing creates forecasts F_t which are a weighted average between the previous periods forecast and the current demand realization, that is,

$$F_t = \alpha Z_t + (1 - \alpha) F_{t-1}, \tag{15}$$

where $0 < \alpha < 1$. We assume that the forecasts are combined with a safety stock T for an order up-to target.

Muth (1960) shows that the exponential smoothing is optimal for an ARIMA(0,1,1) process. Thus the use of exponential smoothing can be considered to assume this non-stationary perception (see Section 7). However the exponential smoothing approach was designed before the optimality result by Muth was found. Furthermore the use of the exponential smoothing approach reflects less of an intended application to non-stationary process since there are specific approaches based on exponential smoothing designed to deal with trends and seasonality, see Holt (1957), Winters (1960), and Holt (2004). We propose therefore a stationary perception associated with the use of the exponential smoothing approach which can be determined as follows. Expanding (15) above we get,

$$\begin{aligned} F_t &= \alpha Z_t + (1 - \alpha)(\alpha Z_{t-1} + (1 - \alpha) F_{t-1}) \\ &= \alpha \sum_{i=1}^{\infty} (1 - \alpha)^{i-1} Z_{t-i+1} \\ &\approx \alpha \sum_{i=1}^{m} (1 - \alpha)^{i-1} Z_{t-i+1} \end{aligned}$$

where the last approximation follows since there is some m for which $(1 - \alpha)^m$ and higher powers can be considered negligible. This implies the following claim for consistent misaligned perceptions which categorize the use of exponential smoothing for setting inventory policies. Lemma 6.4 shows that the perception is stationary and invertible.

Claim 2 *The perception implied by using exponential smoothing with a weighting factor* α *is the following ARMA*(m) *process for both interpolations and extrapolations:*

$$\tilde{\phi}(B)(Z_t - \mu) = a_t$$

where $\tilde{\phi}(B) = 1 - \sum_{i=1}^{m} \alpha (1-\alpha)^{i-1} B^i$.

Lemma 6.4 *The time series*

$$\tilde{\phi}(B)(Z_t - \mu) = a_t$$

where $\tilde{\phi}(B) = 1 - \sum_{i=1}^{m} \alpha (1-\alpha)^{i-1} B^i$ *is stationary and invertible.*

Note here that the mean of the demand process assumed in the perception is the same as the actual mean, that is $\tilde{\mu} = \mu$, as should be expected since exponential smoothing intuitively does not incorporate a preconceived mean. Here the explicit form used could be either of the three explicit forms since the perceptions implied by interpolations and extrapolations are the same. For completeness we provide the results for inventory levels and orders that are associated with the perceptions in Claim 2 and are corollaries of Theorems 4.2 and 4.3:

Proposition 6.5 *The use of exponential smoothing with weighting factor* α *is associated with a time series of inventory* I_t *which is ARMA* (p, q^I) *with the form*

$$\phi(B)(I_t - T) = \theta^I(B) a_t^I$$

and:

$a_t^I = K^I a_t$	$q^I = m + q$
$K^I = -1$	$\theta^I(B) = \frac{-1}{K^I}\left(\tilde{\phi}(B)\theta(B)\right)$

and associated with a time series of orders O_t *which is ARMA*(p, q^O) *with the form*

$$\phi(B)(O_t - \mu) = \theta^O(B) a_t^O$$

and:

$a_t^O = K^O a_t$	$q^O = m + q$
$K^O = 1 + \alpha$	$\theta^O(B) = \left(\frac{-\phi(B)(1-B)}{BK^O}\right)\theta(B)$

6.3 Discussion

The above examination reveals a couple of interesting points about our perceptions framework. Firstly, from our examination of the use of the moving average forecasting, we see that it is very plausible for a setting where the perceptions implied by the extrapolation and interpolation are different. The setting also suggests reasons why the perceptions could be different as here, extrapolations seem directly related to the form of the inventory policy leaving interpolations to be more focused on estimating demand shocks. Secondly from our examination of exponential smoothing, there might be more than one perception which "fits" a particular extrapolation and interpolation combination, as the exponential smoothing is optimal for a non-stationary demand process. However we claim that its use suggests a stationary perception which we provide and show the expected orders and inventory as a result of this perception.

7 Perceptions Framework for Suboptimal policies: Consistent Misaligned Non-stationary Perceptions

In this section we complete the presentation of our perceptions framework with a treatment of demand modeled as non-stationary demand processes. For brevity, we will only consider consistent misaligned perception. As in Section 4, by characterizing the stochastic processes for the inventory levels and order levels given the same misaligned perception in both interpolations and extrapolations, we establish this perception as the basis for the link between the process characteristics of the system. After the results provided below, we describe the more interesting relationships between the categorizing perception and the process results. This misaligned perception is the following ARIMA$\left(\tilde{p}, \tilde{d}, \tilde{q}\right)$ process:

$$\tilde{\varphi}(B)(Z_t - \tilde{\mu}) = \tilde{\theta}(B)\,\tilde{a}_t \tag{16}$$

where $\tilde{\varphi}(B) = \tilde{\phi}(B)\nabla^{\tilde{d}}$ and the implied ARMA$(\tilde{p}, \tilde{q})$ process on the dth difference, $\tilde{\phi}(B)\left(\nabla^{\tilde{d}}(Z_t - \tilde{\mu})\right) = \tilde{\theta}(B)\,\tilde{a}_t$, is stationary and invertible.

We describe the approaches for extrapolation and interpolation which are analogous to their descriptions in section 4. Consider extrapolations, that is our one period ahead forecasts $\hat{Z}_t^e$, from origin t. As with stationary misaligned perceptions, $\hat{Z}_t^e$ can be defined in the following three ways given the three explicit forms of the forecasts for an ARIMA process:

$$\hat{Z}_t^e(1) = \tilde{\mu} + \frac{1 - \tilde{\varphi}(B)}{B}(Z_t - \tilde{\mu}) + \frac{\tilde{\theta}(B) - 1}{B}\tilde{a}_t \tag{17}$$

or

$$\hat{Z}_t^e(2) = \tilde{\mu} + \frac{\tilde{\psi}(B) - 1}{B}\tilde{a}_t \tag{18}$$

where $\tilde{\psi}(B) = 1 + \sum_{j=1}^{\infty}\tilde{\psi}_j B^j = \tilde{\theta}(B)/\tilde{\varphi}(B)$ or

$$\hat{Z}_t^e(3) = \tilde{\mu} + \frac{1 - \tilde{\pi}(B)}{B}(Z_t - \tilde{\mu}) \tag{19}$$

where $1-\sum_{j=1}^{\infty}\tilde{\pi}_j B^j=\tilde{\pi}(B)=\tilde{\varphi}(B)/\tilde{\theta}(B)$. Note that in the first two explicit forms of $\hat{Z}_t^e$ we use the random shock terms $\tilde{a}_t$ to reflect interpolations with misaligned perceptions. Also note that the third form does not use any interpolations. Also recall that for an ARIMA time series Z_t, although $\nabla^d Z_t$ is stationary and invertible, Z_t is not stationary though it is invertible. Therefore the infinite sum $\tilde{\psi}(B)$ above, is not summable and so does not converge in any sense. Alternatively we can rewrite (18) in terms of the dth difference which does have a converging sum of random shocks. As a notational device, we will refer to this infinite random shock form to represent the forecast based on infinite random shock form of the dth difference.

Consider now the perceptions for interpolations. Recall that it is the difference between the actual realization of the dth difference and the one period ahead forecast of the dth difference that is used to generate the realization for the demand errors $\tilde{a}_t$. We can again let $\hat{Z}_t^i$ denote the one period ahead forecast of demand used to create the one period ahead forecast of the dth difference and denote three forms for $\hat{Z}_t^i$ as $\hat{Z}_t^i(1)$, $\hat{Z}_t^i(2)$ and $\hat{Z}_t^i(3)$ which again coincide sequentially with the three explicit forms listed in Section 2.1. Again as with Proposition 4.1, we can write $\tilde{a}_t=\tilde{\pi}(B)(Z_t-\tilde{\mu})$ where $\tilde{\pi}(B)=\tilde{\varphi}(B)/\tilde{\theta}(B)$ for either of the three forms $\hat{Z}_t^i(j)$. Theorems 7.1 and 7.2 below characterize the stochastic processes for the inventory levels and orders.

Theorem 7.1 *A misaligned implied perception is associated with a time series of inventory I_t which is ARMA $\left(p^I,d^I,q^I\right)$ with the form*

$$\varphi^I(B)\left(I_t-T^I\right)=\theta^I(B)\,a_t^I$$

and:

$a_t^I=K^I a_t$	$d^I=\max(0,d-\tilde{d})$	$\varphi^I(B)=\tilde{\theta}(B)\,\phi(B)\,\nabla^{\max(0,d-\tilde{d})}$
$K^I=-1$	$q^I=\tilde{p}+\max(0,\tilde{d}-d)+q$	$T^I=T+\left(\frac{1-B-\tilde{\pi}(B)}{B}\right)(\mu-\tilde{\mu})$
$p^I=\tilde{q}+p$	$\theta^I(B)=\frac{1}{K^I}\left(-\theta(B)\,\tilde{\varphi}(B)\right)$	

This time series is independent of the explicit forecasting form used for either interpolations or extrapolations and the d^Ith difference of inventory time series I_t is stationary. The time series is also invertible if the demand process is invertible.

Theorem 7.2 *A misaligned perception attributed to both the interpolations and the extrapolation is associated with a time series of orders O_t which is ARMA$\left(p^O,d,q^O\right)$ with the form*

$$\varphi^O(B)\left(O_t-\mu\right)=\theta^O(B)\,a_t^O$$

and:

$a_t^O = K^O a_t$		$\varphi^O(B) = \varphi(B)\tilde{\theta}(B)$
$K^O = 1 + \tilde{\varphi}_1 - \tilde{\theta}_1$	$q^O = \max\left(\tilde{p} + q + \tilde{d}, q + \tilde{q} - 1\right)$	
$p^O = \tilde{q} + p$	$\theta^O(B) = \frac{\tilde{\varphi}(B)\theta(B)}{K^O} + \theta(B)\frac{\left(\tilde{\theta}(B) - \tilde{\varphi}(B)\right)}{BK^O}$	

This time series is independent of the explicit form used for either interpolations or extrapolations and the dth difference of the order time series O_t is stationary but not necessarily invertible.

Perceptions Perspective on the Inventory Level Process: Theorem 7.1 shows that for consistent misaligned perceptions, the inventory level will follow a non-stationary stochastic process if the perception represents a lower difference than that of the demand, irrespective of accuracy of the autoregressive and moving average operators of the perception. These operators of the perception still determine the autoregressive and moving average operators of the stochastic process for the inventory level.

Perceptions Perspective on the Order Process: Theorem 7.2 shows that for consistent misaligned perceptions, orders will exhibit the same number of differences as demand. The order of the moving average operator for the order process is increasing in the number of differences in the perception $\tilde{d}$ provided $\tilde{d}$ is large enough.

8 Conclusion

In this paper we proposed a perceptions framework for categorizing a range of inventory decision making settings in a single-stage supply chain. The perceptions framework is based on forecasting with Auto-Regressive Integrated Moving Average (ARIMA) time series models. In order to categorize these settings, we make the *strong* assumption that such settings can be grouped based on the combination of inventory levels, orders and demand persistently exhibited within the system, despite these settings being otherwise considered different from each other. These settings are then categorized by conceptual perceptions of demand which underpin an analytic link between inventory levels, orders and actual demand, a link that we establish in the paper. Our perceptions perspective relies on both interpolations and extrapolations using these ARIMA models. Optimal inventory policies can be categorized as perceptions that are aligned with reality and deviations from such polices as perceptions that are not aligned. With respect to misaligned perceptions, there are two variations. Consistent misaligned perceptions have the same perception for both interpolations and extrapolations while inconsistent misaligned perceptions do not. In support of our framework we categorize commonly used forecasting approaches using our perceptions perspective.

For stationary processes we examined the inventory level process and the order process resulting from misaligned perceptions. In order to examine these effects, we separately examined autoregressive based misalignments and moving average based misalignments. Any misalignment in perception increases the variance of inventory level process over that of aligned perceptions. Monotonic properties related to change in parameters of misaligned perception depended on actual demand. For autoregressive based misalignment and moving average based misalignment, there was some support for variance of inventory levels increasing with difference in coefficients of operators between perception and actual demand. For moving average based misalignment, monotonic properties for the variance of orders were found for changing the parameters of actual demand and keeping the misaligned perception constant, in particular, variance increased with an increase in the coefficients of the moving average operator of demand. For autoregressive based misalignments, variance of orders increased with an increase in the coefficients of the autoregressive operator of the perception.

We expect our proposed perceptions framework will be useful for more robust examinations of supply chain management dynamics. In particular we believe we help broaden the language that can be used surrounding inventory decision making. So called optimal policies are a very restricted class of inventory dynamics both analytically and conceptually. A categorization, such as ours, which incorporates some latitude in describing and modeling inventory policies independent of costs, creates the opportunity for addressing a wider range of real-world situations where optimality is not an objective, either because costs are not known, are subjectively perceived or the goals of the organization are construed and pursued differently.

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9 Appendix:

9.1 Proofs of Theorems 3.1, 3.2, 4.2, 4.3 and Proposition 3.3.

In this appendix we provide selected proofs and only parts of said proof so as to give the reader a sense of the mechanics used to get our results. These mechanics characterize all of the proofs for the results in this manuscript. All proofs can be found in an online supplement at http://www.people.hbs.edu/nwatson/.

Proof of Theorem 3.1

By definition we have

$$I_t = T + \hat{Z}_{t-1} - Z_t.$$

Recall

$$\hat{Z}_{t-1} = \mu + \frac{1-\varphi(B)}{B}(Z_{t-1}-\mu) + \frac{\theta(B)-1}{B}a_{t-1}.$$

We also have that

$$Z_t = \mu + \frac{1-\varphi(B)}{B}(Z_{t-1}-\mu) + \theta(B)\,a_t.$$

This gives

$$\begin{aligned} I_t - T &= \frac{\theta(B)-1}{B}a_{t-1} - \theta(B)\,a_t. \\ &= -a_t. \end{aligned}$$

Proof of Theorem 3.2

Recall

$$\hat{Z}_t = \mu + \frac{1-\varphi(B)}{B}(Z_t - \mu) + \frac{\theta(B)-1}{B}a_t.$$

Now $O_t = Z_t + \hat{Z}_t - \hat{Z}_{t-1}$ which gives

$$O_t - \mu = Z_t - \mu + \frac{1-\varphi(B)}{B}(1-B)(Z_t-\mu) + \frac{\theta(B)-1}{B}(1-B)a_t.$$

Since $(Z_t - \mu) = (\theta(B)/\varphi(B))\, a_t$ we get

$$\begin{aligned}
\varphi(B)(O_t-\mu) &= \theta(B)a_t + \frac{1-\varphi(B)-B+B\varphi(B)}{B}\theta(B)a_t + \frac{(\theta(B)-1)(1-B)}{B}\varphi(B)a_t \\
&= \frac{B+1-\varphi(B)-B+B\varphi(B)}{B}\theta(B)a_t + \frac{(\theta(B)-1)(1-B)}{B}\varphi(B)a_t \\
&= \left(\frac{\theta(B)}{B} - \frac{\theta(B)\varphi(B)(1-B)}{B} + \frac{\varphi(B)(\theta(B)-1)(1-B)}{B}\right)a_t \\
&= \left(\frac{\theta(B)-\varphi(B)}{B} + \varphi(B)\right)a_t.
\end{aligned}$$

The above order process can be rewritten

$$\phi(B)\nabla^d(O_t-\mu) = \left(\frac{\theta(B)-\varphi(B)}{B} + \varphi(B)\right)a_t,$$

with autoregressive order p, difference order d and moving average order $\max\{p+d, q-1\}$. Note first term in polynomial $\frac{\theta(B)-\varphi(B)}{B} + \varphi(B)$ is $1+\varphi_1-\theta_1$ so the above can be normalized as

$$\phi(B)\nabla^d(O_t-\mu) = \left(\frac{\theta(B)-\varphi(B)}{BK^O} + \frac{\varphi(B)}{K^O}\right)a_t^O,$$

where $K^O = 1+\varphi_1-\theta_1$ and $a_t^O = K^O a_t$.

Proof of Proposition 3.3

We have

$$\begin{aligned}
Var(Z_t-\mu) &= Var\left(\frac{\theta(B)}{\phi(B)}a_t\right) \\
&= \sigma^2\left(1+\sum_{i=1}^{\infty}\psi_i^2\right)
\end{aligned}$$

while

$$\begin{aligned}
Var(O_t-\mu) &= Var\left(\frac{\theta(B)-\phi(B)}{\phi(B)BK^O} + \frac{\phi(B)}{\phi(B)K^O}a_t^O\right) \\
&= Var\left(\frac{\theta(B)}{\phi(B)BK^O} - \frac{1+B}{BK^O}a_t^O\right) \\
&= \sigma^2\left(K^O\right)^2\left(1+\sum_{i=1}^{\infty}\bar{\psi}_i^2\right)
\end{aligned}$$

From algebra we have that $K^O = 1 + \psi_1$ and $K^O \bar{\psi}_i = \psi_{i+1}$ for $i \geq 1$. This implies that $Var(Z_t - \mu) < Var(O_t - \mu) \Leftrightarrow \psi_1 = \phi_1 - \theta_1 > 0$.

Proof of Theorem 4.2

We only show the proof for extrapolating with the second explicit form of forecasting, i.e., $\hat{Z}^e_{t-1}(2)$. From the definition of the inventory policy, we have

$$\tilde{I}_t = T + \hat{Z}^e_{t-1}(2) - Z_t.$$

Since

$$\hat{Z}^e_{t-1}(2) = \tilde{\mu} + \frac{\tilde{\psi}(B) - 1}{B}\tilde{a}_{t-1},$$

and we also have that

$$Z_t = \mu + \psi(B) a_t,$$

this gives

$$\tilde{I}_t = T + \frac{\tilde{\psi}(B) - 1}{B}\tilde{a}_{t-1} - \psi(B) a_t - (\mu - \tilde{\mu}).$$

Since $\tilde{a}_t = \tilde{\pi}(B)(Z_t - \tilde{\mu})$, $(Z_t - \mu) = \psi(B) a_t$, $\tilde{\pi}(B) = \tilde{\phi}(B)/\tilde{\theta}(B)$ and $\tilde{\psi}(B) = \tilde{\theta}(B)/\tilde{\phi}(B)$, if we let again $\breve{\mu} = \left(\frac{1-B-\tilde{\pi}(B)}{B}\right)(\mu - \tilde{\mu})$, we get our claim for $J = 2$:

$$\begin{aligned}
\left(\tilde{I}_t - T\right) &= \frac{\tilde{\psi}(B) - 1}{B}\tilde{\pi}(B)(Z_{t-1} - \tilde{\mu}) - \psi(B) a_t - (\mu - \tilde{\mu}) \\
&= \frac{1 - \tilde{\pi}(B)}{B}(Z_{t-1} - \mu) + \left(\frac{1 - \tilde{\pi}(B)}{B}\right)(\mu - \tilde{\mu}) \\
&- \psi(B)(B) a_t - (\mu - \tilde{\mu}) \\
\left(\tilde{I}_t - T - \breve{\mu}\right) &= \frac{1 - \tilde{\pi}(B)}{B}\psi(B) a_{t-1} - \psi(B) a_t \\
&= -\tilde{\pi}(B)\psi(B) a_t \\
\tilde{\theta}(B)\phi(B)\left(\tilde{I}_t - T - \breve{\mu}\right) &= -\tilde{\phi}(B)\theta(B) a_t.
\end{aligned}$$

For all forms $J = 1, 2, 3$ the process for $\tilde{I}_t$ is: $\tilde{\theta}(B)\phi(B)\left(\tilde{I}_t - T - \left(\frac{1-B-\tilde{\pi}(B)}{B}\right)(\mu - \tilde{\mu})\right) = -\tilde{\phi}(B)\theta(B) a_t$ where the order of $\tilde{\theta}(B)\phi(B)$ is $p + \tilde{q}$ while the order of $\tilde{\phi}(B)\theta(B)$ is $\tilde{p} + q$. Note that if both demand and the perception are assumed stationary and invertible, $\tilde{I}_t$ is also stationary and invertible. Note first term in polynomial $-\tilde{\phi}(B)\theta(B)$ is $-\left(1 - \tilde{\phi}_1 - \tilde{\theta}_1\right)$ which we can use to normalize the process as in Theorem 3.2.

Proof of Theorem 4.3

We only show the proof for $J = 3$. Consider $\hat{Z}^e_t(3)$,

$$\hat{Z}^e_t(3) = \tilde{\mu} + \frac{1 - \tilde{\pi}(B)}{B}(Z_t - \tilde{\mu}).$$

Now $\tilde{O}_t = Z_t + \hat{Z}^e_t(3) - \hat{Z}^e_{t-1}(3)$ which gives

$$\begin{aligned}
\tilde{O}_t &= Z_t + \frac{1-\tilde{\pi}(B)}{B}(1-B)(Z_t - \tilde{\mu}) \\
\tilde{O}_t - \mu &= Z_t - \mu + \frac{1-\tilde{\pi}(B)}{B}(1-B)(Z_t - \tilde{\mu}) \\
&= \frac{1-\tilde{\pi}(B)(1-B)}{B}(Z_t - \mu) \\
&\quad + \left(\frac{1-\tilde{\pi}(B)}{B}(1-B)\right)(\mu - \tilde{\mu})
\end{aligned}$$

Since $\left(\frac{1-\tilde{\pi}(B)}{B}(1-B)\right)(\mu - \tilde{\mu}) = 0$, $(Z_t - \mu) = (\theta(B)/\phi(B))\,a_t$ and $\tilde{\pi}(B) = \tilde{\phi}(B)/\tilde{\theta}(B)$ we get

$$\begin{aligned}
\tilde{O}_t - \mu &= \frac{1-\tilde{\phi}(B)/\tilde{\theta}(B)(1-B)}{B}(\theta(B)/\phi(B))\,a_t \\
\tilde{\theta}(B)\phi(B)\left(\tilde{O}_t - \mu\right) &= \left(\frac{\tilde{\theta}(B) - \tilde{\phi}(B)(1-B)}{B}\right)\theta(B)\,a_t.
\end{aligned}$$

For all forms $J = 1, 2, 3$, the process for $\tilde{O}_t$ is the same, that is, $\tilde{\theta}(B)\phi(B)\left(\tilde{O}_t - \mu\right) = \left(\frac{\tilde{\theta}(B)-\tilde{\phi}(B)(1-B)}{B}\right)\theta(B)\,a_t$ where the order of $\tilde{\theta}(B)\phi(B)$ is $p+\tilde{q}$ while the order of $\left(\frac{\tilde{\theta}(B)-\tilde{\phi}(B)(1-B)}{B}\right)\theta(B)$ is $\max\{\tilde{p}+q, q+\tilde{q}-1\}$. Note that if both demand and the perception are assumed stationary and invertible, $\tilde{O}_t$ is also stationary but not necessarily invertible. Note first term in polynomial $\left(\frac{\tilde{\theta}(B)-\tilde{\phi}(B)(1-B)}{B}\right)\theta(B)$ is $1 + \tilde{\phi}_1 - \tilde{\theta}_1$.