An introduction to censored, truncated or sample-selected data

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1 Compare three different cases

A sample is truncated if some observations are systematically excluded from the sample. For example, suppose we are interested in relationship between people's income(y) and education(x). If we have observations of both y and x only for people whose income is above \$20,000 per year, then we have a truncated sample. A sample is censored if no observations have been systematically excluded but some of information contained in them has been suppressed. In other words, in a truncated sample, you don't have observations which have been truncated; while in a censored sample, you do have observations but you don't have full information. The idea of "censoring" is that some data above or below a threshold are mis-reported at the threshold. If we use the same example of people's income, if we don't observe people's income (y) and education(x) with income lower than \$20,000 per year, then we have a truncated sample. If we have observations of x for people whose income above AND below that limit, but no information on exactly how much their income is for those whose income below the limit (only thing we know is that it's lower than \$20,000 per year), then we have a censored sample.

The problem of sample selection arises when no observations have been systematically excluded but some information has been suppressed, just like censoring. However, in sample selection problems, we have information on how the "selection" happens. In censoring, usually a constant "threshold" is set. Observations on dependent variable for those above (or below) that criteria are missing. Using the income example again, the threshold is set to be \$20,000, we have no information on exactly how much their income is for those whose income below the limit (only thing we know is that it's lower than \$20,000 per year). When we have a third variable that determines the censoring situation, then we call it sample selection. For example, we have observed only union member's information on income, but we observe information on education for everybody, including union members or non-members. In this example, union membership is an additional variable that "selects" the sample which are involved in the estimation of effect of education on income.

Richard Breen has a nice table in his book:

Table 1: Comparison of censored, truncated, and sample-selected cases. (borrowed from Richard Breen)

Sample	Y Variable	X Variable	Example
Censored	y is known exactly only if some criterion defined. in terms of y is met.	x variables are observed for the entire sample, regardless of whether y is observed exactly	Determinants of income; income is measured exactly only if it above the poverty line. All other incomes are reported at the poverty line
Sample Selected	y is observed only if a criteria defined. in terms of some other random variable (Z) is met.	x and w (the determinants of whether Z =1) are observed for the entire sample, regardless of whether y is observed or not	Survey data with item or unit non-response
Truncated	y is known only if some criterion defined in terms of y is met.	x variables are observed only if y is observed.	Donations to political campaigns.

2 An illustration for truncated or censored data

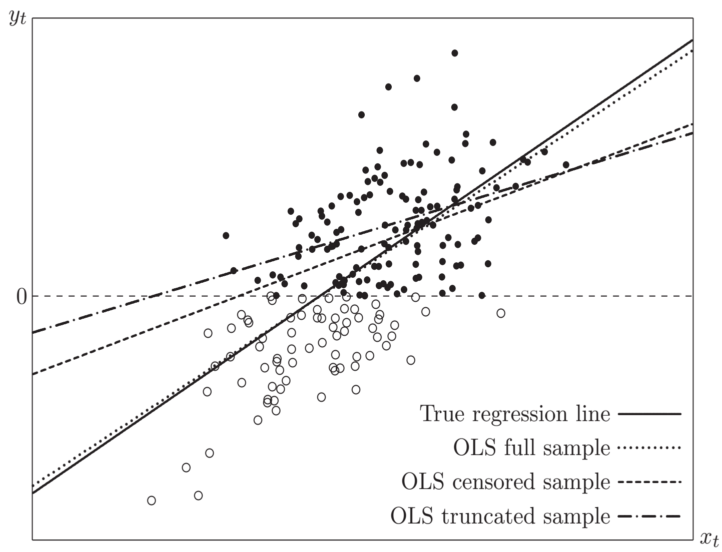
Consider the model

$$y_t^0 = \beta_1 + \beta_2 x_t + u_t,\ u_t \sim NID(0, \sigma^2), \tag{1}$$

where y_t^0 is a latent variable. We actually observe y_t, which differs from y_t^0 , because it's either truncated or censored.

Suppose that censorship or truncation occurs whenever y_t^0 is less than 0. Clearly, the larger the error term u_t, the larger is y_t^0 and thus the greater must be the probability that $y_t^0 \geq 0$. This probability also depends on x_t. So for the sample we observe, u_t does not have conditional mean 0 and is not uncorrelated with x_t. OLS using truncated or censored samples (OLS of y_t on x_t) yields biased and inconsistent estimators. Normally we would like to draw inference on the population which is represented by the full sample. Shown in figure 1, ideally we would have the OLS regression line (if we had the full sample), which is very close to the "true" regression line (which is the mechnism generates the data). We would have the "OLS censored sample" line if we run OLS on the censored sample; we would have "OLS truncated sample" if we run OLS on the truncated sample. It shows that both of them are severely biased from the "true" regression line.

Figure 1: Effects of truncation and censoring from Davidson and MacKinnon



3 Truncated Models

For truncated data, a consistent estimator comes from maximum likelihood estimator (MLE). If we assume the error terms in the latent variable model has a known distribution, then MLE estimator can be applied. The most popular choice is Gaussian.

$$\begin{aligned} \Pr(y_t^0 \geq 0) &= \Pr(\mathbf{X}_t\beta + u_t \geq 0) \\ &= 1 - \Pr(u_t/\sigma < -\mathbf{X}_t\beta/\sigma) \\ &= 1 - \Phi(-\mathbf{X}_t\beta/\sigma) = \Phi(\mathbf{X}_t\beta/\sigma) \end{aligned} \tag{2}$$

The density of y_t is proportional to the density of y_t^0 when $y_t^0 \geq 0$ and y_t is observed. It is 0 elsewhere. The factor of proportionality, which is needed to ensure that the density integrates to unity, is the inverse of the probability that $y_t^0 \geq 0$. The density of y_t can be written as

$$\frac{\sigma^{-1}\phi((y_t - \mathbf{X}_t\beta)/\sigma)}{\Phi(\mathbf{X}_t\beta/\sigma)} \tag{3}$$

This implies the log-likelihood function,

$$\ell(\mathbf{y}, \beta, \sigma) = -\frac{n}{2}\log(2\pi) - n\log(\sigma) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}(y_t - \mathbf{X}_t\beta)^2 - \sum_{t=1}^{n}\log\Phi(\mathbf{X}_t\beta/\sigma). \tag{4}$$

This can be estimated by MLE. In Stata, truncreg is the command to do truncated regression model.

4 Censored Models

The most popular censored model is Tobit model.

$$\begin{aligned} y_t^0 &= \mathbf{X}_t + u_t, \quad u_t \sim NID(0, \sigma^2) \\ y_t &= y_t^0 \text{ if } y_t^0 > 0;\; y_t = 0 \text{ otherwise.} \end{aligned} \tag{5}$$

We see that

$$\begin{aligned} \Pr(y_t = 0) &= \Pr(y_t^0 \leq 0) = \Pr(\mathbf{X}_t\beta + u_t \leq 0) \\ &= \Pr(u_t/\sigma < -\mathbf{X}_t\beta/\sigma) \\ &= \Phi(-\mathbf{X}_t\beta/\sigma) \end{aligned} \tag{6}$$

The contribution to the log-likelihood function made by observations with $y_t = 0$ is

$$\ell_t(y_t, \beta, \sigma) = \log\Phi(-\mathbf{X}_t\beta/\sigma). \tag{7}$$

If y_t is positive, the contribution to the log-likelihood is the logarithm of the density,

$$\log(\frac{1}{\sigma}\phi((y_t - \mathbf{X}_t\beta)/\sigma)). \tag{8}$$

The log-likelihood function of the tobit model is

$$\sum_{y_t=0} \log \Phi(-\mathbf{X}_t\beta/\sigma) + \sum_{y_t>0} \log(\frac{1}{\sigma}\phi((y_t - \mathbf{X}_t\beta)/\sigma)) \tag{9}$$

This can be estimated by MLE. In Stata, tobit or intreg can be used for censoring models.

5 Sample Selection

The sample selection models differ from censored model in that it involves a different variable (from y itself) to determine the censorship (selection).

Suppose that y_t^0 and z_t^0 are two latent variables, generated by the bivariate process

$$\begin{bmatrix} y_t^0 \\ z_t^0 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_t\beta \\ \mathbf{W}_t\gamma \end{bmatrix} + \begin{bmatrix} u_t \\ v_t \end{bmatrix}, \begin{bmatrix} u_t \\ v_t \end{bmatrix} \sim NID(\mathbf{0}, \begin{bmatrix} \sigma^2 & \rho \\ \rho & 1 \end{bmatrix}) \tag{10}$$

We observe y_t and z_t:

$$\begin{aligned} y_t &= y_t^0 \text{ if } z_t^0 > 0;\ y_t \text{ unobservable otherwise;} \\ z_t &= 1 \text{ if } z_t^0 > 0;\ z_t = 0 \text{ otherwise.} \end{aligned} \tag{11}$$

There are two types of observations, ones we observe $y_t = y_t^0$ and $z_t = 1$, along with both $\mathbf{X_t}$ and $\mathbf{W_t}$, and ones we observe only $z_t = 0$ and $\mathbf{W_t}$.

Each observation contributes to the likelihood function by

$$I(z_t = 0)\Pr(z_t = 0) + I(z_t = 1)\Pr(z_t = 1)f(y_t^0|z_t = 1), \tag{12}$$

Under the normality assumption,

$$u_t = \rho v_t + e_t \tag{13}$$

where e_t is independent of $v_t \sim N(0, 1)$. A useful fact about the standard normal distribution is that

$$\mathrm{E}(v_t|v_t > -x) = \lambda(x) = \frac{\phi(x)}{\Phi(x)} \tag{14}$$

and the function $\lambda(x)$ is called the inverse Mills ratio.

The log-likelihood function can be shown to be

$$\sum_{z_t=0} \log \Phi(-\mathbf{W}_t\gamma) + \sum_{z_t=1} \log(\frac{1}{\sigma}\phi((y_t - \mathbf{X}_t\beta)/\sigma)) + \sum_{z_t=1} \log \Phi(\frac{\mathbf{W}_t\gamma + \rho(y_t - \mathbf{X}_t\beta)/\sigma}{(1-\rho^2)^{1/2}}) \tag{15}$$

So this model can be estimated by MLE.

However, it is popular to use Heckman's two-step method.

Heckman's method is based on the fact that the original latent model can be rewritten as

$$y_t = \mathbf{X}_t\beta + \rho v_t + e_t \tag{16}$$

Here the error term u_t is divided into two parts, one perfectly correlated with v_t, and one independent of v_t.

In the first step, an ordinary probit model is used to obtain consistent estimates $\hat{\gamma}$ of the parameters of the selection equation.

In the second step, the unobserved v_t is replaced by the selectivity regressor $\frac{\phi(\mathbf{W}_t\hat{\gamma})}{\Phi(\mathbf{W}_t\hat{\gamma})}$.

Therefore, the regression becomes

$$y_t = \mathbf{X}_t\beta + \rho\frac{\phi(\mathbf{W}_t\hat{\gamma})}{\Phi(\mathbf{W}_t\hat{\gamma})} + e_t \tag{17}$$

Or,

$$y_t = \mathbf{X}_t\beta + \rho\hat{\lambda}_t + e_t \tag{18}$$

One thing to note in Heckman model is that in many situations, $\hat{\lambda}$ is believed to be highly collinear with $\mathbf{X_t}$ (See Olsen), if $\mathbf{W_t}$ is the same as $\mathbf{X_t}$. However, the model can still be estimated due to the nonlinearity of the model. Nevertheless, it becomes a regular practice to require $\mathbf{W_t}$ contains at least one extra variable than $\mathbf{X_t}$, which is sometimes called exclusion restriction. In many situations, $\mathbf{W_t}$ contains all $\mathbf{X_t}$ variables, and at least one more variable. The reason to contain all $\mathbf{X_t}$ variables is because the selection is "endogenous" in the sense that y_t is a factor in determining selection; this is modeled by including all the $\mathbf{X_t}$'s.

In Stata, heckman is the command to do sample selection models. It has options to do a Heckman two-step estimation or MLE estimation. Usually MLE is preferred.

6 Switching Regression (Treatment-Effects Model)

There is another situation that is similar to sample selection model: we observe y for $z = 1$ and $z = 0$. In the example of effect of education on income, we observe both union member's income and non-member's income. In this case, we have switching regression model or treatment-effect model. The treatment effect model estimates the effect of an endogenous binary treatmentz_t (treatment, program participation, etc.) on a continuous, fully-observed variable,y_t, conditional on the independent variables x_t and w_t.

Suppose z_t^0 is a latent variable.

$$\begin{bmatrix} y_t \\ z_t^0 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_t\beta \\ \mathbf{W}_t\gamma \end{bmatrix} + \delta\begin{bmatrix} z_t \\ 0 \end{bmatrix} + \begin{bmatrix} u_t \\ v_t \end{bmatrix}, \begin{bmatrix} u_t \\ v_t \end{bmatrix} \sim NID(\mathbf{0}, \begin{bmatrix} \sigma^2 & \rho \\ \rho & 1 \end{bmatrix}) \tag{19}$$

We observe y_t and z_t:

$$z_t = 1 \text{ if } z_t^0 > 0;\ z_t = 0 \text{ otherwise.} \tag{20}$$

Notice that the only difference between the switching regression and selection model is that we observe y_t when $z_t = 0$ and $z_t = 1$. It's not a selection, but a regime switching.

Both Stata (treatreg) and SAS (qlim) can estimate the switching regression. The default is by MLE.