Generalized Biform Games

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Abstract

How to extend the use of value-based strategy models to situations with large quasi-rents shared among multiple actors, such as ecosystems? How to consider how players understand competition in value-based models? How to overcome some limitations of these models such as lack of uniqueness of solutions? In this paper, we extend the reach of value-based strategy by revisiting the celebrated biform games model to answer these questions. Operationally, we make players evaluate their payoff from the cooperative stage of the game according to a generalized expectation over their value capture. Our solution has several advantages: (i) It subsumes the original biform framework and seamlessly integrates recent works providing bounds to value capture (ii) It allows solving issues such as the possible non-uniqueness of solutions and invariance to the competitive environment structure while maintaining the role of competition in determining value capture (iii) It remains axiomatically justified on behavioral grounds (iv) It permits richer preferences representations that, for example, can include subjective distortions of objective chances of value capture (v) It further leads the way to the use of generalized preference representations in the value-based framework.

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1 Introduction

The study of the relations between value creation and value capture and their implications on firms' behavior is central to the modern analysis of competitive strategy. These treatments have their theoretical bases in the seminal work of Brandenburger and Stuart (1996; 2007) which structure the problem of value creation and appropriation in an hybrid game theory setup. In particular, Brandenburger and Stuart (2007) propose a procedure for evaluating multistage games which have a non-cooperative stage followed by a cooperative stage (*biform games*). Their procedure prescribes that every player backward inducts his payoff from the cooperative stage and *evaluates it as a convex combination of the extremes of that player's coordinate projection on the core* (Gillies, 1953), the intervals of value capture considered in their work.

Recent work in strategic management considers the use of alternative solution concepts for the cooperative stage of biform games (Gans and Ryall, 2017; Ross, 2018) and their relative advantages and disadvantages with respect to the one proposed by Brandenburger and Stuart (2007). The main critiques to the use of the core boil down to two arguments: on the one hand, the core *may not exist*, frustrating analysis beyond this finding, whereas, on the other hand, when it exists, *it may be too big*, suggesting an uncomfortable indeterminacy.

Dealing with non-existence has been the focus of analysis of a series of papers that studied either more general intervals of value appropriability (MacDonald and Ryall, 2004; 2018) or general conditions for the existence of the core in classes of games of interest to strategy scholars (Chatain and Zemsky, 2007; Stuart, 1997; 2004). Relative to the issue of non-uniqueness of the equilibrium, which is characteristic of the core, Ross (2018) proposes, for multi-player settings, the use of alternative point solutions, most notably the Shapley value (Shapley, 1962; Gale and Shapley; 1953) and the nucleolus (Schmeidler, 1969). Yet, putting aside the computational burden that they entail, these point solutions have features that may be undesirable in a strategic management setting. For instance, the Shapley value may not abide to minimal requirements of rationality in strategic thinking for large classes of games as it may allocate value to some players while others would have the ability to block them from capturing it, whereas the nucleolus can be interpreted as a "pessimistic" point solution.

Another set of issues comes from the approach taken by Brandenburger and Stuart (2007), who provide a point solution to the problem of evaluating value when the core does not provide a unique allocation. This is one of the main virtues of their apparatus but it comes at a cost. Indeed, the use of a convex combination of the extremes of core projections can be problematic since, as already noted by the authors themselves (Brandenburger and Stuart, 2007: Appendix B.ii), it implies that *a player doesn't distinguish between two cores that yield the same projection for that player*. This characteristic of the original biform games setup presents issues in some applied contexts where very different strategic situations can give rise to similar core projections, although intuition suggests that a reasonable decision maker should not be indifferent between these situations. For instance, in business ecosystems, value can be created as the result of complementarities between several types of players, all necessary to value creation.

To fix ideas, let us consider a situation in which three types of players are necessary for value creation (e.g. hardware manufacturers, operating systems, and application software) and that there is only one player of each kind. Each player's added value is equal to the total value created and the projection of the core for each player is an interval between zero and the total value of the game. In the classic Brandenburger and Stuart (2007) framework, each player evaluates the benefit as a combination of the best and worst points in that interval, not necessarily egalitarian. Now, assume that a new hardware manufacturer enters the game, and is a perfect substitute (clone) to the one already in the game. Obviously, value captured by the hardware manufacturers drops to zero due to competition. However, in this new game, the upper and lower bounds for value appropriation of the operating system and the application software remain similar to those in the original game and, according to the classic framework, their envisaged value capture should remain the same. Yet, before entry, the full value had to be split three ways, while after entry, it only has to be split between two players who, intuitively, would probably prefer the second game to the first. This is because while the projection of the core does not change for each of these two players, the geometry of the set of core allocations is different in the second game, and arguably more favorable to them, as it now assigns zero value capture to the manufacturers while initially the unique manufacturer could appropriate a wide range of value. This creates a substantial problem for the usefulness of the classic biform games framework in modeling such situation: it seems impossible to recognize the effect of the time-honored strategy of commoditizing complementers in ecosystems in order to capture more value. Conceptually, this example suggests that it may be useful to base the evaluation of value capture (the *confidence indices* in Brandenburger and Stuart, 2007) on the full set of value capture possibilities across all players rather than just on the projection of these along one single axis.

In this paper we aim at addressing the issues raised above. In particular, we propose a generalization of the biform games of Brandenburger and Stuart (2007) which embeds richer implications of the structure of the competition on the chances of appropriation of value for each player. Operationally, this translates into players evaluating their payoff from the cooperative stage of the game by an expected value of their value capture that takes into account the full structure of the constraints on value capture, in addition to the upper and lower bounds on value capture.

Given the structure of the adopted solution concept (which can be the core, or any other set of constraints on value capture), this evaluation can be decomposed into two fundamental constituents. The first one is objective and is given by the expected value of the interval of value capture, where the probability of each possible value captured by a player is given by the objective chances that such value is allocated to the player, given the structure of the game. In other words, each player, when considering the range of possible values that he can capture when joining a coalition, weighs these values by the relative share of allocations that assign such value to him. The other component instead allows to incorporate subjective distortions of such objective expected value by incorporating subjective weights over the lower and the upper bounds of value appropriation.

To this end, we provide an extended definition of biform games, in which we replace the confidence index of Brandenburger and Stuart (2007) with a more general preference relation over payoffs. Relatedly, our embedding of a generic structure of preferences in the biform game definition leads the way to the use of richer payoff evaluation criteria, as advocated by Gans and Ryall (2017).

Our solution has several advantages. First, it embeds the biform games' structure of

Brandenburger and Stuart (2007) as a special case, and thus preserves all of their results when we restrict to their framework. In addition, our formal structure allows to consider more general sets of outcomes, that do not necessarily coincide with the core. In particular, we can apply our results to the recent developments in the value-based strategy literature that focused on the implications of different strategic requirements on intervals of value capture (MacDonald and Ryall, 2004; Montez, Ruiz-Aliseda, and Ryall, 2017). We argue that this allows to further deal with the case of empty cores. Finally, considering more general outcome sets makes it possible to reconcile the use of a point-selection from a set of constraints on value capture with the use of the Shapley value, under an overarching setup which frames both choices as special cases of a unified evaluation criterion. We also note that different strategic constraints on the solution set can further be interpreted as different limits to strategic cognition — more elaborate cognitive capacities correspond to more requirements and constraints. Specifically, this helps translating explicitly the impact of bounded cognition in strategic reasoning over value creation into different perceived intervals of value capture.

2 Examples

This work takes the hybrid noncooperative-cooperative biform game setup of Brandenburger and Stuart (2007) and adopts a different criterion for players to evaluate their cooperative stage payoffs. In particular, they prescribe that players will evaluate the value of each cooperative stage sub-game by a weighted combination of the maximum and the minimum outcome that a player can obtain in the core of each sub-game. Specifically, they write: $u_i^{BS07} = (1 - \alpha_i)\pi_i^{\min} + \alpha_i \pi_i^{\max}$ where $\pi_i^{\min}$ and $\pi_i^{\max}$ are respectively the lower and upper bounds on value capture implied by the core given a particular strategy profile and α_i is the confidence index such that $\alpha_i \in [0, 1]$.

Our criterion differs in two ways. First, it features an additional element: the expected value that a given player can capture in the cooperative game. This expectation is assessed by the player by considering the relative frequency with which a given allocation is assigned to him, among those possible, conditional on the competitive structure of the cooperative game that is played. Second, it allows for considering different solution concepts other than the core. This formulation, that embeds as a special case the original biform game structure, allows to incorporate the implications of the fact that business strategies shape the competitive environment while leaving unconstrained the free-form competition that is characteristic of the cooperative-stage game. We thus write $u_i = \gamma_i \pi_i^{\min} + (1 - \gamma_i - \delta_i)\mathbb{E}(\pi_i) + \delta_i \pi_i^{\max}$, with $\mathbb{E}(\pi_i)$ the expected value capture over the set of possible allocations across all players, and $\gamma_i + \delta_i \in [0, 1]$ the indices representing respectively the subjective weight given to the most pessimistic outcome (γ_i) and the subjective weight given to the most optimistic outcome (δ_i) while $(1 - \gamma_i - \delta_i)$ is the weight given to the more objective probabilistic outcome. Together, they allow to flexibly represent different levels of overconfidence (high δ_i), underconfidence (high γ_i), while accounting for the full geometry of the set of possible allocations ($\mathbb{E}(\pi_i)$).

The following examples show how the use of a refined evaluation criterion for the cooperative stage leads the way to the consideration of novel behavioral insights together with an embedding of the effects of competition on strategic choices.

Example 1. *"A behavioral twist on Brandenburger and Stuart (2007)"*

Let us consider the biform analysis of the "branded ingredient" strategy in Brandenburger and Stuart (2007). In the game, there are two firms such that each can produce a single unit of a given product. In the economy, there is also only one supplier that can supply input to at most one firm, and the cost of this necessary input is \$1. Finally, there are numerous buyers, each demanding at most one unit of product, from either of the two firms. Buyers all have the same tastes but the value that they attach to the product is firm-specific. This situation describes, for example, the case in which two firms produce the same product but in different qualities. It is assumed that buyers are willing to pay up to \$9 for the product sold by Firm 1 and up to \$3 for the product sold by Firm 2. It is further assumed that the supplier has the option of incurring an upfront cost of \$1 to increase the buyers' willingness to pay for Firm 2's product up to \$7, by branding the product of Firm 2. This situation can be modeled as a biform game in which the supplier is the only player who can move in the (non-cooperative) first-stage of the game and can choose whether or not to incur the upfront cost for branding the product of Firm 2. It is easy to observe that, in both second-stage (cooperative) games, all buyers and Firm 2 have zero added value.

If we focus on the use of the core as a solution concept for our cooperative sub-games, then this prescribes that neither the buyers nor Firm 2 is ever able to appropriate positive value. Instead, in the status-quo sub-game, Firm 1 can appropriate value in a range that goes from a minimum of \$0 up to a maximum of \$6 whereas the supplier can appropriate value in a range that goes from a minimum of \$2 to a maximum of \$8. Whereas, in the branded-ingredient sub-game, Firm 1 can appropriate value in a range that goes from a minimum of \$0 up to a maximum of \$2 whereas the supplier can appropriate value in a range that goes from a minimum of \$5 to a maximum of \$7. Remark that, although the supplier can secure at least \$2 in the status-quo game and 5\$ in the branded ingredient strategy, the relative preference for one strategy or the other is completely determined in Brandenburger and Stuart (2007) by the players' confidence indices and nothing can be said ex-ante. Similar considerations apply to Firm 1. Reasoning in terms of expected appropriation within the core, instead, tells us that, abstracting from confidence considerations, in the status-quo sub-game, the average allocation in the core is \$5 for the supplier and 3\$ for Firm 1, whereas, in the branded strategy sub-game, the supplier obtains on average \$6, whereas Firm 1 is allocated \$1, on average. This suggests another feature of our solution: the observation of deviations from "objective" expected appropriation reasoning can be immediately related with over- or under-confidence considerations. In other words, expected allocation reasoning provides the possibility of isolating an objective component of the players' assessment of the value of a coalition, which is computed by considering the relative frequency with which any give value is allocated to a player, from a subjective component, which embeds subjective distortions of appropriation chances. □

Abstracting from behavioral considerations, the previous example shows that our solution also addresses the critiques related to the set-valued nature of the core solution. In the following examples we show how our approach can be used to solve some issues that may arise using the Brandenburger and Stuart (2007) setup.

Example 2. *"Invariance to the competitive structure: Wide cores are not equivalent"*

We reformulate and elaborate on the example developed in the introduction. Consider a status-quo cooperative game in which there are three players: a supplier of operating system A, a hardware manufacturer B, and a developer of application software C. Together they can create a total value of 3, but no value can be created by any other combination of players. This game has non-empty core and, at the status-quo, each player has the same added value, which is equal to \$3. Hence, the lower bound on value creation for each player is \$0 while the upper bound is \$3. It is immediate to compute the expected allocation vector for all players (i.e. the core-center of the status-quo game) which prescribes that, on average, a value of \$1 is allocated to each player. Therefore, according to expected value reasoning, the operating system provider foresees to appropriate, on average, a value of \$1 in the status quo game.

Let us now assume that the operating system provider A can help another hardware manufacturer D entering the game at no extra cost. This case corresponds to a different coalitional game. Let us assume that this game is such that the total value created does not vary as the hardware clone joins the coalition of all players. Now all coalitions that are composed by the operating system, one or two hardware manufacturer, and the application software developer produce a total value of \$3 while all other coalitions create no value. In the core of this game, the operating system can capture value between \$0 and \$3, the application software developer can capture between \$0 and \$3, and the manufacturers (original and clone) capture \$0, as they are identical and have no added value individually.

From the view point of the operating system provider, the upper and lower bounds on value capture in the new game are identical to those in the status-quo game. In the classic Brandenburger and Stuart (2007) framework, this implies that, for fixed confidence indices, the manufacturer should be indifferent between helping or not helping the clones enter the market. Yet, in the standard biform games' setup no direction is given regarding the determinants of the confidence index. In particular, it does not vary with the shape of the core. The simple but crucial observation in this case is that, although the range of possible values that the manufacturer can capture in this game did not change, there is a change in what can be seen as the manufacturer's objective chances of value appropriation. Indeed, in the new coalitional game, only two players, namely the manufacturer and the buyer, can appropriate positive value since they are the only two players whose added value is non-zero.

This implies that, after the clone enters the market, value will be shared among fewer players. In this case, the operating system's expected value capture increases from \$1 to $\$\frac{3}{2}$, making the action of helping the entrance of additional suppliers in the market preferred to the status-quo, from the manufacturer's point of view, in accordance with intuitive reasoning. Bringing in substitutes to complementors (or *commoditizing*) is a widespread strategy in ecosystems. Yet, the classic framework cannot always capture the rationale for this strategy, as in our example.

This simple example shows how expected allocation considerations help embedding considerations about the competitive structure of the game also in those cases in which they do not imply differences in the intervals of value capture for some players in different coalitional games. We also remark that, in this case, the Shapley value of the 4-players game would give some strictly positive value is allocated to the hardware manufacturers notwithstanding the fact that their added value is null. □

The next example shows how our framework can help dealing with the case of empty cores. While there are large classes of games for which cores are not empty, this possibility can still arise in applications and motivated part of the value-based literature to adopt other solution concepts from cooperative game theory, such as the Shapley value and, more rarely, the nucleolus. Yet, these solution concepts rely on properties of allocations that are not compatible with intuition of market competition.

Example 3. *"The core may not exist"*

Consider the following 3-players coalitional transferable-utility game. All players are such that they cannot create positive value on their own but when they all join together they can create a value of \$10. In addition, the players can create positive value from forming 2-players coalitions as follows: if player 1 and player 2 work together they can create a value of \$7, while if player 2 and 3 work together they can create a value of \$6, finally when player 1 and 3 work together they can create a value of \$8.

This simple coalitional game has an empty core as there is no allocation of value satisfying individual rationality for all coalitions. This feature hinders the application of the biform games solution of Brandenburger and Stuart (2007), which relies on the use of the core.

By relaxing this requirement, we can shift the focus away from the core to a weaker set of constraints that can still capture the effect of competition while not preventing existence. In this case, we can concentrate on the constraints resulting from the weaker *added-value principle* (Brandenburger and Stuart, 1996) which dictates that no player can appropriate more than their added value. In the example, we can easily compute the added value of each of the player to the grand coalition: the added value of player 1 is \$4, that of player 2 is \$2, while that of player 3 is \$3.

In this game, it is immediate to observe that the sum of the added values is lower than the value created from the whole coalition. Each player can recognize that it can claim its entire added value without fearing any push back from other players. The only question is what happens to the value left after each added value has been subtracted from the value the grand coalition can create. This suggests the possibility of the value created on top of the added value provided by all players be redistributed to the players in the coalition. This quantity represents, for all value-capturing players, the upper bound of their interval of value-capture, whereas its lower bound is zero for all players. As we will observe in Section 4, an equal redistribution of this extra value to all value-creating players results into a unique point of intersection of the interval of the value appropriation regions of the players, such that all players expect to receive their own added value together with an egalitarian share of the leftover value created by the grand coalition. In sum, when there is some extra value after all players already captured their own added value, without any assumption on the relative importance of players in value creation, ex-ante a player will expect the extra value be divided equally among the value-creating players. □

We defer a more detailed discussion on the use of different intervals of value capture to Section 3. We conclude this illustrative section by observing that the use of extensions of value-appropriation intervals that relate players' added value to their possibilities of value capture can be seen as a point of continuity with respect to the original biform games' tradition (Brandenburger and Stuart, 2007), which is based on intervals defined by projections of the core on players' axes, and its recent developments (Montez, Ruiz-Aliseda, and Ryall, 2017).

3 Preliminary definitions

Let us start by considering a multi-stage game where the first-stage is a simultaneous-moves game while the second-stage is a cooperative transferable-utility game.

A *simultaneous-moves game* is represented by the tuple

$$\mathbb{G} = \langle I, Y, (A_i)_{i\in I}, g, (v_i)_{i\in I}\rangle$$

where

- $I = \{i, i = 1, ...n\}$ is the set of *players*. For every player $i \in I$, we denote by $-i$ all the other players except i;
- A_i is the nonempty set of *actions* for player $i \in I$. We denote by $a_i \in A_i$ the generic action of player i, hence $A = A_i \times A_{-i}$ denotes the set of available actions and elements of A are arrays $a = (a_i, a_{-i})$;
- $g : \times_{i\in I} A_i \to Y$ is the *consequence function*, which maps action profiles into consequences $g(a) \in Y$ and captures the essence of the rules of the game, beyond the assumption of simultaneous moves;
- $v_i : Y \to \mathbb{R}$ is the *von Neumann-Morgenstern utility function* of player i.

From the consequence function g and the utility function v_i of player i, we obtain a function that assigns to each $a = (a_j)_{j\in I}$ the utility $v_i(g(a))$ for player i of consequence $g(a)$. This function

$$u_i = v_i \circ g : \times_{i\in I} A_i \to \mathbb{R}$$

is called the *payoff function* of player i.

In order to make explicit the dependence of the second stage game upon the action profile chosen at the first stage we introduce a notion of conditional cooperative game.

A *conditional*[1] *transferable utility (TU) cooperative game evaluated in the action profile*

[1]Although this terminology is not used in Brandenburger and Stuart (2007), there is no conceptual difference between the second stage cooperative games considered in their work and those studied here.

a is a pair

$$G|a = \langle I, W_a \rangle$$

where

- $I = \{i : i = 1, ..., n\}$ is a set of players,
- $W_a : 2^I \to \mathbb{R}$ is the characteristic function of the conditional TU game. It is defined as the section at a of the function $W : 2^I \times A \to \mathbb{R}$ that assigns to every coalition $C \subseteq I$ and action profile $a \in A$ its worth $C, a)$. By convention, we set $W(\emptyset, a) = 0$ for all $a \in A$ and we assume that $W(\{i\}, a) = W(\{i\}, a')$ for all $a, a' \in A$.

A conditional TU game $G|a$ is *zero-normalized* if and only if $W_a(\{i\}) = 0$ for each $i \in I$. Whereas, a conditional TU game $G|a$ is *convex* if and only if, for all $S, T \subseteq I$ it holds that $W_a(S) + W_a(T) \leq W_a(S \cup T) + W_a(S \cap T)$.

We are now ready to give the definition of our class of biform games.

Definition 4. A *generalized biform game* is a tuple

$$\mathbf{\Gamma} = \left\langle I, (A_i)_{i \in I}, (\succsim_i^*)_{i \in I}, V \right\rangle$$

where

- $I = \{i : i = 1, ..., n\}$ is a set of players;
- A_i is the nonempty set of possible *actions* for player $i \in I$;
- $\succsim_i^*$ denotes player i's preference relation over (cooperative-stage) outcomes;
- $V : A \to \mathbb{R}^{2^I}$ is the value function of the biform game and assigns, to every action profile $a \in A$ the *value* of every coalition $C \subseteq I$ in the cooperative stage of the game.

The value function of the biform game allows to link the non-cooperative stage with the cooperative stage of the game as follows: for every $C \subseteq I$, $V(a)(C) = W(C, a)$, and therefore $V(a)$ corresponds to W_a, that is the section at a of the characteristic function of the conditional TU game $G|a$.

The value created in the conditional cooperative stage game is distributed among the players. A distribution of value (*allocation*) for the conditional game $G|a \in \mathcal{G}^n$ is a profile $\pi^a = \left(\pi_j^a\right)_{j \in C} \in \mathbb{R}_+^n$ where π_j^a is a real number indicating the amount of *value captured* by agent j in return for his participation in the value-creating activities that contribute to the production of $W_a(C)$, the *value created* by coalition C in the conditional game $G|a$. In particular, $W_a(\{i\})$ denotes the value that player i can create on his own.

The *added value of player i* to a coalition C is defined as

$$av_i(C; a) := W_a(C) - W_a(C \setminus \{i\})$$

that is the difference between the value that the coalition C can create when i belongs to the coalition and the value that the coalition C can create when i does not belong to the coalition, when action profile a is chosen (in the non-cooperative stage).

Depending upon how the value created is distributed among players, allocations are characterized by properties that embed classical Economics considerations. In particular, an allocation π^a is *efficient* (or *feasible* in Gans and Ryall, 2017) if $\sum_{i \in I} \pi_i^a = W_a(I)$, i.e. if all the value created by the full set of players is distributed among the players of the game; π^a is *individually rational* if, for each $i \in I$, $\pi_i^a \geq W_a(\{i\})$, i.e. if each player is allocated at least the value that he could create on his own; whereas π^a is *coalitionally rational* if, for every non-empty coalition $C \subseteq I$, $\sum_{i \in C} \pi_i \geq W_a(\{i\})$.[2] Relatedly, the *imputation set* of $G|a \in \mathcal{G}^n$ is the set of all efficient and individually rational allocations for the conditional game $G|a \in \mathcal{G}^n$, i.e.

$$\mathcal{I}(a) := \left\{ \pi^a \in \mathbb{R}^n : \sum_{i \in I} \pi_i^a = W_a(I), \text{and for each } i \in I, \pi_i^a \geq W_a(\{i\}) \right\}$$

and the elements $\pi^a \in \mathcal{I}(a)$ are called *imputations*. Instead, we will refer to the set of efficient allocations for a conditional game $G|a \in \mathcal{G}^n$ as the *pre-imputation set* of $G|a$. Geometrically, the imputation set of a zero-normalized coalitional n-player game can be represented, w.l.o.g.,

[2]Note that Gans and Ryall (2017) refer to coalitional rationality as *competitive consistency* (or *stability*) condition. Competitive consistency trivially implies individual rationality.

by an $(n-1)$-dimensional standard simplex.[3] Since the techniques we develop here rely on properties of simplices, in what follows we will focus mainly on allocations that are non-negative and efficient. However, as we discuss later on, this fact does not constitutes a limitation to our framework due to the possibility of re-normalizing payoffs in conditional games, as already noted in Brandenburger and Stuart (2007) and Chatain and Zemsky (2011), respectively.

Let us denote by $\chi_i(C; a)$ the *interval of value capture* (or *appropriability interval*) for player $i \in I$, that is, the set of values π_i^a that player i can capture when coalition C is formed and the action profile a has been played in the non-cooperative stage. Further denote by I^+ the set of players such that $\chi_i(C; a) \neq \{0\}$ for all $i \in I^+ \subseteq I$. We have that different strategic requirements have different implications on the definition of $\chi_i(C; a)$.

For example, it is well known that, under regularity conditions,[4] individual rationality and efficiency imply $W_a(\{i\}) \leq \pi_i^a \leq av_i(C; a)$. Hence, minimal strategic requirements suggest the definition of the following candidate interval of value capture:

$$\chi_i^0(I; a) = [W_a(\{i\}), av_i(I; a)]$$

and we refer to it as the *added value capture interval* of player $i \in I$.[5]

Arguably, the most commonly used type of interval of value capture is given by the projections onto the i-th coordinate axis of the core of the conditional cooperative games. We recall that the core of a conditional TU game is the set of imputations that are coalitionally

[3] The m-dimensional standard simplex Δ^m is the convex envelope of the canonical base $e_1, ..., e_{m+1}$ of $\mathbb{R}^{n+1}$. That is, $\Delta^m = \left\{(x_1, ..., x_{m+1}) \in \mathbb{R}^{m+1} : \sum_{i=1}^{m+1} x_i = 1, x_i \geq 0 \, \forall i\right\}$. Note that Δ^n is a regular simplex, i.e. a simplex such that the distance between any two of its vertices is constant.

[4] In particular, let $G|a \in \mathcal{G}^n$ be a zero-normalized conditional TU game such that $av_i(I; a) \geq 0$ for all $i \in I$.

[5] Building on the logic underlying the added value principle, MacDonald and Ryall (2004) further derive for each player $i \in I$ the minimum residual mr_i of player i: a more refined lower bound for players' value capture, which is defined as the difference between the value created by all players and the sum of the added values of all players except i. Formally,

$$mr_i(a) = W_a(I) - \sum_{j \in I \setminus \{i\}} av_j(I \setminus \{i\}, a)$$

and it holds that $mr_i(a) \leq \pi_i \leq av_i(C, a)$. As anticipated in the introduction, we can extend our analysis to the use of these bounds.

rational, that is

$$\mathcal{C}\left(G|a\right)=\left\{\pi^a\in\mathbb{R}^n:\sum_{i\in I}\pi_i^a=W_a\left(I\right),\text{and for every }\emptyset\neq C\subseteq I,\sum_{i\in C}\pi_i^a\geq W_a\left(C\right)\right\}.$$

Then we can denote the resulting *interval of core value capture* by

$$\chi_i^{\infty}(I;a)=\text{proj}_i\left\{\mathcal{C}\left(G|a\right)\right\}.$$

Yet, the core is only one instance of solution concept for cooperative games. Let $\mathcal{G}^n$ denote the set of n-player TU games. A *probabilistic solution concept* ψ is a function which, given a TU game in $\mathcal{G}^n$, selects a probability distribution over $\mathbb{R}^n$, i.e.

$$\begin{aligned}\psi:\mathcal{G}^n &\to\Delta\left(\mathbb{R}^n\right)\\ G &\mapsto\psi(G)\end{aligned}$$

and we denote by $\psi\left(G|a\right)$ the *probabilistic solution set* of the conditional n-players TU game $G|a$. Given a conditional TU game $G|a\in\mathcal{G}^n$, if an allocation $\pi^a\in\mathbb{R}^n$ belongs to the support of $\psi\left(G|a\right)$ then we say that π^a is a *solution* of the conditional game $G|a$ and we denote it by π^{a^*}.

At this point, we make the key conceptual move of remarking that any set of allocations that are solutions of a given TU game in $\mathcal{G}^n$, can be seen as the uniform distribution defined over the set itself. Building on this intuition, we observe that we can identify allocations with n-dimensional vectors of real-valued random variables[6]. Accordingly, we denote by Π_i^a the random allocation of value to player $i\in I$ in the conditional game $G|a\in\mathcal{G}^n$ and by $\Pi^a=(\Pi_1^a,\ldots,\Pi_n^a)$ the corresponding random allocation vector.

Finally, in the present paper, we will work under the assumption that players evaluate their cooperative stage payoffs as follows. Let $\chi_i(I;a)$ be the appropriability interval of

[6] We recall that, given a probability space (Ω,Σ,P) and the measurable space $(\mathbb{R},\mathcal{B})$, a *real-valued random variable* X is a measurable function $X:\Omega\to\mathbb{R}$.

player $i \in I$ in the conditional game $G|a$. Then

$$u_i(\chi_i(I;a)) := \gamma_i \min \ \chi_i(I;a) + (1 - \gamma_i - \delta_i)\, \mathbb{E}\left[\chi_i(I;a)\right] + \delta_i \max \ \chi_i(I;a) \qquad (3.1)$$

where γ_i and δ_i are two non-negative real numbers such that $\gamma_i + \delta_i \in [0,1)$, $\mathbb{E}\left[\chi_i(I;a)\right]$ denotes[7] the expected value of the allocation to player $i \in I$ within the interval of value capture $\chi_i(I;a)$, where expectation is taken with respect to the objective chances of appropriation of each conceivable value to player i and $\min \chi_i(I;a)$ and $\max \chi_i(I;a)$ denote the minimum and the maximum of the interval of value capture $\chi_i\,(I;a)$.
Whenever the player evaluates $\chi_i(I;a)$ according to (3.1) we say that it does so by considering its *generalized expected appropriation*. It is immediate to observe how this representation includes as a limit case the one of Brandenburger and Stuart (2007).

In other words, in the present work, we assume that preferences over cooperative-stage outcomes are represented by the Non-Extreme Outcomes Expected Utility criterion of Webb and Zank (2011). We remark that, although we do not pursue this direction here, our definition of generalized biform games further allows for the embedding of more general preference structures on the set of outcomes of the cooperative stage game. In this context, it is important to note that both this criterion and the one used by Brandenburger and Stuart (2007) can be seen as special cases of preference representations.[8] In fact, the latter can be seen as the requirement that $\succsim^*$ be represented by the Hurwicz (1951) criterion[9] over the core outcomes, that is, by a convex combination of the extremes of the projections of the core, for every player.

Our choice thus generalizes their evaluation criterion in at least two directions: we con-

[7] We use this notation instead of the more precise $\mathbb{E}\left[\Pi_i^a|\Pi^a\right]$ with $\Pi^a\,(\omega) \in R$, for all $\omega \in \Omega$, where R is a generic set of restrictions on players' allocations that embeds the assumptions for the problem under analysis. Although improper, we choose this notation for simplicity and ease of interpretation.

[8] In mathematics, a representation theorem is a theorem that states that every abstract structure with certain properties is isomorphic to another (abstract or concrete) structure. One of the cores of decision theory concerns the representation of preference relations $\succsim$ and studies behavioral conditions for observable preference relations to be equivalent to decision makers choosing as if they are maximizing functionals such as, for example, the one in (3.1).

[9] Hurwicz's (1951) criterion evaluates future outcomes under uncertainty by giving a weight to the worst-case scenario and the complement to the best-case scenario.

sider all outcomes in a value capture interval together with their chances of appropriation and we allow for cooperative-stage solution sets that do not necessarily coincide with the core. In fact, at this level of generality, our framework is silent with respect to the solution concept to be adopted in the cooperative stage of the biform game. This allows the present setup be compatible also with the use of recent developments in the theory of value-based strategy, such as the refinements of intervals of value capture based on different notions of competitive intensity in Montez, Ruiz-Aliseda, and Ryall (2017).

Relatedly, we note that our approach allows to exogenously impose intervals of value capture and derive the restrictions on expected allocations that they imply. This approach can be seen as the converse of the more classical procedure of selecting a solution concept and obtain the implied intervals of value capture that are compatible with the requirements on allocations that characterize the chosen solution concept. Both approaches are compatible with our setup, conditional on maintaining the assumption that allocations belong to the imputation set of the conditional games.

4 Expected allocations

We therefore depart from Brandenburger and Stuart (2007) and assume that each player evaluates his appropriability range by his *(generalized) expected appropriation*, instead of a convex combination of the extremes of the value capture range coinciding, for every player, with the player's coordinate projection of the core. This matters to compute the utility of the players, since it now depends on the entire shape of the solution set, thanks to the use of expected allocations, and thus reflects more fully how competition restricts value capture in the game. Moreover, this criterion allows to neatly separate the objective assessment of chances of value capture, which are represented by the expected value term, from the subjective assessment of these chances, which are reflected by the evaluation weights that players place on the extreme values of their value capture range.

The expected value of the value capture range is computed, for every player, by weighting each outcome in the range by the relative proportion of allocations that are solutions of the conditional game which assign that outcome to the focal player. We label this expectation as *objective* because the expectation term is computed by setting probabilities as relative frequencies of value capture. Given our construction, the difference between the objective expected value capture and the generalized expected appropriation of each player uniquely determines the subjective weights placed on the extremes. This feature of our modeling solution allows to therefore interpret the weights assigned to the extremes values of the interval of value capture as overconfidence indices.

In addition, and more importantly, the consideration of an expected allocation to each player, computed by taking into account objective chances of value capture within the coalition that can appropriate positive value, allows to take into account richer considerations on the structure of the competition in the outcomes of the game. In particular, a variation in the number of value capturing players in a coalition will induce a variation in the chances of appropriation of value of the players in that coalition. This will imply that, coherently with the intuition, a player may not be indifferent between two structures of competition, even when they induce the same interval of value capture for that player.

4.1 Centroids, Core-Center and the Shapley value

In this subsection, we will focus on the interpretation and computation of $\mathbb{E}\left[\chi_i(I;a)\right]$, the expected value of the appropriability interval of player $i \in I$.

As already noted, in what follows we will focus on non-negative efficient allocations. This requirement allows to carry out our analysis of allocation of value of n-players conditional TU games in regular simplices with side length equal to the total value created by the grand coalition of players in the conditional game. Given the observation that we can see the coordinates of the $(n-1)$-dimensional simplex as n random variables, the question of computing $\mathbb{E}\left[\chi_i(I;a)\right]$ is the same as that of computing the expected value of the i-th coordinate of the solution set in $\mathbb{R}^n$, which coincides with the i-th coordinate of its centroid in $\mathbb{R}^n$.[10] Operationally, as we will observe next, computing expected allocations of value to

[10] Recall that the *expected* or *mean value* of a continuous random vector X with joint PDF f_X is the *centroid* of the probability density, i.e.

$$\mathbb{E}\left[X\right] = \int_{-\infty}^{\infty} x f_X(x) dx$$

players is particularly immediate when the solution set reduces to special cases of regular polytopes. In the general case, in order to take into account arbitrary bounds to value capture, we must consider the subset of Δ^{n-1} generated by the restrictions corresponding to the solution concept selected.[11]

In order to compute the above expectation we need to endow our outcomes' set with a probability distribution. In this paper, we consider the use of a (continuous) uniform distribution over regular simplices, although other choices are possible. This initial assumption formalizes the idea that, ex-ante, all allocations in the simplex are equally possible. Given our hypotheses, the coordinates of the simplex are jointly distributed according to a Dirichlet with dispersion parameter equal to 1 for all $i \in I$. The specific choice of solution concept, in turn, induces different probabilities over the set of allocations of value created.

If we consider the set of all coalitionally rational and efficient allocations, i.e. the core, then the expected allocation of value to player i corresponds to the i-th coordinate of the centroid of the core, or *core-center*, whose properties have been studied in González-Díaz and Sánchez-Rodríguez (2003a). A general feature of centroids is the fact that they inherit all properties of the set to which they belong, in addition to properties of balancedness (or fairness) that descend directly from the definition of centroids. This makes the centroid of a set-valued solution a good candidate for a point-solution with given desirable properties.

Expected allocations: the simplicial case

Let us start our analysis of expected allocations by investigating the case of solution sets of conditional n-player games that can be represented by simplices in $\mathbb{R}^n$, while we will address the case of general solution sets next. When the solution set is a simplex, computing its centroid, whose coordinates correspond to the expected allocation for each player, is immediate. In fact, in this case, the centroid is found by simply averaging the vertices' coordinates of the simplex. Formally, let $v_1, \ldots, v_m$ denote the vertices of an $(m-1)$-simplex,

In other words, the centroid is the center of gravity of the distribution of X.

[11]Note that, for any $C \subset I$, the subset of the simplex generated by the restrictions in R_C is a convex polytope, yet not necessarily a simplex.

where each v_j, with $j = 1, \ldots, m$ is an m-dimensional vector in $\mathbb{R}^m$. Then, the centroid C is

$$C = \frac{1}{m} \sum_{j=1}^{m} v_j$$

The next example illustrates the case when each player in the coalition of all players can appropriate value in the whole range between their reservation value and the total value created by the coalition.

Example 5. Let us start, for illustration, by considering the imputation set of a conditional TU 3-players game such that $W_a(\{i\}) = 0$ and $av_i(I) = W_a(I) = 1$ for all $i \in I$. This set coincides with the standard 2-dimensional simplex. Δ^2. Note that, if we request no additional condition other than individual rationality and efficiency, in this case, we obtain that, for every player $i \in I$, the interval of value appropriability χ_i^0 is the interval $[0, 1]$, that is, all allocations within the simplex Δ^2 are possible. Let us further assume that, given this condition, no allocation is objectively more likely than the other. This translates into the fact that our (probabilistic) solution set here coincides with a uniform distribution over Δ^2. Here, since, by hypothesis, mass is uniformly distributed over the whole simplex, the expected allocation of value to the players coincides with the centroid of the triangle in Figure 4.1.

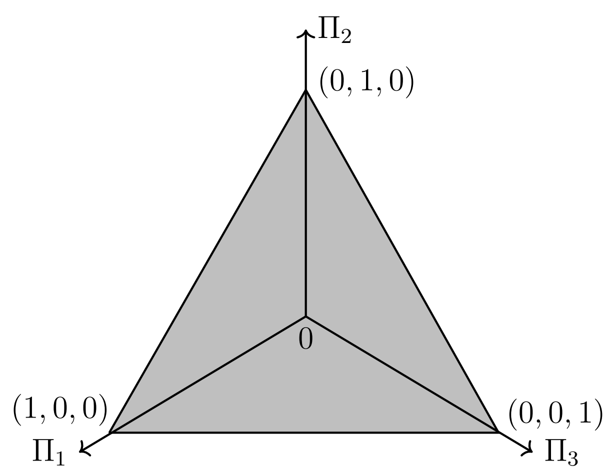

Figure 4.1: The 2-dimensional standard simplex Δ^2

It is immediate to compute that $\mathbb{E}(\chi_i(I; a)) = \frac{1}{3}$ for all $i \in I$. Therefore, in this simple case, where all players can create no value on their own, yet, when coalescing, have the potential of capturing all value created, we have that the expected value of the appropriability interval for each player is equivalent to a fair distribution of the value created. □

Besides the simple case studied in Example 5, there are other relevant cases where restrictions on the allocations' set induce a solution set that is a simplex. Other particularly relevant cases arises when these restrictions imply the solution set be a point, whose coordinates sum up to the total value created in the conditional game. The next proposition, whose proof is trivial and therefore omitted, shows that, when the value created is equal to the sum of the lower bounds of the interval of appropriation of all players, which, under the minimal assumption of individual rationality of allocations, is at least equal to the value that each player can create on his own, then the unique allocation that is a solution of the conditional game is the one for which all players are *at best* indifferent between joining and not joining a coalition with other players.

Proposition 6. *Let us consider a conditional TU game $G|a$, whose @@non-empty@@ solution set is a subset of the imputation set $\mathcal{I}(a)$.*

1. *When the value created in the conditional game is equal to the sum of the upper bounds of the intervals of appropriation of all players, then the expected outcome of the conditional game is the allocation vector that assigns to each player the upper bound of its own interval of appropriation.*

2. *Conversely, when the value created in the conditional game is equal to the sum of the lower bounds of the intervals of appropriation of all players, then the expected outcome of the conditional game is the allocation vector that assigns to each player the lower bound of its own interval of appropriation.*

This result thus generalizes the *adding-up* property of Brandenburger and Stuart (2007) in two directions. On the one hand, we find that the result applies to upper bounds of the appropriation intervals that may differ from added value. On the other hand, Proposition 6 tells us that the adding-up property is true when considering either of the two bounds on players' allocations.

Expected allocations: the general case

Now we move on to the general case where the solution set can be represented by a convex subset of a regular simplex in $\mathbb{R}^n$. This case allows to embed into our analysis the idea that different strategic requirements on allocations imply different restrictions on the set of feasible solutions of a conditional game. In this case, the expected value of the allocation vector π^a is still a centroid, but of the constrained region defined by the restrictions implied by the solution concept on the set of allocations of the conditional game $G|a$.

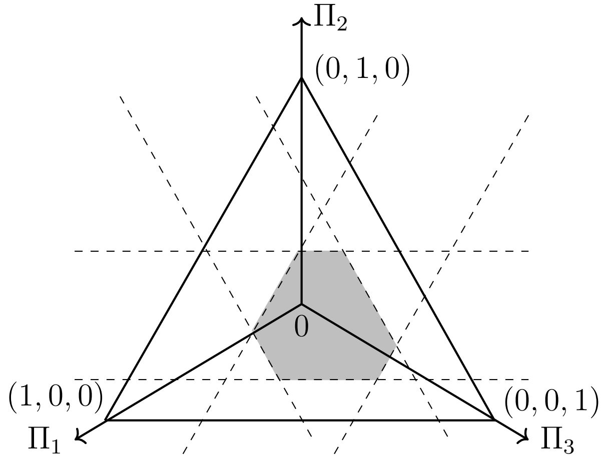

Figure 4.2: A restricted region R in Δ^2

Once the constrained region R is identified, we can obtain its centroid by computing the conditional expected value of the vector coordinates over the constrained region. Formally,

$$\mathbb{E}\left[\Pi_i|\Pi \in R\right] = \frac{\int_R \pi_i f_\Pi(\pi) d\pi}{\int_R f_\Pi(\pi) d\pi} \mathbb{I}_{\left\{\sum_{i=1}^n \pi_i = 1\right\}}(\pi)$$

Geometrically, R is a convex polytope in $\mathbb{R}^n$ given by the finite intersection of the half-spaces generated by the restrictions defined by the solution concept selected, on the pre-imputation set of the conditional game $G|a$. The vertices of this convex polytope can be easily computed by using the McLean and Anderson (1966) algorithm for calculating the coordinates of the extreme vertices of a constrained region (see Appendix). Hence, the expected allocation vector for the conditional game $G|a$ can be alternatively be computed through geometric methods.[12]

While seemingly abstract, the framework hereby developed is deeply rooted in value-based business strategy. In fact, the richer structure imposed on the evaluation of the solution set allows to describe market actors reactions to the full competitive environment where they are embedded into. This view helps reconciling observed behavior in, for example, ecosystem creation (Chatain and Plaksenkova, 2020) that is incompatible with the standard value-based business strategy based on the original biform games' setup.

In addition, our framework allows to smoothly answer questions related to the elasticity of value capture to the competitive environment. In other words, in applied models of value-based business strategy, one central question is that of understanding how value appropriation changes as a parameter of the characteristic function of the game changes. It is immediate to observe that this type of questions can be effectively answered by studying the behavior of the expected value capture of players. Along this line, the next result is, perhaps, the most surprising within the present work. In particular, Proposition 7 shows that there are decreasing marginal relative returns from value creation and that this property is invariant to the number of incumbent value capturing players. That is, the relative share of value added that a player can expect to appropriate is maximal when the value added is smallest. In applications, this property is of fundamental importance in understanding the incentive structure of bottleneck players in ecosystems (Chatain and Plaksenkova, 2020).

Proposition 7. *Let $I = \{1, ..., n\}$, with $n \in \mathbb{N}$ such that $n > 2$, denote the set of players with $i \in I$ denoting the i-th player and $-i$ denote all players $j \in I \setminus \{i\}$, that is all players except the i-th player. Let Π_i further denote the value captured by player i and be such that $a_i \leq \Pi_i \leq b_i$ for all $i \in I$. The set of restriction on players' values appropriation is $R = \times_{i \in I} [a_i, b_i]$ and $\Pi = (\Pi_i)_{i \in I}$ is the (random) vector of value capture for all players in I, with generic realization $\pi = (\pi_i)_{i \in I}$. Let us assume that $a_{-i} = a_i = 0$ and $b_{-i} = 1$. Then, for player i, it holds that*

$$\frac{\mathbb{E}(\Pi_i | \Pi \in R)}{b_i} \longrightarrow \frac{1}{n} \quad as \quad b_i \longrightarrow 1$$

[12]For procedural details, see, for example, Kaiser and Morin (1993).

and

$$\frac{\mathbb{E}\left(\Pi_i | \Pi \in R\right)}{b_i} \longrightarrow \frac{1}{2} \quad as \quad b_i \longrightarrow 0$$

We further remark that the technique used to prove Proposition 7 provides an alternative way of computing the centroid of a solution set, when the latter can be described as a difference between two regular simplices.

Expected allocations under bounded cognition

Solution sets, besides resulting from the application of conditions imposed on allocations of value to players, implicitly incorporate requirements about the players' knowledge of the competitive environment. For example, in strategic behavior settings, positing that solution allocations belong to the core is implicitly equivalent to requesting all players be able to know or anticipate the value created by all possible sub-coalitions in the competitive arena. While this requirement, adhering to the classic Economic tradition, represents a useful benchmark, it may ascribe an unrealistic degree of cognition to players. In particular, it may be the case that some players are not aware of the value created by some of the possible sub-coalitions of players in the game. Solution sets that incorporate these requirements for allocations of value to players will result in super-sets of the core. The analysis hereby developed extends smoothly also to these cases. Although we do not explore this direction in detail here, we remark that incorporating epistemic considerations in biform games, both at the cooperative and non-cooperative stage(s), would represent a meaningful advancement of the present work (Aumann and Brandenburger, 1995: Battigalli and Siniscalchi, 1999; Menon, 2018).

Expected allocations when the core is empty

As previously observed, it is well known that the core of a (conditional) cooperative game does not always exist. This contingency usually compels researchers in value-based business strategy to consider solution concepts that differ from the core (Ross, 2018) . One such possibility, which we will discuss next in relation to the approach developed in the present paper, is to resort to the Shapley value. However, other alternatives are available, that can still be studied within our setup. For example, when the sum of the added value of the players is lower than the total value created by the grand coalition, as in Example 3, a possibility in the direction of attaining efficiency,[13] inspired by the concept of ε-core (Shapley and Shubik, 1966), is to consider a minimal extension to the added value capture intervals that, for each player $i \in I^+$, uses as upper bound $av_i(I; a) + \varepsilon$, where

$$\varepsilon = \frac{W_a\,(I) - \sum_{i \in I} av_i(I; a)}{I^+}.$$

These intervals of value capture incorporate the idea that, in absence of further considerations upon relative bargaining power of the players, all value appropriating players can expect to appropriate, on top of their added value, an egalitarian share of the extra value created, by symmetry considerations.

Expected allocations over Weber sets and the Shapley value

Arguably, whenever the core of a game is empty, the Shapley value is the first alternative solution concept that comes to mind in that this quantity always exists (Ross, 2018). Interestingly, our centroid-based approach to the analysis of biform games reconciles the use of these different solutions. In fact, if we consider the set of marginal vectors of a cooperative TU game, that is, the *Weber set*, then its centroid coincides with the Shapley value of the game (González-Díaz and Sánchez-Rodríguez, 2003b). This result further suggests an additional way to easily compute the centroid of a core, for all those games in which the core and the Weber set coincide, that is, for the class of convex games (Peters, 2015 : Theorem 18.6).[14] More importantly, this perspective allows to add a further strategic justification for the use of the Shapley value, whose characterization is strongly rooted in fairness considerations as opposed to competition dynamics that are instead dominant in the case of the core, in wide classes of games.

[13]We remark that allocations within the regular simplex with side length equal to the value created by the grand coalition are always efficient. As already suggested in Brandenburger and Stuart (2007) in the context of incorporating costs in value creation chains, the side length of the regular simplex describing value created in a conditional game can be renormalized to account for losses in efficiency, as in Chatain and Zemsky (2011).

[14]In general it holds that, for any cooperative TU game, the core is always a subset of the Weber set.

5 Concluding remarks

We provided an extension of the seminal biform games setup of Brandenburger and Stuart (2007) that, while maintaining all of its desirable features, such as the absence of constraints upon the analysis of competition in the coalitional stage, further allows to overcome some of the limitations of the original biform setup. We remark that one main innovation introduced in this paper regards the possibility of adopting a more general representation of preference relations over cooperative-stage payoffs. Hence, our paper complements the work in value based business strategy that focuses on verifying the impact of imposing richer assumptions on the relation between players (Bryan et al, 2019; Chatain and Zemsky, 2011; Ryall and Sorenson, 2007). Here, by adding considerations based on expected value reasoning, we answer to the often raised criticism related to the possible indeterminacy of the solution when the core is used as solution concept. The use of the centroid of the set-valued solution allows to select a point-solution that maintains all desirable properties of the chosen set-valued solution concept, together with additional properties of balancedness. Differently from the use of the Shapley value as point-solution concept, our proposed evaluation criterion can be justified by competitive dynamics instead of fairness concerns. Furthermore, as we have observed both though theory and examples, our solution further addresses issues related with invariance to the competitive structure of the game, which can arise when the seminal biform game setup is used. Our approach can be seen as a genuine generalization of the Brandenburger and Stuart (2007) which arises as a special case of ours, when the core is considered as solution concept and players in their evaluation only focus on extremes outcomes instead of adding expected value considerations. This latter criterion, like the one adopted in the seminal biform games' approach, is still axiomatically justified by behavioral assumptions on the players' preferences as it coincides with a case of the Non-Extreme Outcomes Expected Utility criterion of Webb and Zank (2011). In addition, the possibility of isolating an objective expectation component from the evaluation criterion allows a more transparent analysis of confidence considerations, which represent can be identified as a subjective distortion of appropriation chances on behalf of the players. We further note that our framework does not necessarily prescribe the use of the core as a solution concept. This feature allows, on the one hand, to make use of the recent developments in terms of intervals of value capture that have been studied in the value-based strategy literature. On the other hand, it leads the way to a structured analysis of the effects of different limits to strategic cognition, which has been recently advocated for in the competitive strategy literature (Menon, 2018). Finally, although in the present work we suggest the use of a specific evaluation criterion for cooperative-stage payoffs, the framework that we outlined allows for the use of more general evaluation criteria, through the embedding of a general preference relation within the biform structure. Different behavioral requirements can be imposed upon the preference relation, leading the way to the use of further behavioral considerations, such as attitudes towards risk and ambiguity, as advocated for in Gans and Ryall (2017).

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Appendix A: The McLean and Anderson (1966) algorithm

In this appendix we briefly describe McLean and Anderson's (1966) extreme-vertices algorithm for generating the coordinates of the extreme vertices of a constrained region of a $(q-1)$-dimensional simplex defined by the following upper and lower bounds

$$0 \leq L_i \leq x_i \leq U_i \leq 1 \text{ and } \sum_i^q x_i = 1$$

This algorithm constitutes of the following two-step procedure, as reported in Cornell (2011: 123):

Step 1. List all possible combinations (as in a two-level factorial arrangement) of the values of L_i and U_i for $q-1$ components and leave the value of the remaining component blank. This procedure produces 2^{q-1} combinations. With three components, for example, where L_1, L_2, L_3 are the lower bounds and U_1, U_2, and U_3 are the upper bounds, the component level combinations are $(L_1, L_2, \cdot)$, $(L_1, U_2, \cdot)$, $(U_1, L_2, \cdot)$, $(U_1, U_2, \cdot)$, where the proportion for component 3 is left blank. The procedure is repeated q times, allowing each component to be the one whose level is left blank and is therefore to be computed. This list will consist of $q(2q-1)$ possible combinations.

Step 2. Go through all possible combinations generated in step 1 and fill in those blanks that are admissible; that is, fill in the level (necessarily falling within the constraints of the missing factor) that will make the total of the levels for that treatment combination sums to one. Each admissible combination of the levels of all q components defines an extreme vertex.

Appendix B: Proof of Proposition 7

Let N be a natural number greater than 2. The $(N-1)$-dimensional unit simplex $\Delta^{(N-1)}$, corresponding to the case where $a_j = 0$ and $b_j = 1$ for all $j \in I$ has area $V^1 = \frac{\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}$ and centroid with coordinates $C^1 = \left(\frac{1}{N}\right)_{i=1}^{N}$ in $\mathbb{R}^N$. Instead the polytope corresponding to the complement in $\Delta^{(N-1)}$ of the restriction of $\Delta^{(N-1)}$ generated by adding the constraint $1 \geq b_i > 0$ is a regular $(N-1)$-simplex with side length $(1-b_i)$ and thus area

$$V^2 = \frac{(1-b_i)^{(N-1)}\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}$$

and centroid coordinates

$$C_i^2 = \frac{1}{N}\left(1 + (N-1)b_i\right)$$
$$C_{-i}^2 = \frac{1}{N}\left(1 - b_i\right)$$

Therefore, the centroid of the restricted region is given by

$$C_i^* = \frac{V^1 C_i^1 - V^2 C_i^2}{V^1 - V^2} = \frac{\frac{\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}\frac{1}{N} - \frac{(1-b_i)^{(N-1)}\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}\frac{1}{N}\left(1 + (N-1)b_i\right)}{\frac{\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}} - \frac{(1-b_i)^{(N-1)}\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}}$$

and

$$C_{-i}^* = \frac{V^1 C_{-i}^1 - V^2 C_{-i}^2}{V^1 - V^2} = \frac{\frac{\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}\frac{1}{N} - \frac{(1-b_i)^{(N-1)}\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}\frac{1}{N}\left(1 - b_i\right)}{\frac{\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}} - \frac{(1-b_i)^{(N-1)}\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}}$$

Finally, since $C_i^* = \mathbb{E}\left(\Pi_i | \Pi \in R\right)$, we compute

$$\lim_{b_i \to 1} \frac{\mathbb{E}\left(\Pi_i | \Pi \in R\right)}{b_i} = \frac{1}{N}$$

and

$$\lim_{b_i \to 0^+} \frac{\mathbb{E}\left(\Pi_i | \Pi \in R\right)}{b_i} = \frac{1}{2}$$

■