Contractual Restrictions and Debt Traps*

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Abstract

Microcredit and other forms of small-scale finance have failed to catalyze entrepreneurship in developing countries. In these credit markets, borrowers and lenders often bargain over not only the division of surplus but also contractual flexibility. We build a dynamic model of informal lending and show these lending relationships may lead to endogenous poverty traps for poor borrowers if future income is not pledgeable, yet richer borrowers unambiguously benefit. Improving the bargaining position of rich borrowers can harm poor borrowers, as the lender tightens restrictions and prevents them from growing. The theory rationalizes the low average impact and low demand of microfinance despite its high impact on larger businesses.

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Capital constraints pose a substantial obstacle to small-scale entrepreneurship in the developing world. Experimental evidence from "cash drop" studies paints a remarkably consistent picture: across a broad range of contexts including Mexico, Sri Lanka, Ghana, and India, small-scale entrepreneurs enjoy a *monthly* return to capital in the range of 5% – 10%.[1] Surprisingly, however, many of the entrepreneurs in these studies also had access to credit, from microfinance institutions (MFIs), moneylenders, and a variety of other informal lenders, yet they did not reach their productive capacity until an experimenter gave them cash. Moreover, many experimental evaluations of microfinance find that it has only modest impacts on entrepreneurship.[2] This may be especially puzzling in light of the fact that the interest rates charged for microloans are well below the estimates of marginal return to capital. Why, then, can't small-scale entrepreneurs use available credit to pursue their profitable investment opportunities?

We address this puzzle through a theory that highlights that a profit maximizing lender may offer loan contracts that inhibit borrowers' business growth. Theories that attribute poor borrower outcomes to a misalignment of lender incentives have deep roots in the literature on financial frictions in developing economies (see e.g., Bhaduri (1973), and Bhaduri (1977)), and even cursory searches of popular media unearth numerous such accusations of MFIs and other informal lenders.[3] However this story has fallen out of favor in the recent academic literature, especially about microfinance. In large part, this departure can be traced to early influential work by Braverman and Srinivasan (1981) and Braverman and Stiglitz (1982), who argued that profit-maximizing lenders would always encourage efficient investments in exchange for higher interest rates.

We provide a counterpoint to the above logic by building a dynamic model of informal lending, in which the lender is a principal who cannot extract the benefits of the most efficient investments (e.g. Becker (1962), Tirole (2010)). By embedding this misalignment of incentives in a model faithful to critical features of lending relationships in the developing world, we derive predictions consistent with several stylized facts about microfinance. And by studying a dynamic framework we identify novel economic forces not present in a static environment.

Specifically we highlight two special features of lending relationships in the developing world. First, borrowers cannot credibly pledge the benefits that arise from investment in their businesses. The primary metaphor we will use is that richer borrowers exit the informal lending sector, either because they gain access to cheaper, more formal credit or because they reach self-sufficiency. These borrowers may fully repay their debt, but because they have graduated from the informal sector, their lender is no longer able to extract rents from their relationship. Second, lenders have access to *contractual restrictions* that inhibit borrowers' ability to invest their loans productively. The first

[1] See e.g., De Mel et al. (2008), Fafchamps et al. (2014), Hussam et al. (2017), and McKenzie and Woodruff (2008).

[2] See Banerjee et al. (2015) and Meager (2017) for overviews of the experimental evaluations of microfinance.

[3] See e.g. Flintoff (2010), Melik (2010), Chaudhury and Swamy (2012).

feature reopens the possibility that a profit-maximizing lender would want to stymie his borrower's growth, and the second feature enables him to act on that desire.

While in our model, contractual restrictions are an abstract means for the lender to control the borrower's investment, there are many real-world examples of contractual restrictions that govern the ease with which the borrower can invest her loan to grow her business. For instance, MFIs often impose rigid repayment schedules, requiring that borrowers maintain cash on hand and discouraging long-term investment. Field et al. (2013) demonstrates that by relaxing this restriction, borrowers were able to invest 80% more in their business and, three years later, earned 41% higher profits. Similarly, Barboni and Agarwal (2018) finds that borrowers who benefitted from a similarly relaxed loan contract earned 20% higher sales. Moneylenders commonly require borrowers to work on their land (tying up labor) or to forfeit their own land for the money lender's use (tying up capital).[4] Additionally, both moneylenders and microfinance commonly use guarantors or joint liability. A variety of theory and empirical evidence suggest that guarantors pressure borrowers to eschew profitable but risky investments.[5]

These restrictive features are often attributed to ensuring loan repayment, but, with the exception of a rigid repayment schedule, there is little conclusive evidence that these contractual provisions actually reduce default.[6] Further, we argue that even in the case of repayment rigidity the story may not be clear. Field et al. (2013) reports that on average, borrowers who received a flexible repayment contract defaulted on an extra Rs. 150 per loan. However three years later, these same borrowers earned on average an additional Rs. 450 to Rs. 900 *every week.* If borrowers were able to commit to share these benefits (many of which accrue far in the future), then surely lenders would find it profitable to offer flexible loans. Barboni and Agarwal (2018) presents additional evidence that allowing borrowers to undertake long-term investments reduces the rents their lender can extract, finding that borrowers who receive contracts allowing for flexible repayment are significantly less likely to renew their loans.

Taken together, these findings paint a clear picture. To the extent that MFIs only generate profit from active borrowers, lenders do not wholly benefit from their borrowers' business growth, and therefore whatever costs are imposed on lenders by repayment flexibility are not offset by the much larger gains accruing to borrowers.

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[4]See e.g., Sainath (1996).

[5]See Banerjee et al. (1994), Fischer (2013), Jack et al. (2016), and Maitra et al. (2017).

[6]Gine and Karlan (2014) and Attanasio et al. (2015) each provide experimental evidence that joint liability does not affect the likelihood of default. In contrast, using non-experimental variation, Carpena et al. (2013) finds that joint liability loans are more likely to be repaid.

Model Description

Motivated by these facts, we build a dynamic model of a borrower-lender relationship that captures both the borrower's inability to pledge long-term investments and the lender's ability to inhibit such investments. Specifically, we study a dynamic principal-agent model in continuous time. The principal is an MFI or informal lender, and the agent is a borrower. For concreteness, suppose the borrower is a fruit vendor who operates a mobile cart. The borrower has access to two investment projects: a working capital project (e.g., buying fruits to sell throughout the week), and a fixed capital project (e.g., building a permanent stall to expand her business). The former option generates an immediate payoff, while the latter involves substantial upfront investment, requiring her to forgo immediate consumption but potentially expanding her future business capacity. Following the stochastic games literature, we model the borrower's business expansion through a discrete state space with stochastic transition rate that depends on investment size. For example, in the first state she operates a mobile cart, when she grows to the second state she operates a fixed stall, in the third state she owns several stalls, etc. If she reaches the final state, the game ends, with the borrower receiving a high continuation payoff (resulting from reaching the formal sector or operating at self-sufficiency) and the lender receiving nothing (he has lost his customer). Crucially, beyond committing to repay her debts, the borrower cannot pledge the benefits she accrues after severing ties with her lender.

At each instant, the lender offers to augment the borrower's budget with a loan that specifies both an interest rate and a *contractual restriction*. Specifically, as an abstract representation of the restrictions discussed above, we assume the lender can prevent the borrower from investing in fixed capital if she accepts the loan. We refer to the loan as *restrictive* if it specifies the borrower must invest in working capital, and we refer to the loan as *unrestrictive* if it allows her to invest freely. Accepting a loan is voluntary, so if the borrower does not find the loan's size and interest rate sufficiently attractive to offset the additional contractual restrictions, she can reject it and flexibly allocate her own smaller budget.

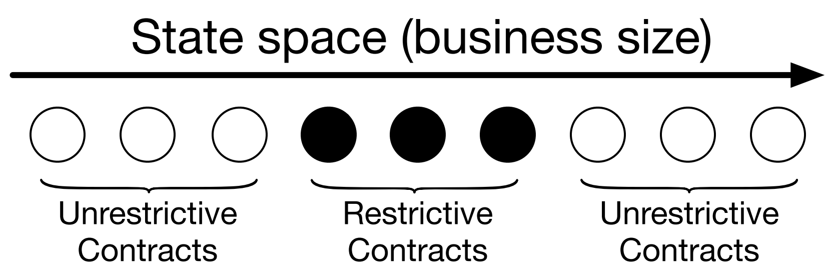
We solve for this game's stationary Markov perfect equilibrium (MPE). In doing so, we assume that both players use strategies that condition only on the borrower's business size. By employing this solution concept we underscore both players' lack of commitment power and preclude the usage of long-term contracts.

In our model the borrower never defaults, but she is exogenously prohibited from committing to long-term contracts that allow her to share the profits after she has terminated the relationship. Thus rather than ensuring repayment, the lender's problem is one of maximizing the rents he extracts from the relationship before she graduates.

Summary of Results

The voluntary nature of the loan contract induces an important tradeoff. For a restrictive loan to surpass the borrower's outside option (i.e. investing her small endowment flexibly), she must be compensated along other dimensions—we focus on the interest rate. The lender's tradeoff informs both when and why we observe contractual restrictions.

Our analysis highlights an asymmetry between restrictive and unrestrictive loans. If the lender is unable to set an interest rate that leaves the borrower with exactly the level of output she would have had in the lender's absence (e.g. due to a running away constraint or asymmetric information about productivity), the borrower will retain more utility from unrestrictive loans in equilibrium. We refer to this asymmetry as the borrower's *expansion rent.* It arises because the borrower cannot commit to sharing the proceeds of business growth and therefore values the investment of her residual income into business growth more highly than alternative (pledgeable) investments. If this asymmetry is severe, the lender only offers restrictive loans, and the borrower remains in poverty. We refer to this phenomenon as a poverty trap because borrowers below a certain wealth level never reach their efficient size, and we also interpret this as a debt trap, because borrowing continues in perpetuity. We show that this trap occurs if and only if the additional surplus the borrower gains from unrestrictive contracts exceeds the additional social welfare generated by business growth. Importantly, the borrower may get stuck in a debt trap even if she would have grown to her efficient size in the absence of a lender. That is, the introduction of a lender may decrease business growth relative to autarky.



We show this game admits a unique MPE, and the probability of a restrictive loan is single peaked in the state. The poorest and richest borrowers receive unrestrictive loans and grow their businesses, but those with intermediate wealth receive restrictive loans and remain at that level indefinitely. Relatively richer borrowers receive unrestrictive loans because they have a strong bargaining position (i.e. their outside option is attractive). The closer a borrower is to the formal sector, the more she values investing in fixed capital, so a lender finds it prohibitively costly to offer her a restrictive loan she would accept. In contrast, the farther a borrower is from the high payoff she enjoys in the formal sector, the less she values investment in fixed capital and therefore the less attractive her outside option is. So the lender finds it profitable to offer restrictive loans that borrowers of intermediate wealth are willing to accept. Finally, the poorest borrowers have a very weak bargaining position. By offering these borrowers unrestrictive loans, the lender benefits from their improved productivity as they grow their businesses but need not fear that their bargaining position will improve too rapidly. The equilibrium contractual structure is depicted in the preceding figure.

The model also yields nuanced comparative statics that shed light on the dynamic interlinkages of wealth accumulation. We show that improving the attractiveness of the formal sector unambiguously improves the welfare of relatively rich borrowers who are close to the formal sector. Yet, because agents are forward-looking, the improved continuation value of richer borrowers in high wealth levels trickles down to earlier states, improving the bargaining position of these borrowers. As lenders anticipate that richer borrowers become more demanding, they react by restricting poor borrowers' investment and hinder their growth in order to ameliorate their improved bargaining position. Notably, fixing any lender behavior, an improvement in the formal sector unambiguously increases the borrower's welfare. It is because of the lender's endogenous response that this improvement harms the borrower.

Our model also offers a counterpoint to the standard intuition that poverty traps are driven by impatience. We show that increasing borrower patience relaxes the poverty trap for rich borrowers, yet higher patience may *amplify* the poverty trap for poorer borrowers, causing them to get trapped at even lower levels of wealth. This is again due to the "trickle down" effect whereby lenders react to richer borrowers becoming more demanding by tightening contractual restrictions on poorer borrowers and preventing their growth.

The poverty trap arises in our model because of the *repeated* nature of lending relationships, as the lender imposes contractual restrictions only to capture future rents. The trap completely disappears if the relationship is short-lived. This result thus also provides a counterpoint to the standard intuition that repeated interactions may help sustain cooperation and improve total surplus; in our model, repeated interactions strictly lower total welfare.

At this point it is useful to contrast our theory to the seminal work of Petersen and Rajan (1995) (PR henceforth), who also study financing with limited commitment and show that improving borrower's future outside option—modeled through intensified competition from other lenders—can make the borrower worse off. Taken at face value, the prediction that improvements in the formal sector may worsen the poverty trap for poor borrowers, seems similar to the message of PR. However, the economic forces in our model differ fundamentally from theirs. In our model, borrowers always reach the formal sector under autarky; yet, in repeated lending relationships, lenders have incentives to create debt traps in order to extract rents. In contrast, borrowers in PR are only "trapped"—and their projects unfinanced—because the lender may refuse to lend. In other words, borrowers are trapped because of the very *presence of lending* in our model and are trapped because of the *lack of lending* in PR.

There are also other fundamental differences: because our model is inherently dynamic and contracts are state dependent, the effect of improvements in future outside option on borrower welfare is state dependent as well, unlike the two-period model of PR.

Our results help to organize a number of findings in the experimental literature on the impacts of microfinance. The low impact of microfinance on the average borrower's income can be explained by the presence of restrictive loans as an equilibrium phenomenon. At the same time, many experiments find that relatively wealthier borrowers do enjoy a high marginal return to microcredit, which is consistent with our model's prediction that wealthier borrowers receive unrestrictive loans.[7] Finally, many experiments find that the demand for microfinance is substantially lower than once expected.[8] In an extension of our model, we show this arises as a natural prediction, as borrowers who receive restrictive loans are near indifference. While this prediction resembles those from many principal-agent models, it stands in sharp contrast to models of credit constrained borrowers, where by definition, borrowers demand more credit than they are offered.

Our theory offers a novel explanation for why credit may have low impact on entrepreneurship. While no one theory is likely to be solely responsible, we argue that our model stands out from existing theories for several reasons.

First, as documented above, the marginal return to capital microentrepreneurs typically demonstrate is substantially above the interest rates charged by microfinance institutions. Because many entrepreneurs have ready access to microcredit, a theory that explains why microcredit has not catalyzed entrepreneurship must appeal to some feature of the loans beyond the transfer of capital and the interest rate. Our theory highlights that microcredit contracts are multidimensional and that constraints imposed by the lender may limit the usefulness of these loans for making long-term investments.

Theories that rationalize contractual restrictions based on deterring default arising from adverse selection and moral hazard can explain these constraints if they serve to distinguish amongst different types of borrowers or discipline their behavior (eg. Ghatak (1999) and Banerjee et al. (1994)), but as we argued above, we don't think the empirical evidence supports these stories. With the exception of rigid repayment schedules, there is little empirical evidence suggesting these restrictions effectively screen or discipline borrowers. And in the case of flexible repayment schedules documented in Field et al. (2013), we note that the positive impact (on borrowers) of relaxing the rigid repayment schedule substantially outweighs the negative impact of additional default. In fact, Barboni and Agarwal (2018) finds no evidence of a negative impact on default resulting from flexible repayment schedules, despite

[7]See Angelucci et al. (2015), Augsburg et al. (2015), Banerjee et al. (2015), Crepon et al. (2015), and Banerjee et al. (2017). This is also consistent with abundant anecdotal evidence that MFIs relax contractual restrictions such as joint liability and rigid repayment schedules for richer borrowers, although this fact may be explained by a number of other theories.

[8]See e.g. Banerjee et al. (2014), and Banerjee et al. (2015).

a marked increase in borrower sales. Therefore it seems that some inability for the borrower to share the benefits of long-term investments is a necessary ingredient in explaining why lenders don't typically offer flexible repayment contracts.

Our paper contributes to the literature on debt traps resulting from limited pledgeability. The topic has a long tradition in the literature on development finance (e.g., Bhaduri (1973), Ray (1998)). The inefficient contractual restrictions in our model arise due to lender's desire and ability to capture future rents; hence, so long as market power is present, the inefficiency we identify and analyze could be relevant in developed financial environments as well (Drechsler et al. (2017), Scharfstein and Sunderam (2017)). The dynamic inefficiency in our model stands in contrast to other papers that study lending inefficiencies arising from information frictions (e.g., Stiglitz and Weiss (1981), Dell'Ariccia and Marquez (2006), Fishman and Parker (2015)), common agency problems (e.g., Bizer and DeMarzo (1992), Parlour and Rajan (2001), Brunnermeier and Oehmke (2013), Green and Liu (2017)), and strategic default (e.g., Breza (2012)). Our work also relates to He and Xiong (2013), who analyze restrictive investment mandates in the context of asset management.

The rest of the paper proceeds as follows. In Section 1 we describe the model. Section 2 characterizes the equilibrium of our game, and Section 3 describes comparative statics. Section 4 discusses the relationship of our results with existing empirical evidence and then concludes. The appendix contains several model extensions, a discussion of institutional features of microfinance that aren't captured by our model, and all of the proofs.

1 The Model

We begin by outlining the model. We discuss several extensions in Appendix A.

Players, Production Technologies, and Contracts We study a dynamic game of complete information and perfectly observable actions. There are two players, a borrower (she) and a lender (he), both of whom are risk neutral. Each period lasts length dt and players discount the future at rate ρ. For analytical convenience we study the continuous time limit as dt converges to 0; a period is therefore an instant. The borrower's business is indexed with a state variable $w \in \{1, \dots, n+1\}$ referred to as her business size. We assume the state space is finite; $n+1$ is therefore the *terminal* state of the game.

In any non-terminal state $w < n+1$, the borrower is endowed with resources e at every instant, and she may obtain additional resources through taking a loan from her lender. The borrower has access to two production technologies: a working capital project and a fixed capital project. Her working capital project transforms resource inputs into consumption goods, which she uses to repay her lender and to eat. The project is a linear technology with capacity constraints, and the maximum scale of the project depends on the business size of the borrower. Specifically, the project in state w generates output according the production function

$$f_w(k) = z \times \min\{k, K_w\},$$

where k is the amount of resources devoted to the working capital project, $z > 1$ is the productivity, and K_w is the capacity constraint in state w. We impose assumptions on K_w such that the maximum capacity always exceeds borrower's endowed resources, and that the capacity increases in the state variable w at a decreasing rate.

Assumption 1. *For all w, 1) $K_w > e$; 2) $K_w > K_{w-1}$; 3) $K_w - K_{w-1} \geq K_{w+1} - K_w$.*

The borrower can also devote resources into a fixed capital project, through which her business grows. Specifically, for a flow of i units of resources into the project, her business moves from state w to $w+1$ according to a Poisson process with arrival rate $\phi \times i$, where $\phi > 0$ is the productivity of the investment project. Her business size remains constant otherwise.[9]

Over each instant, the lender makes a take-it-or-leave-it offer of a loan contract $c = \langle t, R, a\rangle \in \mathscr{C} \equiv \mathbb{R}^+ \times \mathbb{R}^+ \times \{0,1\}$, where t is size of the loan, R is the (contractable) repayment from the borrower to the lender, and a is the contractual restriction.

If the borrower rejects the contract, the borrower flexibly divides her resources e into both projects.

If the borrower accepts the contract, the lender transfers $t > 0$ to the borrower, whose total disposable resources—which is equal to the sum of inputs into both projects, $k+i$—become $e+t$, and, at the end of the period, the borrower repays R to the lender. If the contract specifies $a = 1$, she must devote all resources to the working capital project ($k = e+t$). If instead the contract specifies $a = 0$, then she must devote sufficient resources in the working capital project ($k \geq R/z$) to meet her repayment obligation but is free to invest her residual resources flexibly.[10] We refer to contracts which specify $a = 1$ as *restrictive* loans and those that specify $a = 0$ as *unrestrictive* loans.

Regardless of whether the contract is accepted or rejected, the two players meet again in the next instant and a new contract is proposed unless the terminal state $w = n+1$ is reached.

If the game ever reaches the terminal state $n+1$, both players cease acting. The borrower receives a continuation payoff $U \equiv \frac{u}{\rho}$ (the payoff resulting from reaching the formal sector), and the lender receives a payoff of 0 (having lost his customer). Though the borrower always repays her loans, she

[9]This may be generalized to allow for any transition process in which the probability of transition from w to any other state scales linearly with investment. In Section A.6 we allow the rate at which the borrower grows to be influenced by her business size.

[10]We represent restrictive contracts as those that force the borrower to invest in working capital because, as we show below, when the borrower is free to invest flexibly she will invest in fixed capital.

cannot commit to share the proceeds in state $n+1$.

Repayment Ceiling Even though repayment is contractible—the borrower always pays back R to the lender if the contract is accepted—we impose an additional constraint on repayments to represent limited enforcement. Specifically, for some $h > 0$,

$$R \leq z \times \min\{K_w, e+t-h\} \quad \text{for all } w. \tag{1}$$

The constraint implies that the lender cannot expropriate the entire flow output from the working capital project when all resources are allocated to it. The lender faces the constraint independent of whether restrictive or unrestrictive loans are offered.

The repayment ceiling in (1) is motivated by evidence that loan repayments are not always perfectly enforced in developing economies (e.g., see Breza (2012) for evidence on strategic default). h captures severity of limited enforcement and can be micro-founded in a number of ways. Most straightforwardly, the borrower could renege on her debt with a default cost $z(e+t-h)$, including the cost of hiding her business and income from the lender and the cost of finding a new lender. In this case, the borrower would never repay a debt in excess of this cost. In Appendix A.5, we provide a more elaborate micro-foundation where the borrower has private information about z, the productivity of the working capital project. If the lender does not know the exact productivity of the borrower's working capital project, he cannot set the repayment amount R to fully extract the additional output.

If enforcement is overly weak, no lending takes place because the lender can secure too little repayment. We impose an upper bound on h to rule out this uninteresting case.

Assumption 2. $h < \frac{(z-1)K_1+e}{z}$.

Timing and Summary of Setup

1. The lender makes a take it or leave it offer $c = \langle t, R, a \rangle$ to the borrower, subject to (1).

2. The borrower decides whether to accept the contract.

   (a) If she rejects the contract, she allocates her resources e flexibly among her two projects under the constraint $k+i=e$ (a rejection can thus be equivalently modeled as choosing a contract with $\langle t, R, a \rangle = \langle 0, 0, 0 \rangle$).

   (b) If she accepts the contract, the lender transfers t to the borrower, whose resource constraint becomes $k+i=e+t$. The borrower must invest according to the contractual restriction a:

$$k \begin{cases} = e+t & \text{if } a=1, \\ \geq R/z & \text{if } a=0. \end{cases}$$

The borrower then repays R to the lender.

3. The lender's flow payoff is $R-t$. The borrower's flow payoff is the output of the working capital project net of repayment, $f_w(k)-R$. If the contract is rejected, the lender consumes 0 and the borrower consumes $f_w(k)$. The borrower's business grows from state w to $w+1$ with Poisson rate $\phi \times i$ and remains constant otherwise.

4. If the state $w<n+1$, the period concludes and after discounting the next one begins.

The timing described above can be understood through the lens of the example in our introduction. The borrower is a fruit vendor, and at state w she operates a mobile cart. At the beginning of the week she has a cash endowment e. If she rejects the lender's contract, then she flexibly allocates her endowment between two types of activities. The working capital project represents activities that generate flow income, such as purchasing fruits to sell during the week. The fixed capital project represents activities that expand her business, such as buying raw materials to expand to a market stall from which she may have access to a broader market, thereby relaxing her capacity constraint.

If instead she accepts the contract, the lender transfers resources to the borrower but asks for repayment and contractual restrictions. The borrower's subsequent investment decision then depends on the contractual restriction. An example of a restrictive contract might be one that demands early and frequent repayments. If the lender demands these features, the borrower may be forced to hold cash-in-hand each day to repay a fraction of her loan and may not be able to invest in the longer-term, fixed capital project, which may not generate immediate returns. By the time she has generated sufficient income to repay each installment, she may no longer have sufficient resources to meet the minimum investment in her fixed capital project, as would be the case if she has trouble saving cash from day to day (for instance because she faces pressure from her family to share underutilized assets; see Jakiela and Ozier (2016)). Contracts that require joint liability, or guarantors, might also restrict a borrower's investments, as guarantors might discourage profitable but uncertain projects.[11]

Equilibrium Our solution concept is the standard notion of *Stationary Markov Perfect Equilibrium* (henceforth *equilibrium*)—the subset of the subgame perfect equilibria in which strategies are only conditioned on the payoff relevant state variables. An equilibrium is therefore characterized by the lender's state contingent contractual offers $(t_w, R_w, a_w) \in \mathscr{C}$, the borrower's state and contract-

[11] See eg. Banerjee et al. (1994), Fischer (2013), Jack et al. (2016), and Maitra et al. (2017).

contingent accept/reject decision $d_w : \mathscr{C} \to \{\text{accept, reject}\}$, and the borrower's state, contract and acceptance-contingent investment decision $i_w : \mathscr{C} \times \{\text{accept, reject}\} \to \mathbb{R}$. At all states all strategies must be mutual best responses. We defer discussion of the value functions that pin down these equilibrium response functions to Section 2.

By studying Stationary Markov Perfect Equilibria, we impose that the lender uses an impersonal strategy: borrowers with the same business size must be offered the same (potentially mixed set of) contracts. This may be an especially plausible restriction in the context of large lenders, such as microfinance institutions whose policy makers may be far removed from their loan recipients, thereby rendering overly personalized contract offers infeasible.

Model Discussion The repayment ceiling (1) is an important feature of our analysis, and, as we show, the equilibrium structure depends crucially on whether h, the severity of limited enforcement, exceeds e or not. If $h < e$, constraint (1) never binds, the terminal state $n+1$ is always reached in finite time, and the borrower does not benefit *at all* from having access to the lender. Effectively, the lender exploits the borrower's working capital technology as his own, always demanding large repayments to keep the borrower at her autarky value.

On the other hand, under $h > e$ (i.e., when enforcement is sufficiently weak), the borrower may benefit from having access to the lender—she may enjoy higher utility than in autarky. Perhaps paradoxically, however, the very same condition ($h > e$) creates scope for a lending-induced poverty trap, and the borrower may never grow to state $n+1$.

Understanding the nature of such a poverty trap, its existence, and comparative statics, is the focus of our subsequent analysis. While our model is stylized, it serves to isolate the central economic forces we discuss. In Appendix A, we demonstrate that our qualitative results survive under substantial generalizations of our model.[12] Section B discusses the limitations of our model.

2 Equilibrium Structure

We now describe the borrower and lender's equilibrium behavior and our main results regarding equilibrium structure. Section 2.1 describes the borrower's autarky problem and sets forth an assumption that guarantees the borrower will eventually reach the formal sector (state $n+1$) in autarky. Section 2.2 describes the key incentives of the borrower and lender necessary to understand the structure of the equilibrium. Section 2.3 provides our main results: the equilibrium is unique, and the probability that the lender offers a restrictive contract is single peaked in the state. Thus, the lender's

[12] In particular, we allow e, h, z, and ϕ to vary arbitrarily across states; we extend the analysis to a countably infinite number of states; we allow the borrower to flexibly allocate a fraction of her resources independent of contractual restrictions. We also discuss how our analysis relates to non-profit lenders and competition amongst several lenders.

poorest and richest clients may receive unrestrictive contracts and grow faster than they would have in his absence. But borrowers with intermediate levels of wealth receive restrictive contracts every period and find themselves in a poverty trap. Notably, this poverty trap may exist even if the borrower would have reached the formal sector in autarky and even if the discounted utility from expanding to the formal sector is greater than the total welfare generated by investing the total resources in working capital in every state.

2.1 The Borrower's Autarky Problem

First, consider the borrower's autarky problem, without having access to a lender. That is, the economic environment is as specified in Section 1, but the borrower is forced to reject the lender's contract at all times, choosing the contract $\langle 0,0,0\rangle$ at every instant, and can therefore allocate her resources flexibly between the two projects.

The borrower's continuation value in autarky in state w, B_w^{aut}, solves the following HJB equation:

$$\rho B_w^{aut} = \max_{0\leq i\leq e} \; f_w(e-i) + i \times \phi \times \left(B_{w+1}^{aut} - B_w^{aut}\right). \tag{2}$$

The first term on the right-hand-side represents the consumption payoff from the working capital project; the second term represents change in borrower's continuation value when the fixed capital investment results in a successful expansion of her business size.

Because the right-hand-side of equation (2) is linear in i, the borrower will always choose an extremal level of investment, implying

$$B_w^{aut} = \max\left\{\alpha B_{w+1}^{aut}, \frac{ze}{\rho}\right\}, \tag{3}$$

where $\alpha \equiv \frac{e\phi}{\rho+e\phi} < 1$ and $B_{n+1}^{aut} = U$. Intuitively, αB_w^{aut} is the borrower's value if she invests into fixed capital, and ze/ρ is her value if she chooses to invest into working capital.

Equation (3) implies an attractive, recursive formulation of the borrower's autarky problem: starting from state w, if she invests in fixed capital at every instant until reaching state $w+m$, her value in state w must satisfy $B_w^{aut} = \alpha^m B_{w+m}^{aut}$. This property is an outcome of the Poisson growth process under the fixed capital project,[13] and we will exploit this type of recursive formulation in our later

[13] That is, the borrower's value in state w can be written as a weighted average of her lifetime utility from staying in state w forever and her value of reaching state $w+1$; the weights depend on the Poisson rate of reaching state $w+1$:

$$B_w^{aut} = \max_{i\leq e}\left(1 - \frac{i\phi}{\rho+i\phi}\right) V_w(i) + \frac{i\phi}{\rho+i\phi} B_{w+1}^{aut}.$$

If she invests all resources ($i=e$) into the fixed capital project, then her flow payoff in state w is zero; hence $B_w^{aut} = \alpha B_{w+1}^{aut}$.

analysis of the lending equilibrium.

2.2 Relationship Value Functions

We now outline the borrower and lender's relationship maximization problems and describe their value functions. Let B_w be the borrower's equilibrium continuation utility at the beginning of a period in state w, and let $c_w = \langle t_w, R_w, a_w \rangle$ be the equilibrium contract in state w.

If the contract is unrestrictive ($a_w = 0$), the borrower solves the instantaneous investment problem:

$$v_w(\langle t, R, 0 \rangle) = \max_{\substack{k,i \\ \text{s.t. } k+i \leq e+t \\ zk \geq R}} f_w(k) - R + i\phi(B_{w+1} - B_w). \tag{4}$$

On the other hand, if the contract is restrictive ($a_w = 1$), the borrower must invest all resources into the working capital project, realizing the following instantaneous payoff:

$$v_w(\langle t, R, 1 \rangle) = f_w(e+t) - R.$$

Given equilibrium contract $\langle t, R, a \rangle$ in state w, the borrower's value function B_w satisfies the HJB equation:

$$\rho B_w = \max\left\{ v_w^{Rej}, v_w(\langle t, R, a \rangle) \right\},$$

where v_w^{Rej} is the borrower's instantaneous payoff from rejecting the contract:

$$v_w^{Rej} = \max_{0 \leq i \leq e} f_w(e-i) + i\phi(B_{w+1} - B_w).$$

Intuitively, the borrower always has the option of rejecting the contract, ensuring a minimum flow payoff v_w^{Rej}. If a contract is accepted, the borrower optimally chooses how to allocate her resources subject to contractual restrictions, and her flow payoff depends on two terms: 1) the residual consumption—after repayment R—from the working capital project; 2) the change in continuation payoff due to investments in fixed capital.

The lender's value function L_w in state w satisfies the HJB equation

$$\rho L_w = \max_{\langle t,R,a \rangle \in \mathbb{R}^+ \times \mathbb{R}^+ \times \{0,1\}} R - t + i^*(\langle t, R, a \rangle)\phi(L_{w+1} - L_w),$$

subject to borrower's individual rationality constraint

$$v_w^{Rej} \leq v_w(\langle t, R, a \rangle), \tag{5}$$

the repayment ceiling (1), and $i^*(\langle t,R,a\rangle)$ solves borrower's investment choice problem subject to contractual restriction a:

$$i^*(\langle t,R,a\rangle)\begin{cases}\text{is the maximizer of (4)} & \text{if } a=0,\\ =0 & \text{otherwise.}\end{cases}$$

2.3 Analysis of the Equilibrium

Our model is a dynamic contracting game with state transitions governed by an endogenous stochastic process. When the equilibrium contract in state w is restrictive, i.e. the borrower must invest in the working capital project, state w becomes absorbing and the borrwer's business never grows to state $w+1$. Our subsequent analysis studies the property of such a poverty trap: why it exists and how it is shaped by features of the economic environment.

The model is tractable precisely because of our recursive and modular formulation. We first characterize the optimal lending contract in each state, taking future value functions as given; we then, in a second step, solve for the borrower and lender's equilibrium value as functions of the terminal payoff U, on which we impose the following assumption for expositional simplicity.

Assumption 3. *$z\times\max\{e,h\}\leq\alpha^n\rho U$, where $\alpha\equiv\frac{e\phi}{\rho+e\phi}$.*

Assumption 3 guarantees that the terminal payoff U is sufficiently attractive, so that the borrower always prefers to invest her resource into fixed capital and reach for the terminal payoff, rather than accepting a maximally extractive restrictive contract that delivers a constant flow consumption value of $z\times\max\{e,h\}$ and no business growth. Note that the assumption also guarantees that the the terminal state is always reached under autarky.

Our goal is to highlight that introducing a lender may cause a poverty trap to emerge, despite the borrower's desire to enter the formal sector and that reaching the formal sector maximizes the joint payoff. That is, business growth may slow down *because of* access to credit, even though it is socially efficient to grow. Assumption 3 merely serves to simplify the exposition, as it rules out an uninteresting case in which U is unattractive and the borrower has little desire to grow her business.

We now analyze the equilibrium structure of lending contracts. We first separately characterize the optimal restrictive and unrestrictive contracts; we then study which type of contract should be offered.

Restrictive Contracts Should be Maximally Extractive

First consider restrictive contracts, which, once accepted, dictate that all resources must go into the working capital project. The lender's flow payoff is linear in loan size t_w; hence, t_w should always be chosen to maximize the output of the working capital project, $t_w=K_w-e$. On the other hand, because the lender's flow payoff is increasing in repayment R, he chooses the highest R possible, subject to the repayment ceiling (1)—which, if binding, leaves the borrower a constant flow payoff h—and the borrower's individual rationality constraint (5).

Assumption 3 implies that the borrower would always prefer to reject a contract that offers flow payoff h in favor of investing in fixed capital and growing at her autarkic rate $e\phi$. Hence, optimal restrictive contracts must extract the most repayment that the borrower finds *acceptable;* she is always just indifferent between accepting and rejecting the offer. These results are summarized into the following lemma.

Lemma 1. *If restrictive contracts are offered in state w, then*

$$B_w = \alpha B_{w+1}, \text{ where (recall) } \alpha \equiv \frac{e\phi}{\rho + e\phi}.$$

Moreover, the loan size t_w and repayment R_w must satisfy

$$\begin{cases} t_w = K_w - e, \\ R = zK_w - \rho B_w. \end{cases}$$

Lemma 1 pins down the repayment amount the lender must offer if the borrower is to accept a restrictive contract. The lender's decision between restrictive and unrestrictive contracts is a trade-off between offering the borrower a lower repayment amount and suspending her growth instead of offering a loan with a higher repayment amount that allows her to invest as she pleases.

Unrestrictive Contracts Leave Rents to the Borrower

Now consider unrestrictive contracts. If the lender provides a loan of size $t_w > h - e$, then, because of the repayment constraint, the borrower can always guarantee herself at least the *value* from investing h into fixed capital. The following lemma shows the premise $t_w > h - e$ always holds, and that this logic establishes a lower bound on borrower's value under unrestrictive contracts.

Lemma 2. *The transfer t_w of an unrestrictive contract must satisfy $t_w > h - e$, and borrower's continuation value satisfies*

$$B_w \geq \beta B_{w+1}, \text{ where } \beta \equiv \max\left\{\alpha, \frac{h\phi}{\rho + h\phi}\right\}.$$

Lemma (2) illuminates an important force in our model. Because $\beta \geq \alpha$, borrower's value is always weakly higher under an unrestrictive contract than under a restrictive one, and, in the case with a severe repayment ceiling ($h > e$), the inequality is strict.

The difference between the borrower's continuation value upon receiving an unrestrictive versus restrictive contract is at least $(\beta-\alpha)B_{w+1}$. We refer to this bound as the *expansion rent* in state w. This asymmetry arises because of the repayment ceiling. When $h>e$, upon accepting an unrestrictive loan and allocating the appropriate resources to the working capital project for repayment, the borrower necessarily has a larger residual endowment to allocate to either project than she would have had on her own, and is thus left with strictly higher utility than her outside option. In contrast, upon accepting a restrictive loan, the borrower is forced to invest in working capital, and the repayment amount is chosen such that even though she has a larger set of resouces for her own investment, she is left indifferent between making a large investment in working capital versus making her outside option investment.

Note that Lemma 2 establishes a lower bound on the borrower's utility if she receives an unrestrictive contract. The borrower's equilibrium utility may be strictly higher than βB_{w+1} in states in which the lender actively supports the borrower's growth (i.e. any state w in which $L_{w+1}-L_w \geq \frac{1}{\phi}$). In equilibrium, the lender offers unrestrictive contracts with lower than necessary repayment when the increase in the borrower's output resulting from growth outpaces the corresponding increase in the borrower's outside option. As we discuss below, this condition may be met for borrowers in the lowest business states.

Optimal contract in state w

Observe that the borrower's expansion rent, $(\beta-\alpha)B_{w+1}$, is increasing in her value at the next business size. Consequently, the borrower's welfare at any business size—under either type of contract—is a reflection of the quality of her outside option (growing to the next business size at the autarkic rate), and borrowers with attractive outside options are more demanding on the lender. The size of the expansion rent is an important determinant of when the lender offers restrictive loans to the borrower to slow her growth.

Lemma 3. *The lender offers a restrictive contract with probability one in state w if and only if*

$$\underbrace{\beta\left((L_{w+1}+B_{w+1})-\frac{(z-1)K_w+e}{\rho}-1/\phi\right)}_{\substack{\text{"efficiency gain"}\\ \text{from investing in fixed capital}}} \leq \underbrace{(\beta-\alpha)B_{w+1}}_{\substack{\text{"expansion rent"}\\ \text{captured by the borrower}}} .$$

The left hand side of the above inequality represents the "efficiency gain"—the change in the sum of the lender and borrower's value functions—from investing in fixed capital at the slowest feasible rate under a flexible contract, max $\{e,h\}$, relative to investing everything into working capital. Investing in fixed capital leads to the possible expansion of borrower's business size to $w+1$; the expansion leads to a joint continuation value of $L_{w+1}+B_{w+1}$ but the players forgo the consumption they could have enjoyed in state w, $\frac{(z-1)K_w+e}{\rho}$, and the cost they incur from expansion is $1/\phi$. The expansion happens at Poisson rate $\phi \max\{e,h\}$; hence, in present value terms, the valuation gain from expansion is discounted by $\beta \equiv \frac{\phi \max\{e,h\}}{\rho+\phi \max\{e,h\}}$.

The right hand side of the inequality is the borrower's "expansion rent": the least level of additional surplus she commands from unrestrictive contracts relative to restrictive ones. The lender's welfare gain from business expansion is the difference between the total social welfare gain and the borrower's expansion rent. Thus, if expansion rent exceeds the efficiency gain from business expansion, the lender will offer only restrictive contracts, pinning the borrower to the current state.

In Appendix C in the proof of Propostion 5 we provide a numerical example in which we solve for the equilibrium value functions and contractual offers.

Equilibrium Contract Structure

In any state, the optimal contract as well as borrower's acceptance decision depends on continuation values. The recursive structure implies that the dynamic game is solvable by backward induction: taking values in higher states as given, the lender optimally chooses whether to offer a maximally extractive, yet acceptable, restrictive contract or an unrestrictive contract that enables the borrower to grow. Our model thus admits a unique equilibrium.[14]

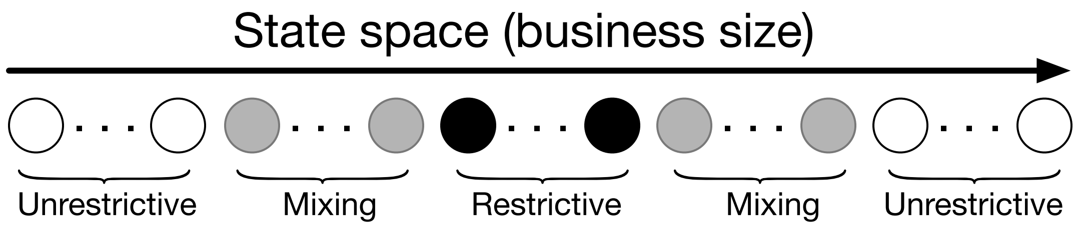
Proposition 1. *An equilibrium exists and is generically unique.*

Our next result provides a transparent characterization of the type of equilibrium contracts.

Proposition 2. *The probability the lender offers a restrictive contract, p_w, is single peaked in w.*

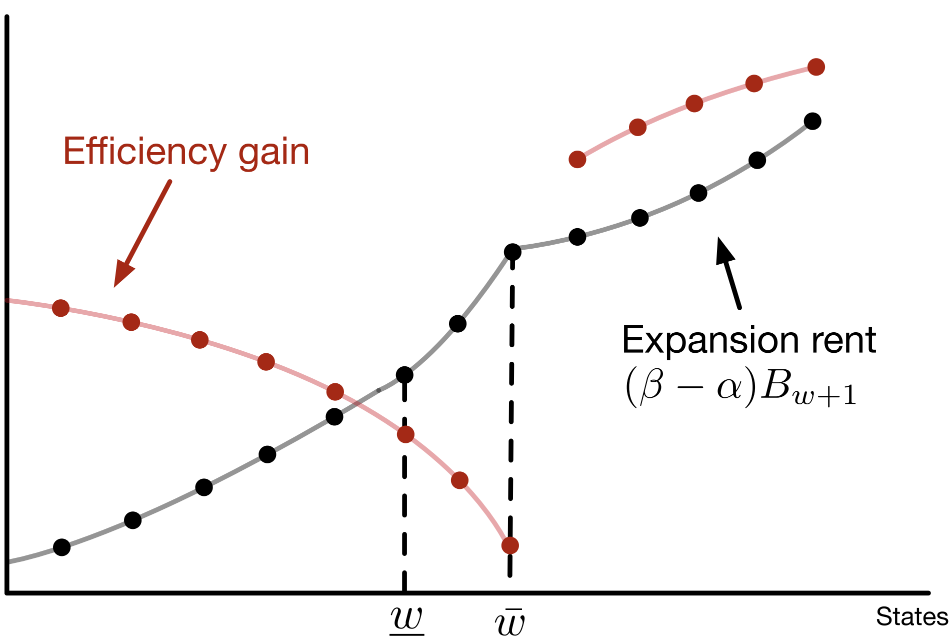
This result implies that in equilibrium the states can be partitioned into three regions: an initial region with only unrestrictive contracts, an intermediate region in which both kinds of contracts are possible, and a final region in which only unrestrictive contracts are offered. In the intermediate region, the probability a restrictive contract is offered is first increasing and then decreasing. Borrowers who arrive at a state in which only restrictive contracts are offered never grow beyond it. This is depicted in the figure below, wherein white circles denote unrestrictive states ($p_w = 0$), black circles denote restrictive states ($p_w = 1$), and grey circles denote mixing states ($p_w \in (0,1)$).

[14] A potential subtlety arises from the fact that the borrower's value in state w is increasing in the probability the lender offers an unrestrictive contract in w. The more frequently the borrower anticipates unrestrictive contracts in w, the higher is her value in state w, and the *less demanding* she will be of restrictive contracts. Hence, the unique equilibrium might involve mixed strategies, with the lender offering unrestrictive and restrictive contracts with interior probabilities.



The proposition implies that the lending relationship may introduce endogenous poverty traps, despite the fact that the borrower always grows to her efficient size in autarky. Moreover, the poverty trap only applies to relatively poor borrowers. Richer borrowers—those in the last, unrestrictive region—unambiguously benefit from the lending relationship and grow at a faster rate than in autarky.

Why is p_w single-peaked? The figure below demonstrates, state-by-state, the trade-off between the "efficiency gain" and "expansion rents" as highlighted in Lemma 3. $\underline{w}$ and $\bar{w}$ in the figure respectively denote the first and last state in which restrictive contracts are offered with probability one. By definition, borrowers at state $w < \bar{w}$ never reach the formal sector. In conjunction with the fact that the working capital project exhibits decreasing returns to scale (Assumption 1), this implies that the efficiency gain of business expansion is decreasing in the state for all states $w < \bar{w}$. For borrowers in states $w > \bar{w}$, the efficiency gain may begin to increase in the state, as these borrowers reach the formal sector in finite time. The expansion rent is everywhere increasing, as the borrower's continuation value increases in the state. When the expansion rent exceeds the efficiency gain, the lender offers restrictive contracts.



While the comparison of the expansion rent to the efficiency gain determines whether a restrictive contract is offered, both the expansion rent and the efficiency gain are determined in equilibrium, as continuation values are a function of whether the borrower ever reaches the formal sector. Therefore, to further understand the logic underlying the equilibrium contractual structure, we anlayze equilibrium behavior moving backwards in the state. Borrowers near the formal sector receive unrestrictive contracts because as the borrower approaches the formal sector it is increasingly costly for the lender to offer a restrictive contract. A borrower in state n has to be compensated with a flow consumption of $\alpha\rho U$ in order to accept a restrictive contract, and when the formal sector is sufficiently attractive (U sufficiently high), this could be prohibitively costly for the lender. However, as the borrower becomes poorer and his business further away from the terminal state, it becomes exponentially cheaper to offer her a restrictive contract. For a borrower who is in state w and expects unrestrictive contracts in all future states, she would accept any restrictive contract that carries consumption over $\alpha\beta^{n-w}\rho U$. Put in terms of the efficiency gain from expansion, as the borrower moves farther from the formal sector, the efficiency gain from expansion may decline. This occurs because the discounted time the borrower spends in the formal sector is less sensitive to the rate of business expansion the farther the borrower is from the formal sector.

In the intermediate region, the trade off highlighted in Lemma 3 becomes important. When the borrower's expansion rent exceeds the gain in joint surplus from business expansion, the lender offers only restrictive contracts, keeping her business inefficiently small. This poverty trap is created by the presence of the lender, as the autarkic borrower would always invest in fixed capital under Assumption 3.

Last, the poorer is the borrower, the smaller is her expansion rent $(\beta-\alpha)B_{w+1}$, as it is tied to her continuation value in the next state. One way to understand this is that the lender has only a weak incentive to stymie the growth of his poorest borrowers, because the rate at which these borrowers' outside option improves is comparatively slow. Moreover, because of the decreasing returns of output from the working capital investment, the joint surplus increase from expansion may become larger as the borrower becomes poorer. When the joint surplus increase is sufficiently large, the lender may even offer excessively generous unrestrictive contracts, actively supporting the borrower's growth.

We close this section with a discussion of the source of this poverty trap.

Proposition 3. *If $h<e$, then flexible contracts are offered $(p_w=0)$ whenever it is socially efficient to invest in fixed capital (i.e. whenever $\beta\left((L_{w+1}+B_{w+1})-\frac{(z-1)K_w+e}{\rho}-1/\phi\right)>0$).*

Under $h<e$, the repayment ceiling constraint (1) never binds under either type of contract; instead, the borrower's individual rationally constraint always binds. Since the borrower always demands the same continuation value αB_{w+1} under either type of contract, the lender always chooses the contractual type that maximizes joint value, offering a flexible contract whenever it is socially efficient to

enable the borrower to grow.

Even though the poverty trap disappears under $h < e$, it is important to note that the equilibrium still features inefficiently slow business expansion relative to the social optimum. A natural question, then, is what contractual flexibility is required to reach the first best level of investment in fixed capital? It is straightforward to verify that *long-term debt or equity contracts*—contracts that allow the borrower to commit a fraction of her formal sector flow payoff to the lender in exchange for favorable unrestrictive contracts—are sufficient to guarantee first best investment. Under such arrangements, the lender's interests are aligned with supporting efficient borrower growth. However arrangements like this are often infeasible in informal financial markets as participants rarely have the capacity to commit to long-term contracts.

3 Model Implications

In this section, we exploit the tractability of our recurisve formulation and characterize how the poverty trap and welfare respond to changes in the economic environment. Our analysis sheds light on a number of economic forces that affect the severity of lending traps, including the attractiveness of outside option, borrower patience, duration of relationships, and competition from other lenders. Throughout the analysis, we highlight an important "trickle down" effect: because the lending relationship is dynamic and forward-looking, changing economic fundamentals can have nuanced impacts on equilibrium contracts and welfare that vary depending on the borrower's business size. We summarize our findings below.

First, we show that an increase in the attractiveness of the formal sector (higher U) unambiguously benefits richer borrowers who are close to the formal sector. Yet, because the borrower's *future* prospect trickles down to the present, the lender offers a different contract in equilibrium, tightening the poverty trap for poorer borrowers. When the effect is sufficiently strong, higher borrower value in the future may even result in the welfare loss of poorer borrowers. These seemingly perverse effects are entirely due to lender's strategic response: holding the type of equilibrium contracts fixed, higher U unambiguously benefits all borrowers. These results are formalized in Section 3.1.

A standard economic explanation for the existence of poverty traps is impatience (Azariadis (1996)), i.e. that more patient entrepreneurs should have stronger incentive to invest and grow. Section 3.2 examines this argument and shows that this intuition does not always hold in our model: in the presence of strategic lenders, increasing borrower patience (while holding lender patience fixed) tends to relax poverty trap for richer borrowers but may tighten it for poorer borrowers. This is, once again, due to the "trickle down" effect. Anticipating the fact that a more patient borrower has stronger incentive to grow and has higher future values, the lender switches to restrictive contracts for poorer borrowers to stymie the improvements in her bargaining position. This result is formalized in Proposition 6.

Our results thus far highlight a lender-induced poverty trap. In Section 3.3, we further clarify that the poverty trap exists not because of *lending*—having access to external funds strictly expands production sets—but because of the *repeated* nature of lending relationships. The poverty trap exists only because the lender has an incentive to capture future rents, and the trap completely disappears if the relationship is short-lived. This result thus provides a counterpoint to the standard intuition that repeated interactions may help sustain coorperation and improve total surplus; in our model, repeated interactions strictly lower total welfare.

Lastly, Section 3.4 analyzes how competition from other lenders affects the nature of equilibrium contracts. We consider both dynamic and instantaneous aspects of competition. Dynamic competition takes the form of stochastic breakage of relationships, and instantaneous comeption is modeled through varying the severity of limited enforcement h—motivated by the microfoundation of h as the cost of finding a new lender. Standard intuitions suggest that competition among lenders should benefit borrowers. We show, because of lender's endogenous response in changing contractual types, competition has nuanced effects on both equilibrium contracts and borrower value. Poverty traps may worsen, and the borrower may become worse off as competition intensifies.

Throughout this section, let $\underline{w}$ and $\bar{w}$ respectively denote the first and the last state in which restrictive contracts are offered with probability one:

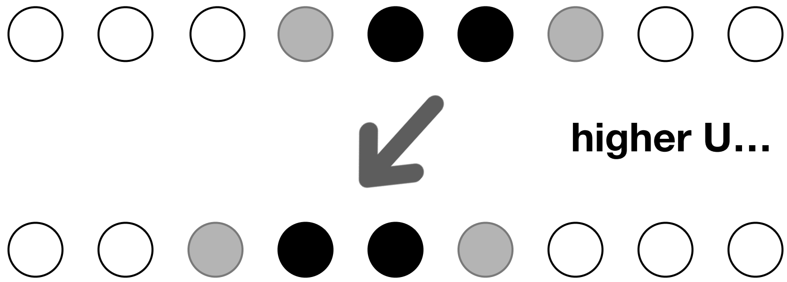
$$\underline{w} \equiv \operatorname*{argmin}_{w} \{w : p_w = 1\}, \quad \bar{w} \equiv \operatorname*{argmax}_{w} \{w : p_w = 1\}.$$

Between $\underline{w}$ and $\bar{w}$ is the region in which restrictive contracts are offered with probability one. Our comparative statics results for contractual type and welfare in state w depend on whether w is to the left or to the right of this region.[15]

3.1 An Attractive Formal Sector Worsens the Poverty Trap

What happens when the formal sector becomes more attractive? We show that under a higher U, the entire restrictive region shifts leftward, meaning the poverty trap is relaxed for rich borrowers but *tightened* for poor borrowers.

[15]For full generality, under the parameter range so that $p_w < 1$ for all w, our theoretically statements hold by letting $\bar{w} = \underline{w} = 1$.



To understand this, first consider a sufficiently rich borrower in the final unrestrictive region. As the formal sector becomes more attractive, the borrower has stronger desire to grow, and, in order for her to accept any restrictive contracts, she must be compensated with a higher flow surplus. In other words, higher U makes it more expensive for the lender to offer restrictive contracts to richer borrowers. Consequently, the lender has a stronger incentive to offer unrestrictive contracts, thereby relaxing the poverty trap for rich borrowers.

On the other hand, the exact same force causes the lender to tighten the reins on poorer borrowers and trap them at even lower levels of wealth. As we show, holding the contractural type fixed, increasing U improves the borrower's continuation value in all states—the "trickle down" effect. Because the gain is smaller for relatively poor borrowers who are more distant from the formal sector, increasing U increases the lender's preference for poor borrowers (relative to rich borrowers). Hence the lender responds to higher U by offering more restrictive contracts to poor borrowers in order to stymie their growth.

Put another way, improving the borrower's continuation value at her current wealth level is always good for her. But increasing the borrower's future continuation value at higher wealth levels may harm her, as the forward-looking lender responds strategically and shifts to offering more restrictive contracts, slowing down the borrower's business growth.

These results are formalized in the propositions below.

Proposition 4. *Generically, increasing the attractiveness of the formal sector relaxes the poverty trap for rich borrowers with $w \geq \bar{w}$ but tightens it for poorer borrowers with $w < \underline{w}$:*

$$\frac{dp_w}{dU} \begin{cases} \leq 0 & \textit{for } w \geq \bar{w}, \\ = 0 & \textit{for } \bar{w} > w \geq \underline{w}, \\ \geq 0 & \textit{for } w < \underline{w}. \end{cases}$$

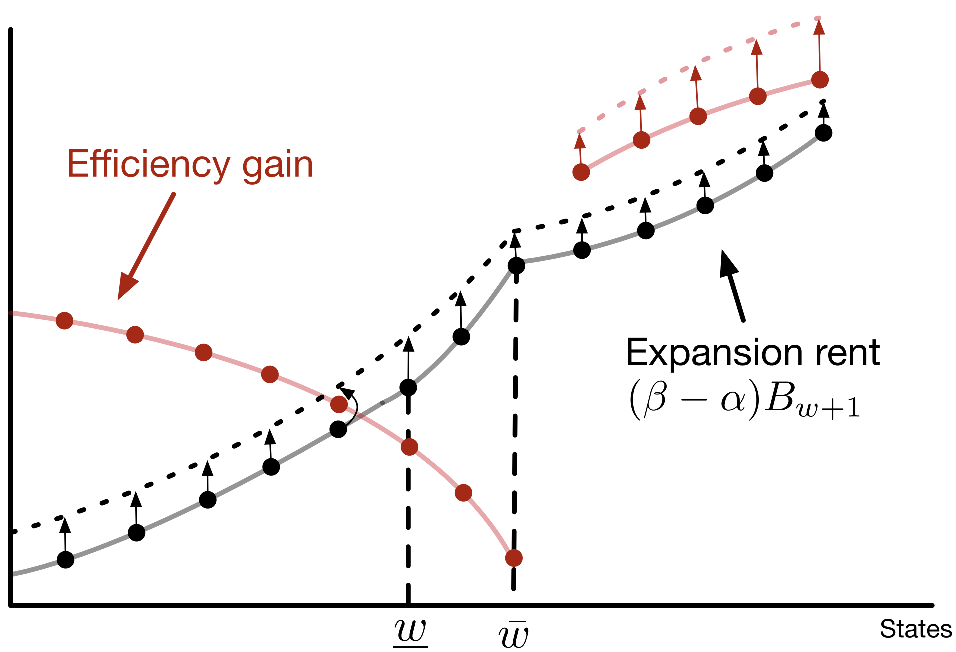
Inreasing the attractiveness of the formal sector lowers the lender's continuation value in all states ($dL_w/dU \leq 0$), strictly so for $w \leq \bar{w}$. Increasing U strictly increases the borrower's continuation utility in states $w \geq \underline{w}$ but can decrease it in states $w < \underline{w}$ particularly if the borrower is impatient:

$\frac{dB_w}{du} < 0$ *for states* $w < \underline{w}$ *if* $p_{\underline{w}-1} > 0$ *and* $\rho > \bar{\rho}$ *for some constant* $\bar{\rho}$.

Proposition 5. *Start with an equilibrium in which restrictive contracts are offered with probability* p_w *and consider a marginal increment in the attractiveness of the formal sector. Suppose the lender has to hold the type of contractual offers fixed (holding* p_w *constant) and can respond only by adjusting demanded repayment so that all contracts are still accepted by the borrower. In this case, increasing* U *strictly improves the borrower's continuation utility in all states,* $\left.\frac{dB_w}{dU}\right|_{\{p_w\}\ constant} > 0$.

That contracts become more restrictive for poor borrowers (those in states $w < \underline{w}$) can also be understood in terms of the trade-off in Lemma 3, i.e., between the "efficiency gain" from business growth and the "expansion rent" retained by the borrower. Note that borrowers in states $w < \underline{w}$ will never grow beynd state $\underline{w}$. Hence, increasing U has no impact on the efficiency gain of growth in these states. However, increasing U always raises the borrower's share of surplus from business expansion due to the trickle-down effect (i.e. her expansion rent increases in U). Because her expansion rent increases, but the efficiency gain does not, the lender shifts towards restrictive contracts, thereby slowing down the borrower's growth in these states.

State-by-state changes in "efficiency gain" and "expansion rent" in response to higher U



To summarize, higher U unambiguously benefits richer borrowers but may harm poorer borrowers especially if they are impatient. The mechanism operates through lender's strategic response and changes in equilibrium contractual types.

3.2 More Patient Borrowers May Face Worse Poverty Traps

Standard intuition suggests that poverty traps are generated by impatience (see, for instance, Azariadis (1996)). Yet in this model increasing the borrower's patience has a very similar effect to increasing formal sector attractiveness, and hence can tighten the poverty trap and make the borrower *worse off* at some levels of wealth. For full generality, suppose the borrower and the lender to have different discount rates, ρ^B and ρ^L respectively. Lemma 3 demonstrates that, under $\rho^B = \rho^L$, the key trade-off underpinning the optimal contractual type is the comparison between the "efficiency gain" from fixed capital investment and the "expansion rent" captured by the borrower. In Appendix A.3, we show a similar condition for optimal contractual type still holds when the two players have different discount rates.

Proposition 6. *Increasing the borrower's patience (lowering ρ^B) relaxes the poverty trap for relatively rich borrowers ($\frac{dp_w}{d\rho^B} \geq 0$ for $w > \bar{w}$) but may tighten it for poorer borrowers.*

For rich borrowers above the highest pure restrictive state ($w > \bar{w}$), the comparative static on ρ^B works similarly to the comparative static on U. Increasing the borrower's patience increases how much she values business expansion. This causes her to demand more surplus if she were to accept restrictive contracts, thereby raising the lender's incentive to offer unrestrictive contracts. Thus, in all states $w > \bar{w}$, the rate at which borrowers reach the formal sector increases, and so does her continuation value B_w.

The effect trickles down to states $w \leq \bar{w}$ through raising borrower's continuation utility B_w. However, the comparative statics of raising borrower patience are more nuanced than the comparative statics on raising U, as there are now two competing effects. On the one hand, higher patience leads to higher borrower continuation value B_{w+1}, which always raises the expansion rent more than it raises the efficiency gain. This trickle down effect is also present when we analyzed the effect of higher U.

On the other hand, higher borrower patience also generates an additional effect. Recall the borrower's expansion rent in state w is $\max\left\{0, \frac{h\phi}{\rho^B+h\phi} - \frac{e\phi}{\rho^B+e\phi}\right\} \times B_{w+1}$ and is jointly determined by the continuation value B_{w+1} and the difference in expansion rate under flexible contract ($h\phi$) and under autarky ($e\phi$). A very patient borrower perceives expanding at fast and slow rates to be little different. In other words, holding B_{w+1} constant, higher borrower patience lowers the expansion rent and making her less demanding when being offered restrictive contracts.

Which of these two forces dominates is in general ambiguous, but we show in the proof of Proposition 6, via numerical example, that these forces can resolve in favor of increasing the expansion rent. Thus, in contrast to standard intuition of poverty traps, increasing borrower patience can worsen this poverty trap precisely due to lender's strategic response.

3.3 Poverty Traps Arise Due to Repeated Interactions

Our results thus far highlight a lender-induced poverty trap that is otherwise non-existent under autarky. We now further clarify that the poverty trap exists not because of *lending*—having access to external funds always expands production sets—but because of the *repeated* nature of lending relationships. The lender has an incentive to trap the borrower only because of future rents, and the trap completely disappears if the relationship is short-lived.

To demonstrate this, we extend the model such that in any state the borrower and lender cease interacting with Poisson intensity q. When such a shock arises, the lender's continuation payoff is 0, but the borrower is matched to a new, identical lender. Our baseline model corresponds to the case $q = 0$, i.e. the Poisson rate of relationship breakage is zero. At the other extreme, as $q \to \infty$, the model becomes a continuous-time game between a long-run player (the borrower) and an infinite stream of short-run players (lenders). Each lender interacts only once with the borrower. Discussion of the formal setup can be found in Appendix A.4.

Proposition 7. *As $q \to \infty$, the lender always offers unrestrictive contracts.*

The lender in our model always myopically prefers to offer unrestrictive contracts, as the borrower will accept higher interest rates when contracts are unrestrictive. The lender's incentive to offer restrictive contracts comes from the prospect of interacting with the borrower in the future. Increasing q diminishes such dynamic considerations and raises the lender's incentive to offer unrestrictive contracts in any state. At the limit $q \to \infty$, each lender only interacts with the borrower for a vanishing amount of time, and, consequently, only unrestrictive contracts are offered in equilibrium.

This result provides a counterpoint to the standard intuition that repeated interaction enables cooperation and promotes socially efficient behavior. Our model highlights that, under motives to maintain dynamic rents, repeated interaction strictly lowers welfare.

3.4 Competition and Its Nuanced Effects on the Poverty Trap

Our final set of results analyze how competition from other lenders affects the nature of equilibrium contracts. We consider both dynamic and instantaneous aspects of competition. Standard intuitions suggest that competition among lenders should benefit borrowers. We show, because of the lender's endogenous response in changing contractual types, competition has nuanced effects on both equilibrium contracts and borrower value. Poverty traps may worsen, and the borrower may become worse off as competition intensifies.

Dynamic Competition To analyze dynamic competition, we extend the analysis in Section 3.3 and allow for the Poisson rate of relationship breakage to be state dependent. We interpret higher q_w as state w becoming more dynamically competitive: any lending relationship for businesses of size

w breaks with higher intensity, and after any breakage the incumbent lender's continuation payoff becomes zero as the borrower meets a new, identical lender.

Introducing state-dependent dynamic competition has nuanced effects on equilibrium contracts.

Proposition 8. *Start from an equilibrium under arbitrary $\{q_w\}$. The equilibrium contractural structure may feature disjoint regions of restrictive contracts. Raising $q_{w'}$ in any state w' reduces the probability of restrictive contracts in w', and may increase or decrease the probability of restrictive contracts in prior states.*

In state w', the lender responds to an increase in $q_{w'}$ by reducing the probability he offers restrictive contracts; the increase in competition reduces the value he extracts from borrowers in w' relative that of borrowers in all other states. The lender is made worse off in w' but the borrower's welfare improves, as she receives more unrestrictive contracts. Moving backwards, in state $w'-1$ the change in equilibrium contracts is more nuanced. On the one hand, the lender's incentive to offer unrestrictive contracts diminishes with his value in state w'. On the other hand, the repayment amount the borrower demands from restrictive contracts increases as her value in state w' increases. In general, either of these forces can dominate.

Instantaneous Competition We interpret changes in h, the severity of limited enforcement, as varying instantaneous competition. This interpretation is motivated by the fact that h can be microfounded as the cost of abandoning the current lender and finding a new one instantaneously. Alternatively, we interpret h as the borrower's bargaining power within each instantaneous interaction.

We find that increasing h—the amount of the loan the borrower may keep to herself at every instant—can paradoxically harm her by inducing the lender to tighten contractual restrictions. In effect, the borrower's inability to commit not to exercise her improved bargaining power reduces the lender's desire to offer unrestrictive contracts.

Proposition 9. *Increasing h increases the likelihood of restrictive contracts in state n. In states $w < n$ increasing h may increase or decrease the likelihood of restrictive contracts. How borrower's value changes in response to h is ambiguous in any state.*

Increasing h raises the rate at which the borrower reaches the formal sector when she is offered an unrestrictive loan, without influencing her ability to reach the formal sector on her own. In the final state n, this force makes unrestrictive loans strictly less attractive. Thus, the lender shifts toward restrictive contracts, and his continuation value in state n declines.

In general, how the borrower's continuation value varies with h is ambiguous in all states. In state n, this is because under a higher h, unrestrictive contracts, which yield higher continuation utility than restrictive ones, become even more attractive but are less forthcoming due to lender's endogenous response. The ambiguity in the borrower's continuation value trickles down to all earlier states, leaving the contractual response in these states ambiguous as well.

4 Empirical Evidence and Concluding Remarks

Our model formalizes the intuition that, because informal lenders may not be able to offset the costs of supporting borrower growth by extracting the benefits in the future, they may impose contractual restrictions that inhibit long-term, profitable investments.

Our simple theory is able to organize many of the established facts about microfinance. First, the model reconciles the seemingly inconsistent facts that small-scale entrepreneurs enjoy very high return to capital yet are unable to leverage microcredit and other forms of informal finance to realize those high returns, despite moderate interest rates charged by the lenders (see Banerjee et al. (2015) and Meager (2017) for overviews of the experimental evaluations of microfinance, and see, for instance, De Mel et al. (2008), Fafchamps et al. (2014), Hussam et al. (2017), and McKenzie and Woodruff (2008) for evidence that microentrepreneurs have high return to capital). In our model, firms that borrow from the informal lender may see their growth stalled, and remain in the relationship indefinitely, even though they would have continued to grow in the absence of a lender. Put simply, in this model, having access to a lender can *reduce* business growth.

While the experimental studies cited above find, on average, low marginal returns to credit, a number of them find considerable heterogeneity in returns to credit across borrowers of different size. In particular, they consistently find that treatment effects are higher for businesses that are more established (see Angelucci et al. (2015), Augsburg et al. (2015), Crepon et al. (2015), and Banerjee et al. (2017)). Notably, Banerjee et al. (2015) finds the quantile-treatment effect of micro-credit on business profit is U-shaped. Our model sheds light on this heterogeneity. Very small and very large firms grow faster in the presence of a lender than without, whereas intermediate sized firms may not grow at all due to the lending trap.

One puzzling fact in the microfinance literature is that, despite the fact that loan products carry low interest rates relative to the returns to capital, demand for microcredit contracts is low in a wide range of settings (See e.g. Banerjee et al. (2014), and Banerjee et al. (2015)). Our model offers a novel explanation. Despite low interest rates, contractual restrictions that impose constraints on business growth push borrowers in the restrictive region exactly to their individual rationality constraints; these borrowers do not benefit at all from having access to informal lending. In Appendix A.1, we formalize low take-up rate of informal loans through a model extension in which the lender is incompletely informed about the borrower's outside option. While our argument is intuitive, it stands in sharp contrast to standard intuitions based on borrower-side financial constraints, which predict that credit constrained borrowers should have high demand for additional credit at the market interest rate.

Our theory also offers nuanced predictions on the poverty trap as the lending environment changes. Increasing formal sector attractiveness improves the bargaining position of rich borrowers, increases their welfare, and relaxes the poverty trap. However, the same improvement may harm the welfare of poorer borrowers; anticipating that rich borrowers have improved bargaining positions, the lender shifts towards restrictive loans for poor borrowers to prevent them from reaching higher levels of wealth and exploiting their improved positions. Similarly and counter to standard intuitions, increasing the borrower's patience (and hence her value for business expansion) can make relatively poor borrowers worse off and tighten the poverty trap. Also, shutting down the repeated nature of borrower-lender interactions unambiguously eliminates the poverty trap, whereas marginally introducing competition from other lenders may not help borrowers and may worsen the poverty trap.

In addition to the theories cited in the introduction, it is worth contrasting our theory with two other classes of theories prominent in development economics. The first might sensibly be labeled "blaming the borrower." These theories allude to the argument that many borrowers are not natural entrepreneurs and are primarily self-employed due to a scarcity of steady wage work (see e.g., Schoar (2010)). While these theories have some empirical support, they are at best a partial explanation of the problem as they are inconsistent with the large impacts of cash grants, as cited in our introduction.

Second are the theories that assign blame to the lender for not having worked out the right lending contract. These theories implicitly guide each of the experiments that evaluate local modifications to standard contracts (see e.g. Gine and Karlan (2014), Attanasio et al. (2015), and Carpena et al. (2013) on joint liability, Field et al. (2013) on repayment flexibility, and Feigenberg et al. (2013) on meeting frequency). While many of these papers contribute substantially to our understanding of how microfinance operates, none have so far generated a lasting impact on the models that MFIs employ.

Our theory, in contrast, assumes that borrowers have the competence to grow their business and that lenders are well aware of the constraints imposed on borrowers by the lending paradigm. Instead we focus on the rents that lenders enjoy from retaining customers and the fact that sufficiently wealthy customers are less reliant on their informal financiers. Part of this theory's value, therefore, may very well be its distance from the main lines of reasoning maintained by empirical researchers.

References

Angelucci, Manuela, Dean Karlan, and Jonathan Zinman (2015). Microcredit impacts: Evidence from a randomized microcredit program placement experiment by compartamos banco. *American Economic Journal: Applied Economics* 7(1), 151–182.

Attanasio, Orazio, Britta Augsburg, Ralph De Haas, Emla Fitzsimons, and Heike Harmgart (2015). The impacts of microfinance: Evidence from joint-liability lending in mongolia. *American Economic Journal: Applied Economics* 7(1), 90–122.

Augsburg, Britta, Ralph De Haas, Heike Harmgart, and Costas Meghir (2015). The impacts of microcredit: Evidence from bosnia and herzegovina. *American Economic Journal: Applied Economics 7*(1), 183–203.

Azariadis, Costas (1996). The economics of poverty traps part one: Complete markets. *Journal of Economic Growth*.

Banerjee, Abhijit, Timothy Besley, and Timothy W. Guinnane (1994). Thy neighbor's keeper: the design of a credit cooperative with theory and a test. *Quarterly Journal of Economics*, 491–515.

Banerjee, Abhijit, Emily Breza, Esther Duflo, and Cynthia Kinnan (2017). Do credit constraints limit entrepreneurship? heterogeneity in the returns to microfinance. *Working Paper*.

Banerjee, Abhijit, Esther Duflo, Rachael Glennerster, and Cynthia Kinnan (2015). The miracle of microfinance? evidence from a randomized evaluation. *American Economic Journal: Applied Economics 7*(1), 22–53.

Banerjee, Abhijit, Esther Duflo, and Richard Hornbeck (2014, May). Bundling health insurance and microfinance in india: There cannot be adverse selection if there is no demand. *American Economic Review 104*(5), 291–297.

Banerjee, Abhijit, Dean Karlan, and Jonathan Zinman (2015). Six randomized evaluations of microcredit: Introduction and further steps. *American Economic Journal: Applied Economics 7*(1), 1–21.

Barboni, Giorgia and Parul Agarwal (2018). Knowing what's good for you: Can a repayment flexibility option in microfinance contracts improve repayment rates and business outcomes? *Working Paper*.

Becker, Gary (1962). Investment in human capital: A theoretical analysis. *Journal of Political Economy 70*(5), 9–49.

Bhaduri, Amit (1973). A study in agricultural backwardness under semi-feudalism. *The Economic Journal 83*(329), 120–137.

Bhaduri, Amit (1977). On the formation of usurious interest rates in backward agriculture. *Cambridge Journal of economics 1*(4), 341–352.

Bizer, David S. and Peter M. DeMarzo (1992). Sequential banking. *Journal of Political Economy*, 41–61.

Braverman, Avishay and Thirukodikaval N. Srinivasan (1981). Credit and sharecropping in agrarian societies. *Journal of Development Economics 9*(3), 289–312.

Braverman, Avishay and Joseph E. Stiglitz (1982). Sharecropping and the interlinking of agrarian markets. *American Economic Review 72*(4), 695–715.

Breza, Emily (2012). Peer effects and loan repayment: Evidence from the krishna default crisis. *Working Paper*.

Brunnermeier, Markus K. and Martin Oehmke (2013). The maturity rat race. *The Journal of Finance 68*(2), 483–521.

Carpena, Fenella, Shawn Cole, Jeremy Shapiro, and Bilal Zia (2013). Liability structure in small-scale finance: Evidence from a natural experiment. *World Bank Economic Review 27*(3), 437–469.

Chaudhury, Lakita and Anand Swamy (2012). Microfinance and predatory lending: The same old story?

Crepon, Bruno, Florencia Devoto, Esther Duflo, and William Pariente (2015). Estimating the impact of microcredit on those who take it up: Evidence from a randomized experiment in morocco. *American Economic Journal: Applied Economics 7*(1), 123–150.

De Mel, Suresh, David McKenzie, and Christopher Woodruff (2008). Returns to capital in microenterprises: evidence from a field experiment. *Quartery Journal of Economics 123*(4), 1329–1372.

Dell'Ariccia, Giovanni and Robert Marquez (2006). Lending booms and lending standards. *The Journal of Finance 61*(5), 2511–2546.

Drechsler, Itamar, Alexi Savov, and Philipp Schnabl (2017). The deposit channel of monetary policy. *Quarterly Journal of Economics*.

Fafchamps, Marcel, David McKenzie, and Christopher Woodruff (2014). Microenterprise growth and the flypaper effect: Evidence from a randomized experiment in ghana. *Journal of Development Economics 106*, 211–226.

Feigenberg, Benjamin, Erica Field, and Rohini Pande (2013). The economic returns to social interaction: Experimental evidence from microfinance. *Review of Economic Studies 80*(4), 1459–1483.

Field, Erica, Rohini Pande, John Papp, and Natalia Rigol (2013). Does the classic microfinance model discourage entrepreneurship? experimental evidence from india. *American Economic Review 103*(6), 2196–2226.

Fischer, Greg (2013). Contract structure, risk sharing, and investment choice. *Econometrica 81*(3), 883–939.

Fishman, Michael J. and Jonathan Parker (2015). Valuation, adverse selection and market collapses. *The Review of Financial Studies 28*(9), 2575 – 2607.

Flintoff, Corey (2010). India's poor reel under microfinance debt burden.

Ghatak, Maitreesh (1999). Group lending, local information, and peer selection. *Journal of Development Economics 60*(1), 27–50.

Gine, Xavier and Dean Karlan (2014). Group versus individual liability: Short and long term evidence from philippine microcredit lending groups. *Journal of Development Economics 107*, 65–83.

Green, Daniel and Ernest Liu (2017). Growing pains in financial development: Institutional weakness and investment efficiency. *Working Paper*.

He, Zhiguo and Wei Xiong (2013). Delegated asset management, investment mandates, and capital immobility. *Journal of Financial Economics*.

Hussam, Reshma, Natalia Rigol, and Benjamin N. Roth (2017). Targeting high ability entrepreneurs using community information: Mechanism design in the field. *Working Paper*.

Jack, William, Michael Kremer, Joost de Laat, and Tavneet Suri (2016). Joint liability, asset collateralization, and credit access: Evidence from rainwater harvesting tanks in kenya. *Working Paper*.

Jakiela, Pamela and Owen Ozier (2016). Does africa need a rotten kin theorem? experimental evidence from village economies. *Review of Economic Studies*.

Maitra, Pushkar, Sandip Mitra, Dilip Mookherjee, Alberto Motta, and Sujata Visaria (2017). Financing smallholder agriculture: An experiment with agent-intermediated microloans in india. *Journal of Development Economics 127*, 306–337.

Maskin, Eric and Jean Tirole (2001). Markov perfect equilibrium: I. observable actions. *Journal of Economic Theory 100*(2), 191–219.

McKenzie, David and Christopher Woodruff (2008). Experimental evidence on returns to capital and access to finance in mexico. *World Bank Economic Review 22*(3), 457–482.

Meager, Rachael (2017). Aggregating distributional treatment effects: A bayesian hierarchical analysis of the microcredit literature. *Working Paper*.

Melik, James (2010). Microcredit 'death trap' for bangladesh's poor.

Parlour, Christine and Uday Rajan (2001). Competition in loan contracts. *American Economic Review*.

Petersen, Mitchell A. and Raghuram G. Rajan (1995). The effect of credit market competition on lending relationships. *Quartery Journal of Economics*, 407–443.

Ray, Debraj (1998). *Development Economics*. Princeton University Press.

Sainath, Palagummi (1996). *Everybody loves a good drought: stories from India's poorest districts*. Penguin Books India.

Scharfstein, David and Ali Sunderam (2017). Market power in mortgage lending and the transmission of monetary policy. *Working Paper*.

Schoar, Antoinette (2010). The divide between subsistence and transformational entrepreneurship. In Josh Lerner and Scott Stern (Eds.), *Innovation Policy and the Economy, Volume 10*, Volume 10, pp. 57–81. University of Chicago Press.

Stiglitz, Joseph E. and Andrew Weiss (1981). Credit rationing in markets with imperfect information. *American Economic Review 71*(3), 393–410.

Tirole, Jean (2010). *The Theory of Corporate Finance*. Princeton University Press.

Additional Material and Proofs (For Online Publication)

Section A provides several extensions to the baseline model, including 1) incorporating borrower private information about outside option; 2) allow the borrower to maintain flexible control over a fraction of her income even when subjected to contractual restrictions; 3) allowing the borrower and lender to have different discount rates; 4) allowing lending relationships to break down stochastically; 5) removing the repayment ceiling to demonstrate that expansion rents can be modeled with the borrower's private information about her productivity; 6) allowing for arbitrary production functions.

In Section B we provide additional discussions on how our modeling assumptions map to the institutional features of microfinance.

All proofs are in Section C.

A Additional Extensions to the Model

A.1 The Borrower is Privately Informed About Her Outside Option

In this section we explore an extension in which the borrower maintains some private information about her outside option. In particular, we augment the model such that the borrower's autarkic endowment is privately known. If she rejects the lender's contract, she receives an endowment of $e+v_t$. Let $v_t \overset{\text{iid}}{\sim} G$ be a random variable, privately known to the borrower and redrawn each period in an iid manner from some distribution G. Further, assume that if the borrower accepts the lender's contract, her endowment is still $e+t_w$. One way to understand this is that in the event that the borrower rejects the lender's contract, a relative will offer her a gift of size v_t, which she can allocate flexibly between her projects. We make the following additional assumption on the range of G to simplify the discussion.

Assumption 4. *Let G have bounded support with minimum 0 and maximum $\bar{v}$ such that $\bar{v} < h$.*

The above assumption guarantees that the borrower will accept any unrestrictive contract. However, if the lender offers the borrower a restrictive contract, he will now face a standard screening problem. Because he would like to extract the maximum acceptable amount of income, borrowers with unusually good outside options will reject his offer. This is encoded in the following proposition.

Proposition 10. *The borrower may reject restrictive contracts with positive probability.*

This intuitive result offers an explanation for the low takeup of microcredit contracts referenced in the introduction. Lenders who offer restrictive contracts to borrowers aim to extract the additional income generated by the loan, but in doing so lenders are sometimes too demanding and therefore fail to attract the borrower. In contrast, because lenders who offer unrestrictive contracts necessarily leave the borrower with excess surplus, demand for these contracts is high.

A.2 The Borrower Flexibly Allocates a Fraction of Her Income

In this section we explore an extension to the model in which, even when subjected to contractual restrictions, the borrower maintains flexible control over a fraction of her income. In doing so we aim to show that our main result is qualitatively robust. Rather than finding that the borrower may remain inefficiently small forever, we now find that having access to a lender may *slow* the borrower's growth relative to her autarkic rate.

Formally, the model is as in Section 1 but after accepting a contract $\langle t, R, a \rangle$, the borrower is free to invest an amount e' of her endowment flexibly, irrespective of the contractual restriction a the lender imposes. Thus we have weakened the lender's ability to influence the borrower's project choice. We maintain all other assumptions from Sections 1 and 2, and make the following observation.

Proposition 11. *For $e' < e$, the lender may offer a restrictive contract on the equilibrium path. In such an equilibrium, the borrower reaches the formal sector in finite time, but will grow more slowly than she would have in autarky.*

A.3 The Borrower and Lender Have Different Discount Rates

In this extension, the model is just as in Section 1, except the borrower and lender have different discount rates, ρ^B and ρ^L. In this variant of the model, the borrower gets stuck in state w if a variant of the inequality identified in Lemma 3 is satisfied. If the borrower receives a restrictive contract with probability 1, her state w value is $\frac{e\phi}{\rho^B+e\phi}B_{w+1}$. If she receives an unrestrictive contract with probability 1 her state w value is $\frac{\max\{e,h\}\phi}{\rho^B+\max\{e,h\}\phi}B_{w+1}$. Following the derivation of Lemma 3 it is straightforward to verify that the lender offers restrictive contracts with probability 1 if and only if

$$\begin{aligned}
&\frac{\max\{e,h\}\phi}{\rho^L+\max\{e,h\}\phi}\left(L_{w+1}-\frac{(z-1)K_w+e}{\rho}-1/\phi\right)+\frac{\max\{e,h\}\phi}{\rho^B+\max\{e,h\}\phi}B_{w+1} \\
\leq\ &\left(\frac{\max\{e,h\}\phi}{\rho^B+\max\{e,h\}\phi}-\frac{e\phi}{\rho^B+e\phi}\right)B_{w+1}.
\end{aligned}$$

A.4 The Borrower and Lender Stochastically Cease Interacting

In this extension, the model is just as in Section 1, except in each period the borrower and lender cease interacting according to a Poisson process with arrival intensity q. In such an event, the lender receives a continuation value of 0 but the borrower is matched to a new lender. This effectively introduces a wedge between the lender's discount rate and the borrower's. The borrower's discount rate remains at ρ while the lender's effective discount rate is now $\rho + q$. The model therefore works in the same way as the extension above with different discount rates.

A.5 Expansion Rents Absent Repayment Ceiling

In this section we provide a model extension where we remove the repayment ceiling in Equation 1. We show that the borrower may still enjoy an expansion rent if she has private information about her productivity. Since the lender cannot perfectly set the repayment amount to extract the full benefit of the loan, the borrower is left with higher welfare from unrestrictive contracts than from restrictive contracts.

Specifically, we set $h = e$ (which is equivalent to removing the ceiling as it never binds) and make the following change to the model in Section 1: we assume z is a random variable, drawn i.i.d. across periods and privately known to the borrower. Let

$$z = \begin{cases} z_1 & \text{with probability } p \\ z_2 & \text{with probability } 1-p \end{cases}$$

with $z_2 > z_1$. For p sufficiently close to 1 the lender's optimal restrictive contract is $\langle K_w - e, z_1 K_w - \rho B_w, 1\rangle$. The borrower's continuation value in state w if she receives restrictive contracts with certainty is

$$\alpha B_{w+1} + (1-p)\frac{(z_2 - z_1) K_w}{\rho}$$

That is, the lender sets the highest repayment amount that the borrower with productivity z_1 would accept, letting the borrower with productivity z_2 enjoy consumption surpassing her outside option.

Similarly, for p sufficiently close to 1, the lender's optimal unrestrictive contract is $\langle K_w - e, z_1 (K_w - e), 0\rangle$. The borrower's continuation value in state w if she receives unrestrictive contracts with probability 1 is

$$\beta' B_{w+1}$$

where $\beta' \equiv \frac{\left(e+(1-p)\frac{z_2-z_1}{z_2}K_w\right)\phi}{\left(e+(1-p)\frac{z_2-z_1}{z_2}K_w\right)\phi+\rho}$. That is, when the borrower receives an unrestrictive contract, her business grows at rate $\left(e+(1-p)\frac{z_2-z_1}{z_2}K_w\right)\phi > e\phi$, as with probability $(1-p)$, she need not invest the lender's full transfer in order to meet her repayment obligation. The borrower's expansion rent is therefore

$$\beta' B_{w+1} - \left(\alpha B_{w+1} + \frac{(1-p)(z_2 - z_1) K_w}{\rho} \right)$$

and when the expansion rent exceeds the change in joint surplus from allowing the borrower to expand at rate β', the lender offers only restrictive contracts in equilibrium.

A.6 Arbitrary Production Functions and Countably Infinite States

In this section we argue that the key intuitions highlighted above survive substantial generalization of the production function and other parameters of the model, and extension to a countably infinite number of states.

First, we abandon our decreasing returns Assumption 1 and so allow K_w to fluctuate arbitrarily across states. We now also index the autarky endowment e_w, the rate of fixed investment growth ϕ_w, and the repayment wedge h_w by the state and make no assumptions about them other than Assumption 3 above, which guarantees that in autarky the borrower reaches the formal sector in finite time.

Structure of the Equilibrium

First, the proof of Proposition 1, which states that the equilibrium is unique, did not rely on any of the above assumptions, and thus goes through unchanged. In this section we discuss the structure of the unique equilibrium. A typical equilibrium is depicted below, with each circle representing a state and shaded circles representing states in which restrictive contracts are offered.

Even though in general we cannot say anything about the organization of restrictive and unrestrictive states, we argue that many of the empirical facts discussed in Section 3 can still be understood through the equilibrium above. In fact, with the exception of heterogeneity in returns to credit, our explanation of the facts in that discussion only depends on the potential for each type of contract to coexist in a single equilibrium. As such, we focus our attention for the remainder of this discussion on the prediction that wealthy borrowers will receive unrestrictive contracts and thus will enjoy high returns to credit.

To do so, we first outline how to transfer the insights in the above model to one with a countably infinite number of states. Given that our results do not depend on the number of states n, or the cost of investment ϕ_w, this is a straightforward task. We define a sequence of games satisfying the above assumptions, each with successively more states.

Let Γ^1 be an arbitrary game with n business states.

For $m > 1$, let Γ^m be constructed in the following way:

- Γ^m has $2^{m-1}n$ business states, and let K_w^m, e_w^m, h_w^m, and ϕ_w^m be the corresponding parameters for game Γ^m.
- If w is even, set $K_w^m = K_{w/2}^{m-1}$, $e_w^m = e_{w/2}^{m-1}$, and $h_w^m = h_{w/2}^{m-1}$.
- If w is odd, set
$$K_w^m \in \left[\min\left\{K_{w-1}^m, K_{w+1}^m\right\}, \max\left\{K_{w-1}^m, K_{w+1}^m\right\}\right]$$
$$e_w^m \in \left[\min\left\{e_{w-1}^m, e_{w+1}^m\right\}, \max\left\{e_{w-1}^m, e_{w+1}^m\right\}\right]$$
and
$$h_w^m \in \left[\min\left\{h_{w-1}^m, h_{w+1}^m\right\}, \max\left\{h_{w-1}^m, h_{w+1}^m\right\}\right]$$
- If w is even, set $\phi_w^m = \frac{\phi_{w/2}^{m-1}}{2}$, and if w is odd, set $\phi_w^m = \frac{\phi_{(w-1)/2}^{m-1}}{2}$.

Thus Γ^m has twice as many states at Γ^{m-1}, and even states in Γ^m correspond to states in Γ^{m-1}. The parameters in odd states take values intermediate to those in the surrounding states. Because the cost of investment in Γ^m is only half of that in Γ^{m-1}, a borrower investing in fixed capital at the same rate in either game would reach the formal sector in the same expected time.

One way to understand Γ^m relative to Γ^{m-1} is that the borrower and lender appreciate more nuanced differences in the borrower's business size. Holding investment rate fixed, it takes the same amount of time to get from w to $w+2$ in Γ^{m+1} as it does to get from $\frac{w}{2}$ to $\frac{w}{2}+1$ in Γ^m, but along the way in Γ^m the borrower and lender realize an intermediate production function change. For $m' > m$, we say $\Gamma^{m'}$ is *descended from* Γ^m if there is a sequence of games $\Gamma^m, \ldots, \Gamma^{m'}$ that can be derived in this manner. We have the following result.

Proposition 12. *For any Γ^m, there is an $\bar{m}$ such that for all $m' > \bar{m}$, the equilibrium in any $\Gamma^{m'}$ descended from Γ^m features a $\tilde{w}$ such that for $w \geq \tilde{w}$ the borrower reaches the formal sector in finite time starting from state w if it is socially efficient to do so.*

The above result says that for any game with sufficiently fine discrimination between states, all sufficiently wealthy borrowers receive unrestrictive contracts in equilibrium and thus realize high returns to credit. The intuition is simple. Because entering the formal sector is efficient, the lender is unable to profitably offer a sufficiently wealthy borrower (i.e. one who is sufficiently near to the formal sector) a restrictive contract she will accept. As the borrower and lender become arbitrarily discerning of different states, there will eventually be business states in which the borrower is indeed sufficiently wealthy.

Comparative statics

As before, we can be fairly precise in describing how the equilibrium changes with respect to various fundamentals of the game. In this section we focus on the comparative static with respect to U.

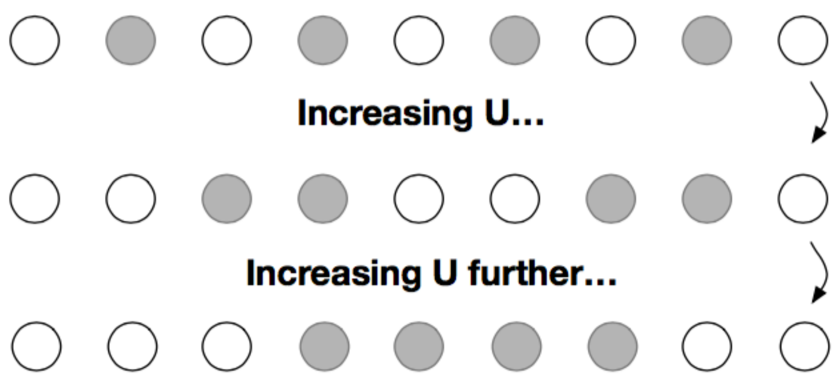
Note that without loss of generality we can identify m disjoint, contiguous sets of states $\{\underline{w}_1, \dots, \bar{w}_1\}$, ..., $\{\underline{w}_m, \dots, \bar{w}_m\}$ such that $\bar{w}_m = \max\{w : p_w = 1\}$, $\underline{w}_m = \max\{w : p_w = 1, p_{w-1} < 1\}$, and in general for $k \geq 1$, $\bar{w}_k = \max\{w < \underline{w}_{k+1} : p_w = 1\}$ $\underline{w}_k = \max\{w \leq \bar{w}_k : p_w = 1, p_{w-1} < 1\}$. An arbitrary set $\{\underline{w}_k, \dots, \bar{w}_k\}$ is a contiguous set of states where restrictive contracts are offered with probability 1, and each pure restrictive state is contained in one of these sets. Let $\kappa_w \equiv e_w \phi_w$ and $\gamma_w \equiv \max\{e_w, h_w\} \phi_w$.

We consider an impatient borrower and establish the following result.

Proposition 13. *For impatient borrowers, the regions of contiguous restrictive states merge together as the formal sector becomes more attractive.*

That is, for $\rho > \max_w \frac{\kappa_w \gamma_w}{\kappa_w + \gamma_w}$, $\frac{dp_w}{du} < 0$ *for* $w \in \{\bar{w}_m + 1, n\}$, $\frac{dp_w}{du} > 0$ *for* $w \in \{\bar{w}_{m-1} + 1, \underline{w}_m - 1\}$, $\frac{dp_w}{du} < 0$ *for* $w \in \{\bar{w}_{m-2} + 1, \underline{w}_{m-1} - 1\}$*, and so on.*

Proposition 13 states that the highest region of pure restrictive states moves leftward, the second highest region moves rightward, and so on. This is depicted in the following figure.



The intuition is as follows. For $(\underline{w}_m \bar{w}_m)$, the analysis exactly follows that of Section 3.1, and hence it shifts leftward as u increases. But recall that for the impatient borrowers to the left of a restrictive state, the leftward shift *lowers* their utility. This is akin to lowering the utility of entering the formal sector, and so for the next set of restrictive states $(\underline{w}_{m-1} \bar{w}_{m-1})$ the analysis reverses, and $\bar{w}_{m-1}$ and $\underline{w}_{m-1}$ shift rightward. The rest follows by backward induction.

As u increases, the contiguous groups of restrictive states merge together. Eventually they either merge into a single contiguous group of restrictive states or disappear altogether. If they merge into one group, the comparative statics work exactly as they did in Section 3.1.

B Microfinance Institutional Features Not Captured by the Model

In this section we provide additional discussions on how our modeling assumptions map to the institutional features of microfinance.

Non-profit MFIs

The debt trap in our model arises when a profit-maximizing lender prolongs the period over which he can extract rents from his borrower. Yet the low returns from microfinance have been observed across a range of microfinance institutions spanning both for-profit and non-profit business models. We believe there are a number of ways in which the forces identified in our model might similarly apply to non-profit MFIs. First, because the two business models share many practices, features that are adaptive for profit-maximizing MFIs may have been adopted by non-profits. A second possibility operates through the incentives of loan officers who are in charge of originating and monitoring loans. Across for-profit and non-profit MFIs many loan officers are rewarded for the number of loans they manage, so losing clients through graduation may not be in their self-interest. Put another way, even in non-profit MFIs, the loan officers often have incentives that make them look like profit-maximizing lenders.

Infinite stream of borrowers

One crucial feature of our debt trap is that when the borrower becomes wealthy enough to leave her lender, the lender loses money. In reality there are many unserved potential clients in the communities in which MFIs operate. Why, then, can't an MFI offer unrestrictive contracts and then replace borrowers who have graduated with entirely new clients? The proximate answer is that unserved clients are unserved primarily because they have no demand for loans (e.g. Banerjee et al. (2014)). This may be unsatisfactory, as demand for microfinance would presumably increase if the lender lifted contractual restrictions and allowed the borrowers to invest more productively. But even in this case, the pool of potential borrowers who would find this appealing is likely limited. For instance, Banerjee et al. (2017) and Schoar (2010) argue that only a small fraction of small-scale entrepreneurs are equipped to put capital to productive use. Thus, it is reasonable to assume that MFIs lose money when a borrower terminates the relationship.

Microentrepreneurs do not enjoy compounded growth

The high marginal return to capital observed in the cash drop studies cited above should not be confused with explosive business growth. Small-scale entrepreneurs often put cash to good use but seem not to reinvest it in their business for compounded growth. If these kinds of investments are unlikely to transform small-scale businesses into businesses large enough for formal loans, perhaps an MFI should be eager to support them. We argue this is not the case.

Recall that entry into the formal sector is merely a metaphor for a broader contracting friction endemic in informal markets. Any investment a borrower makes such that

1. the borrower can't credibly pledge the returns to the lender, and

2. the borrower will become less reliant on the lender with positive probability

will be discouraged in the same way. We argue that both features are satisfied by investments that make the borrower permanently more productive but nevertheless too small to benefit from formal loans.

Competition among microlenders

Important to our model is that the borrower has no other lender to turn to when the monopolist offers her a restrictive contract. Yet many microfinance institutions operate in close proximity to one another. While microfinance may appear to be a highly competitive market in an ex-ante sense (i.e. for new borrowers), we argue that microfinance institutions gain private information about their borrowers over the course of the lending relationship. Indeed this is one of the primary reasons cited for the increasing loan profile across borrowing cycles. The information rents that lenders enjoy over their existing clients induces imperfect competition ex-post. It is straightforward to show that the forces we identify above survive in an (ex-post) imperfectly competitive market.

C Omitted Proofs

Throughout this appendix, we will use $\kappa \equiv e\phi$ to represent the speed at which the borrower invests in fixed capital in autarky and $\gamma \equiv \max\{e,h\}\,\phi$ to denote the speed at which the borrower invests in fixed capital when she receives a maximally extractive unrestrictive contract in equilibrium. We will use $\delta(p) \equiv p\kappa + (1-p)\gamma$ to refer to convex combinations of these rates of growth.

Proof of Lemma 1

Suppose in equilibrium, the lender offers the restrictive contract $\langle t_w, R_w, 1\rangle$ with probability 1 in state w. That $t_w = K_w - e$ follows from the fact that if t_w were less than $K_w - e$ the lender could improve his payoff by increasing t_w by some $\varepsilon > 0$ and increasing R by $z\varepsilon$. This new contract would leave the borrower with the same surplus and would thus be acceptable. That $R_w = zK_w - \rho B_w$ follows from the fact that the lender sets the repayment at the highest level that surpasses the borrower's outside option.

It remains to show that $B_w = \alpha B_{w+1}$. First, note that if (off the equilibrium path) the borrower were to reject the lender's contract, she would invest her entire endowment e in fixed capital. This follows from the fact that if the borrower had a weak preference to invest her autarkic endowment in working capital, the borrower would also accept a restrictive contract with the highest feasible repayment, $R_w = z(K_w - \max\{e,h\})$. In such an equilibrium, the borrower's value would be $B_w = \frac{z\max\{e,h\}}{\rho}$, which, by Assumption 3, would contradict the borrower's weak preference for working capital.

Hence the borrower's outside option is to invest e into the fixed capital project. Because the lender sets repayment R_w such that the borrower is indifferent between accepting the contract versus her outside option, the borrower's continuation utility is $B_w = \alpha B_{w+1}$.

Proof of Lemma 2

We first demonstrate that for an unrestrictive contract $\langle t_w, R_w, 0\rangle$ offered in equilibrium, $t_w > h - e$. Suppose not. Then the lender necessarily sets repayment $R_w = 0$. In order to justify the lender's negative flow-payoff, the borrower must invest her resources in fixed capital and the lender's equilibrium value function L_w must be increasing in the borrower's business size. But in this case, because of the linearity of production, the lender would prefer to make an arbitrarily large transfer to the borrower, contradicting that $t_w < h - e$.

That the borrower's continuation utility $B_w \geq \beta B_{w+1}$ follows from the fact an unrestrictive contract with $t_w > h - e$ leaves the borrower with at least $\max\{h,e\}$ resources that she could invest in the fixed capital project.

Proof of Lemma 3

If the lender offers a restrictive contract with probability 1 in state w, his continuation utility is

$$L_w^R = \frac{zK_w - (K_w - e)}{\rho} - \alpha B_{w+1}$$

That is, by Lemma 1, the lender takes the full surplus of production minus αB_{w+1}. If instead he offers an unrestrictive contract in state w, his continuation payoff is

$$L_w^U = (1-\beta)\frac{(zK_w - (K_w - e + \max\{h,e\}))}{\rho} + \beta L_{w+1}$$

Now,

$$
\begin{aligned}
L_w^R &\geq L_w^U \\
&\iff \\
\frac{zK_w-(K_w-e)}{\rho}-\alpha B_{w+1} &> (1-\beta)\frac{(zK_w-(K_w-e)-\max\{h,e\})}{\rho}+\beta L_{w+1} \\
&\iff \\
-\alpha B_{w+1} &> -(1-\beta)\frac{\max\{h,e\}}{\rho}+\beta\left(L_{w+1}-\frac{zK_w-(K_w-e)}{\rho}\right) \\
&\iff \\
(\beta-\alpha)B_{w+1} &> \beta\left(B_{w+1}+L_{w+1}-\frac{(z-1)K_w+e}{\rho}-\frac{1}{\phi}\right)
\end{aligned}
$$

which completes the proof.

Proof of Proposition 1

The existence of an equilibrium follows standard arguments (see Maskin and Tirole (2001)). In this section we prove that generically the equilibrium is unique. We do so by backward induction on the state. We first consider equilibrium behavior in state n:

Lemma 1 established that the lender never offers the borrower an excessively generous restrictive contract. So, in equilibrium, the lender either offers the borrower the contract $\langle K_w-e+\max\{e,h\},zK_w,0\rangle$ or the contract $\langle K_w-e,R(p),1\rangle$ for some $R(p)$ that pushes the borrower to her outside option. Now conjecture that in equilibrium the lender offers the borrower a restrictive contract with probability p.

Noting that the borrower receives the maximum of the utility from investing her outside option in fixed capital and from consuming $z\max\{e,h\}$ upon receiving a restrictive contract, we have

$$
\begin{aligned}
B_n(p) &= p\left(\max\left\{e^{-\rho dt}\left(\left(1-e^{-\kappa dt}\right)\frac{u}{\rho}+e^{-\kappa dt}B_n(p)\right), z\max\{e,h\}dt+e^{-\rho dt}B_n(p)\right\}\right) \\
&\quad +(1-p)\left(e^{-\rho dt}\left(\left(1-e^{-\gamma dt}\right)\frac{u}{\rho}+e^{-\gamma dt}B_n(p)\right)\right)
\end{aligned}
$$

$$
B_n(p)=\max\left\{\frac{p\left(1-e^{-\kappa dt}\right)+(1-p)\left(1-e^{-\gamma dt}\right)}{1-pe^{-(\rho+\kappa)dt}-(1-p)e^{-(\rho+\gamma)dt}}e^{-\rho dt}\frac{u}{\rho}, \frac{pz\max\{e,h\}dt+(1-p)e^{-\rho dt}\left(1-e^{-\gamma dt}\right)\frac{u}{\rho}}{1-pe^{-\rho dt}-(1-p)e^{-(\rho+\gamma)dt}}\right\}
$$

where, recall, $\kappa\equiv\phi e$, and $\gamma\equiv\phi\max\{e,h\}$. It is straightforward to verify that $\frac{dB_n(p)}{dp}<0$. This is intuitive as a restrictive contract pushes the borrower to her individual rationality constraint (if possible), whereas an unrestrictive contract does not.

The highest possible repayment rate $R(p)$ that can be required for a restrictive contract is pinned down by the borrower's individual rationality constraint

$$(zK_n - R(p))\,dt + e^{-\rho dt} B_n(p) \geq e^{-\rho dt}\left(\left(1 - e^{-\kappa dt}\right)\frac{u}{\rho} + e^{-\kappa dt} B_n(p)\right)$$
$$\Longrightarrow$$
$$zK_n - R(p) = \max\left\{\frac{e^{-\rho dt}}{dt}\left(1 - e^{-\kappa dt}\right)\left(\frac{u}{\rho} - B_n(p)\right), z\max\{e,h\}\right\}$$

The maximal acceptable repayment rate is increasing in $B_n(p)$—this is intuitive as the higher is the borrower's continuation value in state n, the less she values investment.[16]

Now, consider the lender's decision of whether to offer an unrestrictive or restrictive contract. Fixing the borrower's expectation that the lender offers a restrictive contract with probability p, in any period in which the lender offers an unrestrictive contract his utility is

$$(zK_n - (K_n - e) - \max\{e,h\})\,dt + e^{-\rho dt}\left(1 - e^{-\gamma dt}\right) L_n(p)$$

If he offers a restrictive contract his utility is

$$(R(p) - (K_n - e))\,dt + e^{-\rho dt} L_n(p)$$

So he offers a restrictive contract if and only if the following incentive compatibility constraint holds:

$$\begin{array}{rcl} (R(p) - (K_n - e))\,dt + e^{-\rho dt} L_n(p) & \geq & (zK_n - (K_n - e) - \max\{e,h\})\,dt + e^{-\rho dt}\left(1 - e^{-\gamma dt}\right) L_n(p) \\ & \Longleftrightarrow & \\ (zK_n - \max\{e,h\} - R(p))\,dt & \leq & e^{-(\rho+\gamma)dt} L_n(p) \end{array}$$

The left hand side is the additional consumption the lender must forgo to persuade the borrower to accept a restrictive contract, and the right hand side is the discounted expected loss the lender incurs from allowing the borrower to invest in fixed capital.

Note that the lender's continuation utility $L_n(p)$ is weakly decreasing in p. This is so because the set of restrictive contracts the borrower will accept is decreasing in p, while the set of unrestrictive contracts is unchanged.

Thus, the left hand side of the above inequality is weakly increasing in p, and the right hand side is weakly decreasing in p. Given the lender's incentive compatibility constraint, we argue that, generically, there can only be one equilibrium level of p.

[16] Note that by Assumption 3, $zK_n - R(1) > z\max\{e,h\}$ for sufficiently small dt, but in general $zK_n - R(p)$ may equal $\max\{e,h\}$ for some $p < 1$.

If at $p = 0$ (pure unrestrictive), the lender's incentive compatibility constraint for unrestrictive contracts is strictly satisfied, i.e.

$$(zK_n - \max\{e,h\} - R(0))\,dt > e^{-(\rho+\gamma)dt}L_n(0) \tag{6}$$

then it will be strictly satisfied for all higher levels of p, thereby contradicting that any $p > 0$ can be supported in equilibrium.

If at $p = 1$ (pure restrictive) the lender's incentive compatibility constraint for restrictive contracts is strictly satisfied, i.e.

$$(zK_n - \max\{e,h\} - R(1))\,dt < e^{-(\rho+\gamma)dt}L_n(1) \tag{7}$$

then it will be strictly satisfied for all lower levels of p, thereby contradicting that any $p < 1$ can be supported in equilibrium.

If neither of the above inequalities holds even weakly, then by the intermediate value theorem there will be a $\bar{p}$ at which

$$(zK_n - \max\{e,h\} - R(\bar{p}))\,dt = e^{-(\rho+\gamma)dt}L_n(\bar{p})$$

Note that when the borrower believes she will receive a restrictive contract with probability $\bar{p}$, the amount of consumption she demands when given a restrictive contract, $zK_n - R(\bar{p})$, is strictly larger than than zh_n (the minimum feasible consumption the lender can leave the borrower). To see this, note that by assumption

$$(zK_n - \max\{e,h\} - R(0))\,dt < e^{-(\rho+\gamma)dt}L_n(0)\,.$$

Now, supposing that $zK_n - R(\bar{p}) = z\max\{e,h\}$, we'd have $L_n(\bar{p}) = L_n(0)$ (because the borrower is willing to accept all feasible contracts in both cases), which would imply that

$$(zK_n - \max\{e,h\} - R(\bar{p}))\,dt < e^{-(\rho+\gamma)dt}L_n(\bar{p})$$

and would contradict that the lender is indifferent between restrictive and unrestrictive contracts. Therefore we know that $zK_n - R(\bar{p}) = z\max\{e,h\}$ and thus $\frac{dR(\bar{p})}{dp} < 0$. At $p > \bar{p}$ the lender will strictly prefer unrestrictive loans and at $p < \bar{p}$ the lender will strictly prefer restrictive loans, contradicting that any $p \neq \bar{p}$ can be supported in equilibrium.[17]

The backward induction step is similar to the proof for state n. The principal distinction is that in a

[17] Note that since both $R(p)$ and $L(p)$ can both be written in terms of exogenous parameters, it will hold generically that neither inequality (6) nor (7) holds with equality.

general state w the lender may offer an unrestrictive contract with lower than necessary repayment, i.e. $\langle t, R, 0\rangle$ with $R < z(e+t-h)$. Importantly, if an equilibrium exists in which such a contract is offered in state w, then the same contract is offered in state w across all equilibria. Suppose to the contrary that there was a second equilibrium in which the lender offered a restrictive contract (with maximum acceptable interest rate). In the second equilibrium, the borrower would have a lower continuation value in state w and hence demand a lower repayment amount in exchange for the restrictive contract (because such a contract would keep her in state w). In this second equilibrium the lender would find it strictly less attractive to offer a restrictive contract, and all unrestrictive contracts would still be acceptable. Hence the lender would prefer to offer the unrestrictive contract $\langle t, R, 0\rangle$.[18]

The remaining case is the one in which the lender either offers the borrower the contract $\langle K_w - e + \max\{e, h\}, zK_w, 0\rangle$ or the contract $\langle K_w - e, R(p), 1\rangle$ for some $R(p)$ that pushes the borrower to her outside option. The proof proceeds exactly as it did in state n.

Proof of Proposition 2

We aim to show that in equilibrium the probability the lender offers the borrower a restrictive contract in state w, p_w, is single peaked in w. In particular, we show that $p_{\tilde{w}} < p_{\tilde{w}+1} \implies p_{\tilde{w}-m} \leq p_{\tilde{w}-(m-1)}$ for $m \leq k$ with strict inequality if $p_{\tilde{w}-(m-1)} > 0$:

Assume that in equilibrium the borrower invests her outside option in fixed capital in states $\tilde{w}-1, \tilde{w}$ and $\tilde{w}+1$, and that the lender offers either a maximally extractive, acceptable restrictive or unrestrictive contract (we defer the case in which he offers more generous than necessary unrestrictive contracts to the end of the proof). We begin by defining a function that implicitly determines the equilibrium probability $p_{\tilde{w}}$ that the lender offers the borrower a restrictive contract. To do so, we first determine the borrower's value in state $\tilde{w}$ as a function of the probability p she expects a restrictive contract. This allows us to determine the maximal repayment rate $R(p)$ she is willing to accept for a restrictive contract given the probability p she expects the lender to offer a restrictive contract. Finally, $R(p)$ allows us to determine the lender's payoff from offering restrictive contracts, and by comparing this to his payoff from offering unrestrictive contracts, we pin down the equilibrium probability $p_{\tilde{w}}$.

In state $\tilde{w}$, if in equilibrium the borrower receives a restrictive contract with probability $p_{\tilde{w}}$, her

[18] In states in which the lender offers an unrestrictive contract with lower than necessary repayment, the borrower invests in both fixed and working capital projects: if she invested all of her resources in working capital, the lender would charge a higher repayment, and if she were to invest all of her resources in fixed capital the lender would want to offer an infinitely large transfer (due to the linearity of fixed capital investment cost). While the lender's contract and the expected amount of the borrower's investment in fixed and working capital are both uniquely determined, the exact sequence of investment decisions is not. For instance, the borrower could follow a mixed strategy, randomizing each period between investing everything in working capital and investing everything in fixed capital, or she could follow a deterministic strategy whereby she invests a constant fraction of her resources in fixed capital every period. No matter the strategy she follows, the borrower's expected investment in fixed capital is uniquely determined and must be fixed across all periods.

utility is

$$B_{\tilde{w}}(p_{\tilde{w}}) = e^{-\rho dt}\left(p_{\tilde{w}}\left(\left(1-e^{-\kappa dt}\right)B_{\tilde{w}+1}+e^{-\kappa dt}B_{\tilde{w}}\right)+(1-p_{\tilde{w}})\left(\left(1-e^{-\gamma dt}\right)B_{\tilde{w}+1}+e^{-\gamma dt}B_{\tilde{w}}\right)\right)$$

$$\begin{aligned}
B_{\tilde{w}}(p_{\tilde{w}}) &= \frac{p_{\tilde{w}}\left(e^{-\rho dt}-e^{-(\kappa+\rho)dt}\right)B_{\tilde{w}+1}+(1-p_{\tilde{w}})\left(e^{-\rho dt}-e^{-(\gamma+\rho)dt}\right)B_{\tilde{w}+1}}{1-p_{\tilde{w}}e^{-(\rho+\kappa)dt}-(1-p_{\tilde{w}})e^{-(\rho+\gamma)dt}} \\
&\to \frac{p_{\tilde{w}}\kappa B_{\tilde{w}+1}+(1-p_{\tilde{w}})\gamma B_{\tilde{w}+1}}{\rho+p_{\tilde{w}}\kappa+(1-p_{\tilde{w}})\gamma} \\
&= \frac{\delta(p_{\tilde{w}})}{\rho+\delta(p_{\tilde{w}})}B_{\tilde{w}+1}
\end{aligned}$$

where $\delta(p_{\tilde{w}}) \equiv p_{\tilde{w}}\kappa+(1-p_{\tilde{w}})\gamma$.

Further recall that in equilibrium, the maximum repayment $R(p)$ that the borrower would accept is given by

$$\begin{aligned}
(zK_{\tilde{w}}-R(p))\,dt+e^{-\rho dt}B_{\tilde{w}} &= e^{-\rho dt}\left(\left(1-e^{-\kappa dt}\right)B_{\tilde{w}+1}+e^{-\kappa dt}B_{\tilde{w}}\right) \\
R(p)dt &= zK_{\tilde{w}}dt-e^{-\rho dt}\left(\left(1-e^{-\kappa dt}\right)(B_{\tilde{w}+1}-B_{\tilde{w}})\right).
\end{aligned}$$

Now, given the borrower's equilibrium expectation $p_{\tilde{w}}$, we can calculate the lender's payoff from offering a maximally extractive, acceptable restrictive, or unrestrictive contract. Because the problem is stationary, we can determine which contract the lender prefers by comparing the lender's lifetime utility if he were to offer only restrictive contracts or only unrestrictive contracts in state $\tilde{w}$. If the lender were to offer only restrictive contracts in state $\tilde{w}$, his utility would be

$$\begin{aligned}
L^R_{\tilde{w}}(p_{\tilde{w}}) &= (R(p_{\tilde{w}})-(K_{\tilde{w}}-e))\,dt+e^{-\rho dt}L^R_{\tilde{w}}(p_{\tilde{w}}) \\
&= (zK_{\tilde{w}}-(K_{\tilde{w}}-e))\,dt-e^{-\rho dt}\left(\left(1-e^{-\kappa dt}\right)(B_{\tilde{w}+1}-B_{\tilde{w}})\right)+e^{-\rho dt}L^R_{\tilde{w}}(p_{\tilde{w}}) \\
&= \frac{(zK_{\tilde{w}}-(K_{\tilde{w}}-e))\,dt-e^{-\rho dt}\left(\left(1-e^{-\kappa dt}\right)\left(\frac{\rho}{\rho+\delta(p_{\tilde{w}})}B_{\tilde{w}+1}\right)\right)}{1-e^{-\rho dt}} \\
&\to \frac{(zK_{\tilde{w}}-(K_{\tilde{w}}-e))}{\rho}-\frac{\kappa}{\rho+\delta(p_{\tilde{w}})}B_{\tilde{w}+1}.
\end{aligned}$$

On the other hand, if the lender were to offer only unrestrictive contracts in state $\tilde{w}$ his utility would be

$$\begin{aligned}
L^U_{\tilde{w}}\left(p_{\tilde{w}}\right) &= \left(zK_{\tilde{w}}-\left(K_{\tilde{w}}-e\right)-\max\left\{h,e\right\}\right)dt+e^{-\rho dt}\left(\left(1-e^{-\gamma dt}\right)L_{\tilde{w}+1}+e^{-\gamma dt}L^U_{\tilde{w}}\left(p_{\tilde{w}}\right)\right) \\
&= \frac{\left(zK_{\tilde{w}}-\left(K_{\tilde{w}}-e\right)-\max\left\{h,e\right\}\right)dt+\left(\left(e^{-\rho dt}-e^{-(\rho+\gamma)dt}\right)L_{\tilde{w}+1}\right)}{1-e^{-(\rho+\gamma)dt}} \\
&\to \frac{\left(zK_{\tilde{w}}-\left(K_{\tilde{w}}-e\right)-\max\left\{h,e\right\}\right)+\gamma L_{\tilde{w}+1}}{\rho+\gamma} \\
&= \left(1-\beta\right)\frac{\left(zK_{\tilde{w}}-\left(K_{\tilde{w}}-e\right)-\max\left\{h,e\right\}\right)}{\rho}+\beta L_{\tilde{w}+1}.
\end{aligned}$$

Next, consider the function $g_{\tilde{w}}\left(p\right)\equiv L^U_{\tilde{w}}\left(p\right)-L^R_{\tilde{w}}\left(p\right)$. Note that $L^U_{\tilde{w}}\left(p\right)$ is independent of p and $L^R_{\tilde{w}}\left(p\right)$ is decreasing in p, so $g_{\tilde{w}}\left(p\right)$ is increasing. If $g_{\tilde{w}}\left(1\right)<0$, then the unique equilibrium is for the lender to offer a restrictive contract with probability 1. If $g_{\tilde{w}}\left(0\right)>0$, then the unique equilibrium is for the lender to offer an unrestrictive contract with probability 1. Else, as shown in Proposition 1, generically there is a unique $p_{\tilde{w}}\in[0,1]$ such that $g_{\tilde{w}}\left(p_{\tilde{w}}\right)=0$, and the unique equilibrium is for the lender to offer a restrictive contract with probability $p_{\tilde{w}}$.

We now verify that p_w is single peaked in w by considering the following five exhaustive cases:

$0<p_{\tilde{w}}<p_{\tilde{w}+1}<1$

$0<p_{\tilde{w}}<p_{\tilde{w}+1}=1$

In these cases, we aim to verify that $p_{\tilde{w}-1}<p_{\tilde{w}}$

$0=p_{\tilde{w}}<p_{\tilde{w}+1}<1$

$0=p_{\tilde{w}}<p_{\tilde{w}+1}=1$

In this case, we aim to verify that $p_{\tilde{w}-1}=p_{\tilde{w}}=0$ and $g_{\tilde{w}-1}\left(0\right)>g_{\tilde{w}}\left(0\right)$

$0=p_{\tilde{w}}=p_{\tilde{w}+1}$ and $g_{\tilde{w}}\left(0\right)>g_{\tilde{w}+1}(0)$

In this case, we aim to verify that $p_{\tilde{w}-1}=p_{\tilde{w}}=0$ and $g_{\tilde{w}-1}(0)>g_{\tilde{w}}(0)$.

We will provide the proof for the case wherein $0<p_{\tilde{w}}<p_{\tilde{w}+1}<1$ and omit the others as they are all similar.

Because $p_{\tilde{w}+1}$ is interior, we have

$$L_{\tilde{w}+1}=L^R_{\tilde{w}+1}(p_{\tilde{w}+1})=\frac{(zK_{\tilde{w}+1}-(K_{\tilde{w}+1}-e))}{\rho}-\frac{\kappa}{\rho+\delta(p_{\tilde{w}+1})}B_{\tilde{w}+2}$$

and

$$B_{\tilde{w}+1} = \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}+1})} B_{\tilde{w}+2}.$$

Thus, in state w we have

$$\begin{aligned}
g_{\tilde{w}}(p_{\tilde{w}}) &\equiv L^U_{\tilde{w}} - L^R_{\tilde{w}}(p_{\tilde{w}}) \\
&= (1-\beta)\frac{(zK_{\tilde{w}} - (K_{\tilde{w}} - e) - \max\{h,e\})}{\rho} + \beta\left(\frac{(zK_{\tilde{w}+1} - (K_{\tilde{w}+1} - e))}{\rho} - \frac{\kappa}{\rho + \delta(p_{\tilde{w}+1})} B_{\tilde{w}+2}\right) \\
&\quad - \left(\frac{(zK_{\tilde{w}} - (K_{\tilde{w}} - e))}{\rho} - \frac{\kappa}{\rho + \delta(p_{\tilde{w}})} \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}+1})} B_{\tilde{w}+2}\right) \\
&= \beta\frac{(zK_{\tilde{w}+1} - (K_{\tilde{w}+1} - e)) - (zK_{\tilde{w}} - (K_{\tilde{w}} - e))}{\rho} \\
&\quad - (1-\beta)\frac{z\max\{e,h\}}{\rho} - \frac{\kappa}{\rho + \delta(p_{\tilde{w}+1})} B_{\tilde{w}+2}\left(\beta - \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}})}\right) \\
&= 0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
g_{\tilde{w}-1}(p) &= L^U_{\tilde{w}-1} - L^R_{\tilde{w}-1}(p) \\
&= \beta\frac{(zK_{\tilde{w}} - (K_{\tilde{w}} - e)) - (zK_{\tilde{w}-1} - (K_{\tilde{w}-1} - e))}{\rho} - (1-\beta)\frac{z\max\{h,e\}}{\rho} \\
&\quad - \frac{\kappa}{\rho + \delta(p_{\tilde{w}})} \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}+1})} B_{\tilde{w}+2}\left(\beta - \frac{\delta(p_{\tilde{w}})}{\rho + \delta(p)}\right) \\
&= 0.
\end{aligned}$$

Comparing the expression for $g_{\tilde{w}-1}(p)$ to the that of $g_{\tilde{w}}(p)$, we can see that the sum of first two terms is strictly larger in the expression for $g_{\tilde{w}-1}(p)$ (by Assumption 1). That means that in order for p to set $g_{\tilde{w}-1}(p) = 0$ (if possible), we need the third term in $g_{\tilde{w}-1}(p)$ to be strictly smaller than the third term in $g_{\tilde{w}}(p)$. That is,

$$\begin{aligned}
-\frac{\kappa}{\rho + \delta(p_{\tilde{w}})} \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}+1})} B_{\tilde{w}+2}\left(\beta - \frac{\delta(p_{\tilde{w}})}{\rho + \delta(p)}\right) \quad &< \quad -\frac{\kappa}{\rho + \delta(p_{\tilde{w}+1})} B_{\tilde{w}+2}\left(\beta - \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}})}\right) \\
&\iff \\
\frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}})}\left(\beta - \frac{\delta(p_{\tilde{w}})}{\rho + \delta(p)}\right) \quad &> \quad \left(\beta - \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}})}\right)
\end{aligned}$$

Recall that $p_{\tilde{w}+1} > p_{\tilde{w}}$ by . So $\frac{\delta(p_{\tilde{w}+1})}{\rho+\delta(p_{\tilde{w}})} < 1$. Thus,

$$
\begin{aligned}
\left(\beta - \frac{\delta(p_{\tilde{w}})}{\rho+\delta(p)}\right) \quad &> \quad \left(\beta - \frac{\delta(p_{\tilde{w}+1})}{\rho+\delta(p_{\tilde{w}})}\right) \\
&\implies \quad -\frac{\delta(p_{\tilde{w}})}{\rho+\delta(p)} > -\frac{\delta(p_{\tilde{w}+1})}{\rho+\delta(p_{\tilde{w}})} \\
&\implies \quad \frac{\delta(p_{\tilde{w}})}{\rho+\delta(p)} < \frac{\delta(p_{\tilde{w}+1})}{\rho+\delta(p_{\tilde{w}})} \\
&\implies \quad p < p_{\tilde{w}}
\end{aligned}
$$

On the other hand, if there is no $p \geq 0$ such that $g_{\tilde{w}-1}(p) = 0$, then $g_{\tilde{w}-1}(0) > 0$, and the unique equilibrium includes $p_{\tilde{w}-1} = 0$. This completes the argument for this case. As the remaining cases are similar, they are omitted.

The remaining possibility is that in some state w' the lender offers an unrestrictive contract $\langle t, R, 0\rangle$ with certainty, $p_w = 0$, where $R < z(t+e-h)$ (i.e. the unrestrictive contract is more generous than necessary, to support the borrower's growth). In such a case it is straightforward to show that the lender offers unrestrictive contracts with lower than necessary repayment in all states $w < w'$ as well, completing the proof.

Proof of Proposition 3

Suppose $h < e$ and consider the lender's behavior in state n. Fixing a probability p_n that the borrower anticipates a restrictive contract in equilibrium, and recalling that we can consider the lender's continuation utility in state n from a fixed action due to the stationarity of the problem, the lender's utility from offering a restrictive contract is

$$
\begin{aligned}
L_n^R(p_n) \quad &= \quad \frac{zK_n - (K_n - e)}{\rho} - \frac{\kappa}{\rho + \delta(p_n)} \frac{u}{\rho} \\
&= \quad \frac{zK_n - (K_n - e)}{\rho} - \alpha \frac{u}{\rho}
\end{aligned}
$$

where the equality follows from the fact that $h < e$.

On the other hand, his utility from offering an unrestrictive contract is

$$
\begin{aligned}
L_n^U(p_n) \quad &= \quad \frac{(zK_n - (K_n - e) - \max\{h, e\})}{\rho}(1-\beta) \\
&= \quad \frac{(zK_n - (K_n - e) - \max\{h, e\})}{\rho}(1-\alpha)
\end{aligned}
$$

The lender prefers offering an unrestrictive contract over a restrictive one if and only if

$$\frac{(zK_n - (K_n - e) - \max\{h,e\})}{\rho}(1-\alpha) \geq \frac{zK_n - (K_n - e)}{\rho} - \alpha\frac{u}{\rho}$$

$$(1-\alpha)\frac{(zK_n - (K_n - e) - \max\{h,e\})}{\rho} + \alpha\frac{u}{\rho} \geq \frac{zK_n - (K_n - e)}{\rho} \tag{8}$$

The left hand side of inequality 8 is the sum of the borrower and lender's welfares if the borrower invests in fixed capital at the slowest possible rate in the relationship, and the right hand side is the sum of their welfares if the borrower invests in working capital. Thus if it is socially efficient to invest in fixed capital, the lender strictly prefers unrestrictive contracts, irrespective of the borrower's expectation p_n, and thus in equilibrium in state n the lender chooses unrestrictive contracts with probability 1.

Moving backwards, the proof proceeds similarly.

Proof of Proposition 4

Lemma 4. *For any state $w > \bar{w}$, $\frac{dp_w}{du} \leq 0$ with strict inequality if $p_w > 0$.*

Proof. By definition, in states $w > \bar{w}$, $p_w < 1$. Thus, in equilibrium the lender at least weakly prefers offering the borrower an unrestrictive contract. We can therefore write the lender's continuation utility in each such state as the utility he derives from offering an unrestrictive contract at every period (fixing the borrower's expectation at p_w).[19] That is,

$$L_w = L_w^U \equiv (1-\beta)\frac{(zK_w - (K_w - e) - \max\{h,e\})}{\rho} + \beta L_{w+1}$$

On the other hand, if the lender were to offer a minimally generous restrictive contract at every period (again, fixing the borrower's expectation at p_w) he would receive a continuation utility of

$$L_w = L_w^R(p_w) \equiv \frac{zK_w - (K_w - e)}{\rho} - \frac{\kappa}{\rho + \delta(p_w)}B_{w+1}$$

In state n $L_n^U = (1-\beta)\frac{(zK_n-(K_n-e)-\max\{h,e\})}{\rho}$ which is not a function of u. On the other hand $L_n^R(p_n) = \frac{zK_n-(K_n-e)}{\rho} - \frac{\kappa}{\rho+\delta(p_n)}\frac{u}{\rho}$ is decreasing in u. Hence, if in state n $L_n^U > L_n^R(0)$, then $p_n = 0$

[19] Note that in full generality he may offer the borrower an unrestrictive contract with positive transfer in state w. If so his continuation utility is $L_w^U = \left(1 - \frac{\phi\left(t_w+e-\frac{R}{z}\right)}{\phi\left(t_w+e-\frac{R}{z}\right)+\rho}\right)\frac{R-t_w}{\rho} + \left(\frac{\phi\left(t_w+e-\frac{R}{z}\right)}{\phi\left(t_w+e-\frac{R}{z}\right)+\rho}\right)L_{w+1}$, but otherwise the argument proceeds unchanged.

and $\frac{dp_n}{du}=0$. The lender's continuation utility is $L_n=(1-\beta)\frac{(zK_n-(K_n-e)-\max\{h,e\})}{\rho}$ and $\frac{dL_n}{du}=0$. The borrower's utility is $B_n=\beta\frac{u}{\rho}$ so $\frac{dB_n}{du}>0$.

If $L_n^I=L_n^C(p_n)$ for some $p_n\in[0,1]$ then p_n is the solution to

$$g(p_n)\equiv(1-\beta)\frac{(zK_n-(K_n-e)-\max\{h,e\})}{\rho}-\left(\frac{(zK_n-(K_n-e))}{\rho}-\frac{\kappa}{\rho+\delta(p_n)}\frac{u}{\rho}\right)=0$$

Since δ is a decreasing function, it is clear that $\frac{dp_n}{du}<0$. But we still have $L_n=(1-\beta)\frac{(zK_n-(K_n-e)-\max\{h,e\})}{\rho}$, so that $\frac{dL_n}{du}=0$. $B_n=\frac{\delta(p_n)}{\rho+\delta(p_n)}\frac{u}{\rho}$, so

$$\frac{dB_n}{du} = \frac{\left(\frac{d\delta(p_n)}{dp_n}\frac{dp_n}{du}\right)\rho}{(\rho+\delta(p_n))^2}\frac{u}{\rho}+\frac{\delta(p_n)}{\rho+\delta(p_n)}>0.$$

Proceeding backward to any state $w>\bar{w}$, suppose $\frac{dB_{w+1}}{du}>0$, $\frac{dL_{w+1}}{du}=0$. Then the proof proceeds exactly as above. This completes the proof of the lemma. □

We next consider the comparative static in states $w\in\{\underline{w},\dots,\bar{w}\}$.

Lemma 5. *For $w\in\{\underline{w},\dots,\bar{w}\}$, generically $\frac{dp_w}{du}=0$, $\frac{dB_{\bar{w}+1}}{du}>0$ and $\frac{dL_{\bar{w}}}{du}<0$.*

Proof. By definition $p_w=1$ for $w\in\{\underline{w},\dots,\bar{w}\}$. Generically, this preference will be strict and thus $\frac{dp_{\bar{w}}}{du}=0$.

Recall that in Lemma 4 we established $\frac{dB_{\bar{w}+1}}{du}>0$. We also know that $L_{\bar{w}}=L_{\bar{w}}^R(1)\equiv\frac{(zK_{\bar{w}}-(K_{\bar{w}}-e))}{\rho}-\alpha B_{\bar{w}+1}$. Hence generically $\frac{dL_{\bar{w}}}{du}<0$. Further, $B_{\bar{w}}=\alpha B_{\bar{w}+1}$ so $\frac{dB_{\bar{w}}}{du}=\alpha\frac{dB_{\bar{w}+1}}{du}>0$.

For the remainder of the states $w\in\{\underline{w},\dots,\bar{w}\}$, the result follows from straightforward induction. □

We now consider the comparative statics for $w<\underline{w}$ in the following three lemmas.

Lemma 6. *Suppose $p_{\underline{w}-1}=0$. Then generically $\frac{dp_w}{du}=0$, $\frac{dL_w}{du}<0$, and $\frac{dB_w}{du}>0$ for all $w<\underline{w}$.*

Proof. If $p_{\underline{w}-1}=0$ and $L_{\underline{w}-1}^U>L_{\underline{w}-1}^R(0)$, then the lender's preference for unrestrictive contracts is strict, so $\frac{dp_{\underline{w}-1}}{du}=0$. Further, Lemma 5 established that $\frac{dL_{\underline{w}}}{du}<0$ and $\frac{dB_{\underline{w}}}{du}>0$. Therefore, because $L_{\underline{w}-1}=(1-\beta)\frac{\left(zK_{\underline{w}-1}-\left(K_{\underline{w}-1}-e\right)-\max\{h,e\}\right)}{\rho}+\beta L_{\underline{w}}$, we know $\frac{dL_{\underline{w}-1}}{du}<0$. And $B_{\underline{w}-1}=\beta B_{\underline{w}}$ so $\frac{dB_{\underline{w}-1}}{du}>0$. Moving backwards proceeds by straightforward induction. □

The remainder of the proof deals with the case for which $p_{\underline{w}-1}>0$. We split the analysis into two cases based on the players' level of patience.

Lemma 7. *Suppose $p_{\underline{W}-1} > 0$ and $\rho > \frac{\kappa\gamma}{\kappa+\gamma}$. Then $\frac{dp_w}{du} > 0$, $\frac{dL_w}{du} < 0$, and $\frac{dB_w}{du} < 0$ for all $w < \underline{w}$.*

Proof. Consider first state $\underline{w}-1$. We know $p_{\underline{W}-1}$ is the solution to $g\left(p_{\underline{W}-1}\right) = 0$. So

$$\beta\frac{\left(zK_{\underline{w}} - \left(K_{\underline{w}} - e\right)\right) - \left(zK_{\underline{w}\text{-}1} - \left(K_{\underline{w}-1} - e\right)\right)}{\rho}$$
$$-\left(1-\beta\right)\frac{z\max\left\{e,h\right\}}{\rho} - \left(\beta - \frac{\delta\left(p_{\underline{W}}\right)}{\rho+\delta\left(p_{\underline{W}-1}\right)}\right)\frac{\kappa}{\rho+\delta\left(p_{\underline{W}}\right)}B_{\underline{W}+1} = 0$$

$$\Longleftrightarrow$$

$$\beta\frac{\left(zK_{\underline{w}} - \left(K_{\underline{w}} - e\right)\right) - \left(zK_{\underline{w}\text{-}1} - \left(K_{\underline{w}-1} - e\right)\right)}{\rho} - \left(1-\beta\right)\frac{z\max\left\{e,h\right\}}{\rho}$$
$$= \left(\beta - \frac{\delta\left(p_{\underline{W}}\right)}{\rho+\delta\left(p_{\underline{W}-1}\right)}\right)\frac{\kappa}{\rho+\delta\left(p_{\underline{W}}\right)}B_{\underline{W}+1}$$

Note that the left hand side of the above equation is constant in u.[20] Thus

$$\begin{aligned}
0 &= \frac{d\left(\beta\frac{\kappa}{\rho+\delta\left(p_{\underline{W}}\right)} - \frac{\kappa}{\rho+\delta\left(p_{\underline{W}-1}\right)}\frac{\delta\left(p_{\underline{W}}\right)}{\rho+\delta\left(p_{\underline{W}}\right)}\right)B_{\underline{W}+1}}{du} \qquad (9)\\
&\Longrightarrow\\
0 &= -\beta\frac{\kappa\frac{d\delta\left(p_{\underline{W}}\right)}{dp_{\underline{W}}}\frac{dp_{\underline{W}}}{du}}{\left(\rho+\delta\left(p_{\underline{W}}\right)\right)^2}B_{\underline{W}+1} + \frac{\kappa\frac{d\delta\left(p_{\underline{W}-1}\right)}{dp_{\underline{W}-1}}\frac{dp_{\underline{W}-1}}{du}}{\left(\rho+\delta\left(p_{\underline{W}-1}\right)\right)^2}\frac{\delta\left(p_{\underline{W}}\right)}{\rho+\delta\left(p_{\underline{W}}\right)}B_{\underline{W}+1}\\
&\quad -\frac{\kappa}{\rho+\delta\left(p_{\underline{W}-1}\right)}\frac{\left(\frac{d\delta\left(p_{\underline{W}}\right)}{dp_{\underline{W}}}\frac{dp_{\underline{W}}}{du}\right)\left(\rho+\delta\left(p_{\underline{W}}\right)\right) - \delta\left(p_{\underline{W}}\right)\frac{d\delta\left(p_{\underline{W}}\right)}{dp_{\underline{W}}}\frac{dp_{\underline{W}}}{du}}{\left(\rho+\delta\left(p_{\underline{W}}\right)\right)^2}B_{\underline{W}+1}\\
&\quad +\left(\beta\frac{\kappa}{\rho+\delta\left(p_{\underline{W}}\right)} - \frac{\kappa}{\rho+\delta\left(p_{\underline{W}-1}\right)}\frac{\delta\left(p_{\underline{W}}\right)}{\rho+\delta\left(p_{\underline{W}}\right)}\right)\frac{dB_{\underline{W}+1}}{du}\\
&\Longrightarrow\\
0 &> \frac{\kappa\frac{d\delta\left(p_{\underline{W}-1}\right)}{dp_{\underline{W}-1}}\frac{dp_{\underline{W}-1}}{du}}{\left(\rho+\delta\left(p_{\underline{W}-1}\right)\right)}\\
&\Longrightarrow\\
0 &< \frac{dp_{\underline{W}-1}}{du}
\end{aligned}$$

where the second implication follows by removing positive terms from the right hand side and

[20]The right hand side of the above equation can be simplified by noting that $p_{\underline{W}} = 1$, but we leave it in this more general form to economize on notation in the backward induction step.

noting that $p_{\underline{W}} > p_{\underline{W}-1}$, which implies that $\frac{\delta(p_{\underline{W}})}{\rho+\delta(p_{\underline{W}-1})} < \frac{\delta(p_{\underline{W}-1})}{\rho+\delta(p_{\underline{W}-1})} \leq \beta$.

Next, note that

$$L_{\underline{W}-1} = L^U_{\underline{W}-1} = (1-\beta)\,\frac{\left(zK_{\underline{w}-1} - \left(K_{\underline{w}\text{-}1} - e\right) - \max\{e,h\}\right)}{\rho} + \beta L_{\underline{W}} \tag{10}$$

so by Lemma 5, we know that $\frac{dL_{\underline{W}-1}}{du} < 0$.[21]

To find the sign of $\frac{dB_{\underline{W}-1}}{du}$, recall that $\frac{\rho+\delta(p_{\underline{W}-1})}{\delta(p_{\underline{W}-1})}B_{\underline{W}-1} = B_{\underline{W}} = \frac{\delta(p_{\underline{W}})}{\rho+\delta(p_{\underline{W}})}B_{\underline{W}+1}$. Hence, we know that

$$\begin{aligned}
0 &= \frac{d\left(\beta\frac{\kappa}{\rho+\delta(p_{\underline{W}})} - \frac{\kappa}{\rho+\delta(p_{\underline{W}-1})}\frac{\delta(p_{\underline{W}})}{\rho+\delta(p_{\underline{W}})}\right)B_{\underline{W}+1}}{du} \\
&= \frac{d\left(\beta\frac{\kappa}{\delta(p_{\underline{W}})} - \frac{\beta\frac{\kappa}{\delta(p_{\underline{W}})}\rho-\kappa}{\delta(p_{\underline{W}-1})}\right)B_{\underline{W}-1}}{du} \\
&= \frac{d\left(\beta + \frac{\beta\rho-\kappa}{\delta(p_{\underline{W}-1})}\right)B_{\underline{W}-1}}{du} \\
&= \frac{-(\beta\rho-\kappa)\frac{d\delta(p_{\underline{W}-1})}{dp_{\underline{W}-1}}\frac{dp_{\underline{W}-1}}{dU}}{\left(\delta\left(p_{\underline{W}-1}\right)\right)^2}B_{\underline{W}-1} + \frac{dB_{\underline{W}-1}}{du}\left(\beta + \frac{\beta\rho-\kappa}{\delta\left(p_{\underline{W}-1}\right)}\right)
\end{aligned} \tag{11}$$

where the second equality follows from noting that $p_{\underline{W}} = 1$. Reducing, we have

$$\frac{dB_{\underline{W}-1}}{du} = NEG\left(\beta\rho-\kappa\right) \tag{12}$$

where NEG is a negative constant. The one subtle algebraic reduction in this final step is that $\left(\beta + \frac{\beta\rho-\kappa}{\delta(p_{\underline{W}-1})}\right) = \frac{\rho+\delta(p_{\underline{W}-1})}{\delta(p_{\underline{W}-1})}\left[\beta - \frac{\kappa}{\rho+\delta(p_{\underline{W}-1})}\right] > 0$.

Since we have assumed $\rho > \frac{\kappa\gamma}{\kappa+\gamma}$, which is equivalent to $\rho\beta > \kappa$, we have $\frac{dB_{\underline{W}-1}}{du} < 0$.

Moving backward to state $\underline{w}-2$, suppose $p_{\underline{W}-2} > 0$ (or $p_{\underline{W}-2} = 0$, but $L^I_{\underline{W}-2} - L^C_{\underline{W}-2}(0) = 0$).

[21] Note that in full generality the lender may offer the borrower an unrestrictive loan with positive transfer in state $\underline{w}-1$. If so, his continuation utility is $L^U_{\underline{w}\text{-}1} = \left(1 - \frac{\phi\left(t_w+e-\frac{R}{z}\right)}{\phi\left(t_w+e-\frac{R}{z}\right)+\rho}\right)\frac{R-t_w}{\rho} + \left(\frac{\phi\left(t_w+e-\frac{R}{z}\right)}{\phi\left(t_w+e-\frac{R}{z}\right)+\rho}\right)L_{\underline{w}}$, and the interest rate becomes weakly higher but otherwise the argument to follow goes through unchanged.

Then $p_{\underline{w}-2}$ is the solution to $g_{\underline{w}-2}\left(p_{\underline{w}-2}\right)=0$. That is

$$(1-\beta)\frac{zK_{\underline{w}-2}-\left(K_{\underline{w}-2}-e\right)-\max\{e,h\}}{\rho}+\beta L_{\underline{w}-1}-\left(\frac{zK_{\underline{w}-2}-\left(K_{\underline{w}-2}-e\right)}{\rho}-\frac{\kappa}{\rho+\delta\left(p_{\underline{w}-2}\right)}B_{\underline{w}-1}\right)=0$$

Differentiating both sides with respect to u, we see

$$\beta\frac{dL_{\underline{w}-1}}{du}+\left(\frac{-\kappa\frac{d\delta\left(p_{\underline{w}-2}\right)}{dp_{\underline{w}-2}}\frac{dp_{\underline{w}-2}}{du}}{\left(\rho+\delta\left(p_{\underline{w}-2}\right)\right)^2}B_{\underline{w}-1}+\frac{\kappa}{\rho+\delta\left(p_{\underline{w}-2}\right)}\frac{dB_{\underline{w}-1}}{du}\right)=0 \tag{13}$$

We know that $\frac{dL_{\underline{w}-1}}{du}<0$ and $\frac{dB_{\underline{w}-1}}{du}<0$. Hence, $\frac{dp_{\underline{w}-2}}{du}>0$. Further,

$$L_{\underline{w}-2}=(1-\beta)\frac{zK_{\underline{w}-2}-\left(K_{\underline{w}-2}-e\right)-\max\{e,h\}}{\rho}+\beta L_{\underline{w}-1}$$

so $\frac{dL_{\underline{w}-2}}{du}<0$. And $B_{\underline{w}-2}=\frac{\delta\left(p_{\underline{w}-2}\right)}{\rho+\delta\left(p_{\underline{w}-2}\right)}B_{\underline{w}-1}$ so

$$\frac{dB_{\underline{w}-2}}{du}=\frac{\frac{d\delta\left(p_{\underline{w}-2}\right)}{dp_{\underline{w}-2}}\frac{dp_{\underline{w}-2}}{du}\rho}{\left(\rho+\delta\left(p_{\underline{w}-2}\right)\right)^2}B_{\underline{w}-1}+\frac{\delta\left(p_{\underline{w}-2}\right)}{\rho+\delta\left(p_{\underline{w}-2}\right)}\frac{dB_{\underline{w}-1}}{du}<0 \tag{14}$$

If instead we had $p_{\underline{w}-2}=0$ and $L^U_{\underline{w}-2}>L^R_{\underline{w}-2}(0)$, then $\frac{dp_{\underline{w}-2}}{du}=0$. $L_{\underline{w}-2}=(1-\beta)\frac{zK_{\underline{w}-2}-\left(K_{\underline{w}-2}-e\right)-\max\{e,h\}}{\rho}+\beta L_{\underline{w}-1}$ so $\frac{dL_{\underline{w}-2}}{du}<0$. $B_{\underline{w}-2}=\beta B_{\underline{w}-1}$ so $\frac{dB_{\underline{w}-2}}{du}<0$.

Because we used only that $\frac{dB_{\underline{w}-1}}{du}<0$ and $\frac{dL_{\underline{w}-1}}{du}<0$, moving backwards from state $\underline{w}-2$ to state 0 is straightforward induction. $\square$

We now complete the proof of Proposition 4 by considering a patient borrower.

Lemma 8. *Suppose $p_{\underline{w}-1}>0$ and $\rho<\frac{\kappa\gamma}{\kappa+\gamma}$. Then $\frac{dp_w}{du}>0$ and $\frac{dL_w}{du}<0$ for all $w<\underline{w}$.*

Proof. In state $\underline{w}-1$ everything follows as it did in Lemma 7 except that $\frac{dB_{\underline{w}-1}}{du}$, determined by equation (12), is positive. In state $\underline{w}-2$, the considerations are similar. $\frac{dp_{\underline{w}-2}}{du}>0$ is determined by equation (9) (reducing all indices by 1), and $\frac{dL_{\underline{w}-2}}{du}<0$ is determined by (10) (reducing all indices by 1). However the sign of $\frac{dB_{\underline{w}-2}}{du}$, determined by (11), is now ambiguous.

Moving back to an arbitrary state $w<\underline{w}$ such that $\frac{dB_{w+1}}{du}>0$, the considerations will be exactly the same as for $\underline{w}-2$. In any state $w<\underline{w}$ for which $\frac{dB_{w+1}}{du}<0$, $\frac{dp_w}{du}>0$ is determined by equation (13), $\frac{dL_w}{du}<0$ is determined by (10), and $\frac{dB_w}{du}<0$ is determined by (14). This completes the proof. $\square$

Together, Lemmas 4 through 8 complete the proof of Proposition 4.

Proof of Proposition 5

Fixing the lender's behavior, the borrower's continuation utility in state n is

$$B_n(p_n) = \frac{p_n\left(1-e^{-\kappa dt}\right)+(1-p_n)\left(1-e^{-\gamma dt}\right)}{1-p_n e^{-(\rho+\kappa)dt}-(1-p_n)\,e^{-(\rho+\gamma)dt}}e^{-\rho dt}\frac{u}{\rho}$$

which increases linearly in u. Moving backward, suppose B_{w+1} is increasing in u. Then, noting that

$$B_w(p_w) = \frac{p_w\left(1-e^{-\kappa dt}\right)+(1-p_w)\left(1-e^{-\gamma dt}\right)}{1-p_w e^{-(\rho+\kappa)dt}-(1-p_w)\,e^{-(\rho+\gamma)dt}}e^{-\rho dt}B_{w+1}$$

increases linearly in B_{w+1} completes the proof.

Proof of Proposition 6

The proof for states $w \geq \underline{w}$ proceeds exactly as in Proposition 4 and is thus omitted. In this section we provide an example in which $\frac{dp_{\underline{w}-1}}{d\rho^B} < 0$ so that making the borrower more patient can strengthen the poverty trap.

We prove this result with a three state model $w \in \{1,2,3\}$ in which the game ends if the borrower ever reaches state 3. We take $e = .15$, $z = 2$, $\phi = 2$, $h = 50$, $K_1 = 642$, $K_2 = 1000$, $\frac{u}{\rho^B} = 2000$, and $\rho^B = \rho^L = 1$. It is easily verified that Assumption 3 hold in states 1 and 2. That is,

$$\alpha^2\frac{u}{\rho^B} = \left(\frac{.3}{1.3}\right)^2 2000 > \frac{zh}{\rho^B} = 100$$

Now we verify that in state 2 the lender offers the borrower a restrictive contract with probability 1. If the borrower expects a restrictive contract with probability 1, then the lender gets the following continuation utility if he offers the borrower a restrictive contract in state 2:

$$L_2^R(1) = \frac{zK_2-(K_2-e)}{\rho^L} - \alpha\frac{u}{\rho^B} = 1000.15 - \frac{.3}{1.3}2000 \approx 539$$

If instead the lender offers the borrower an unrestrictive contract at every period in state 2, his continuation utility is

$$L_2^U = \frac{zK_2-(K_2-e)-h}{\rho^L}(1-\beta) \approx 9.4$$

Because the lender finds it *least* appealing to offer a restrictive contract when the borrower expects restrictive contracts with probability 1, we conclude that in the unique equilibrium the lender offers the borrower a restrictive contract with probability 1.

We next verify that, in equilibrium, the lender mixes between restrictive and unrestrictive contracts in state 1.

First, consider the lender's continuation utility in state 1 from offering the borrower a restrictive contract with probability 1 *when she expects an restrictive contract with probability p:*

$$
\begin{aligned}
L_1^R(p) &= \frac{zK_1 - (K_1 - e)}{\rho^L} - \max\left\{\frac{zh}{\rho^L}, \frac{\kappa}{\rho^B + \delta(p)} B_2\right\} \\
&= 642.15 - \max\left\{100, \left(\frac{.3}{1 + .3p + 100(1-p)}\right)\left(\frac{.3}{1.3}\right) 2000\right\}
\end{aligned}
$$

Note that the repayment rate the lender must set is the larger of $z(K_w - h)$ and what the lender must set so that the borrower achieves the utility she would have received from investing e in fixed capital.

If instead the lender were to offer an unrestrictive contract with probability 1, his state 1 continuation utility would be

$$
L_1^U = \frac{zK_1 - (K_1 - e) - h}{\rho^L}(1-\beta) + \beta L_2 \approx \frac{592.15}{101} + \frac{100}{101} 539
$$

It is easily verified that $L_1^R(0) > L_1^U > L_1^R(1)$ and hence the unique equilibrium in state 1 involves the lender mixing between restrictive and unrestrictive contracts. The probability p_1 that the lender offers a restrictive contract is determined by the following equation.

$$
\begin{aligned}
\frac{zK_1 - (K_1 - e) - h}{\rho^L}(1-\beta) + \beta L_2 \quad &= \quad \frac{zK_1 - (K_1 - e)}{\rho^L} - \frac{\kappa}{\rho^B + \delta(p_1)} B_2 \\
&\Longrightarrow \\
\frac{592.15}{101} + \frac{100}{101} 539 \quad &\approx \quad 642.15 - \frac{.3}{1 + .3p_1 + 100(1-p_1)}\left(\frac{.3}{1.3} 2000\right) \\
&\Longrightarrow \\
p_1 \quad &\approx \quad .9995
\end{aligned}
$$

Now, recall the investment rent in state 1 is

$$
\left(\beta - \frac{\kappa}{\rho^B + \delta(p_1)}\right) B_2 \approx \left(\frac{100}{101} - \frac{.3}{1 + .3p_1 + (1-p_1)100}\right) B_2
$$

We have
$$\frac{d\left(\beta-\frac{\kappa}{\rho^B+\delta(p)}\right)B_2}{d\rho^B}=\frac{d\left(\beta-\frac{\kappa}{\rho^B+\delta(p)}\right)}{d\rho^B}B_2+\left(\beta-\frac{\kappa}{\rho^B+\delta\left(p\right)}\right)\frac{d}{d\rho^B}B_2$$
Now,
$$\begin{aligned}\frac{d}{d\rho^B}B_2 &= \frac{d\frac{.3}{\rho^B+.3}\frac{u}{\rho^B}}{d\rho^B}\\ &\approx -\frac{.3}{\left(\rho^B+.3\right)^2}\frac{u}{\rho^B}-\frac{.3}{\rho^B+.3}\frac{u}{\left(\rho^B\right)^2}\\ &\approx -\frac{.3}{\left(1.3\right)^2}2000-\frac{.3}{1.3}2000\\ &\approx -816.56\end{aligned}$$

And,
$$\begin{aligned}\frac{d}{d\rho^B}\left(\frac{100}{\rho^B+100}-\frac{.3}{\rho^B+.3p+(1-p)\,100}\right) &\approx \left(-\frac{100}{\left(\rho^B+100\right)^2}+\frac{.3}{\left(\rho^B+.3p+(1-p)\,100\right)^2}\right)\\ &\approx .05\end{aligned}$$

So,
$$\frac{d}{d\rho^B}\left[\left(\beta-\frac{\kappa}{\rho^B+\delta}\right)B_2\right]\approx -675.87<0$$
Therefore, making the borrower more patient increases the investment rent and reduces p_1.

Proof of Proposition 7

In the limit, as $q\to\infty$ the lender behaves as a short-run player, while the borrower continues to care about the future. The lender receives the maximum feasible flow payoff $zK_w-(K_w-e)-\max\{h,e\}$ from an unrestrictive contract, and weakly less than that from a restrictive contract. Hence he always offers an unrestrictive contract in equilibrium.

Proof of Proposition 10

Lemma 9. *The borrower may reject a restrictive contract on the equilibrium path.*

Proof. To prove this lemma we need only find an example in which the borrower rejects a restrictive contract with positive probability. To do so, we modify the example from the proof of Proposition 6. Specifically, consider the one state example in which we take $e=.15$, $z=2$, $\phi=2$, $h=50$, $K_1=642$, $K_2=1000$, $\frac{u}{\rho^B}=2000$, and $\rho^B=\rho^L=1$. We define the distribution G such that $\nu=0$ with probability

$1-\varepsilon$, and $\nu = 45$ with probability ε.

We verified in the proof of Proposition 6 that this example satisfies Assumption 3 and that in equilibrium the lender offers the borrower a restrictive contract with probability 1. Clearly, for sufficiently small ε, the lender would prefer to offer the least generous restrictive contract that borrowers of type $\nu = 0$ would accept. The loss the lender suffers from being rejected with probability ε is vanishing. In contrast, if the lender offers a contract that both types of borrowers would accept, he incurs a first order loss in order to compensate the high type borrower for the $\nu = 45$ additional forgone investment. $\square$

Proof of Proposition 11

Define $\alpha' \equiv \frac{e'\phi}{\rho+e'\phi}$, and suppose $e' < e$. Because the borrower can invest e' flexibly regardless of the contractual restriction, if the lender offers a restrictive contract he will set the interest rate such that the borrower is indifferent between investing e in fixed capital versus investing only e' in fixed capital, and consuming the residual output that the lender leaves her.

It is straightforward to show that the borrower's expansion rent is still $(\beta-\alpha)B_{w+1}$. Further, as in Proposition 3, in equilibrium the borrower receives a restrictive contract with certainty in state w if and only if

$$(\beta-\alpha)B_{w+1} \geq (\beta-\alpha')\left(B_{w+1}+L_{w+1}-\frac{(z-1)K_w+e}{\rho}-\frac{1}{\phi}\right).$$

For $e' < e$, the borrower will grow more slowly in equilibrium than in autarky in any state in which the above condition is satisfied.

Proof of Proposition 12

Fix a game Γ with n states and a cost of fixed investment $\{\phi_w\}$. Then for game Γ^m with $m > 0$, a borrower investing in fixed capital rate i in state $2^m n$ will derive a state $2^m n$ continuation value of

$$\frac{i\phi_{2^m n}}{\rho+i\phi_{2^m n}}\frac{u}{\rho}$$

which converges to $\frac{u}{\rho}$ as m becomes large. If the borrower's equilibrium expectation is that the lender will offer the restrictive contract with probability 1, then the lender's continuation utility in state $2^m n$ from doing so is

$$L^R_{2^m n}(1) = \frac{zK_{2^m n}-(K_{2^m n}-e)}{\rho} - \frac{\kappa_{2^m n}}{\rho+\kappa_{2^m n}}\frac{u}{\rho}$$

which for sufficiently large m will be negative when it is socially efficient to invest.

On the other hand, the lender's continuation utility in state $2^m n$ if he offers an unrestrictive contract in every period is

$$L^U_{2^m n} = \frac{zK_{2^m n} - (K_{2^m n} - e) - \max(h,e)}{\rho} \left(1 - \frac{\gamma_{2^m n}}{\rho + \gamma_{2^m n}}\right)$$

which is positive for all $m > 0$. Thus, for sufficiently high m, the lender will offer an unrestrictive contract with positive probability in state $2^m n$, completing the proof.

Proof of Proposition 13

For states $w > \bar{w}_{m-1}$, the proof closely follows that of Proposition 4. Specifically, for states $w > \bar{w}_m$, the proof follows that of Lemma 4. For states $w \in \{\underline{w}_m, \ldots, \bar{w}_m\}$, the proof follows that of Lemma 5, and for states $w \in \{\bar{w}_{m-1} + 1, \ldots, \underline{w}_m - 1\}$, the proof follows that of Lemma 7.

For $w \in \{\bar{w}_{m-2} + 1, \bar{w}_{m-1}\}$, the logic of Proposition 4 is reversed, as $\frac{dB_{\bar{w}_{m-1}+1}}{du} < 0$. That is, in the state directly beyond the pure restrictive state $\bar{w}_{m-1}$, the borrower's continuation utility is decreasing in u. In contrast, in the state directly beyond the pure restrictive state $\bar{w}_m$, the borrower's continuation utility is *increasing* in u. So, the comparative static for $w \in \{\bar{w}_{m-2} + 1, \bar{w}_{m-1}\}$ comes directly from reversing the signs in Lemmas 5 and 7.

The proof proceeds similarly for all consumption regions $\{\underline{w}_{\tilde{m}} \ldots, \bar{w}_{\tilde{m}}\}$ for $\tilde{m} \leq m - 2$.