HARVARD | BUSINESS | SCHOOL

Estimating Models of Supply and Demand: Instruments and Covariance Restrictions

Alexander MacKay
Nathan H. Miller

Working Paper 19-051

Estimating Models of Supply and Demand: Instruments and Covariance Restrictions

Alexander MacKay
Harvard Business School

Nathan H. Miller
Georgetown University

Working Paper 19-051

Estimating Models of Supply and Demand: Instruments and Covariance Restrictions*

Alexander MacKay
Harvard University†

Nathan H. Miller
Georgetown University‡

First Draft: March 20, 2017
This Draft: May 29, 2019

Abstract

We consider the identifying power of supply-side assumptions in the estimation of models of supply and demand. Pairing supply with demand expands the set of restrictions that identify a general class of models. As is well known, the problem of price endogeneity is resolved with an instrument that shifts supply. We show that, even with imperfect competition, an instrument that shifts demand also obtains identification. Our analysis establishes a fundamental link between the (endogenous) price coefficient and the covariance of unobservable cost and demand shocks. Thus, a covariance restriction on unobserved shocks can achieve identification, even in the absence of instruments. Further, it can be possible to bound the structural parameters using the supply and demand assumptions alone. We develop an estimator based on the covariance restriction that is efficiently estimated by the output of ordinary least squares regression and performs well in small samples. We illustrate the methodology with applications to ready-to-eat cereal, cement, and airlines.

JEL Codes: C13, C36, D12, D22, D40, L10

Keywords: Identification, Demand Estimation, Covariance Restrictions, Instrumental Variables

---

*We thank Steven Berry, Chuck Romeo, Gloria Sheu, Karl Schurter, Jesse Shapiro, Jeff Thurk, Andrew Sweeting, Matthew Weinberg, and Nathan Wilson for helpful comments. We also thank seminar and conference participants at Harvard University, MIT, the University of Maryland, the Barcelona GSE Summer Forum, and the NBER Summer Institute. Previous versions of this paper were circulated with the title "Instrument-Free Demand Estimation."

†Harvard University, Harvard Business School. Email: amackay@hbs.edu.

‡Georgetown University, McDonough School of Business. Email: nathan.miller@georgetown.edu.

1 Introduction

A fundamental challenge in identifying models of supply and demand is that firms can adjust prices to reflect demand shocks. As a result, the empirical relationship between prices and quantities does not represent a demand curve but rather a mixture of demand and supply. Researchers typically address this challenge by using instruments from the supply-side model to estimate demand, employing the formal supply model only in counterfactual simulations. Some applications have estimated demand and supply jointly, noting that their combination can improve efficiency (e.g., Berry et al., 1995). However, our understanding of the identifying power of joint estimation has, thus far, been primarily guided by intuition.[1]

In this paper, we provide a formal analysis of the identifying power of supply-side restrictions in joint estimation. We provide a constructive identification result for the primary coefficient of interest in many empirical studies—the price parameter—when the price reflects demand shocks that are unobserved to the econometrician but observed by firms (Working, 1927). Invoking the supply model expands the set of assumptions that identify the structural parameters. We classify these assumptions into three broad approaches: supply shifters, demand shifters, and covariance restrictions on unobserved shocks.

Supply shifters are the most common form of instruments used in empirical applications, and can be used to obtain demand curves without formalizing supply-side behavior. However, a formal model of supply is often required to construct counterfactuals that address the research question. Supply-side assumptions are made in order to capture the equilibrium response to a policy change (Fowlie et al., 2016), quantify the effect of an economic mechanism (Feenstra and Weinstein, 2017), or understand the nature of competition (Miller and Weinberg, 2017). With these assumptions in hand, it is possible to identify the price parameter using two other approaches: demand shifters and covariance restrictions.

These two approaches are combined in a seminal paper by Hausman and Taylor (1983) and follow-on work. Building on this literature, we show how either approach can be *separately* used to identify demand, even with more general supply-side behavior. Perhaps surprisingly, covariance restrictions alone can identify models of imperfect competition, even though such assumptions do not allow for the construction of an "instrument" to use in estimation.[2] The core intuition behind our results is that the supply-side model dictates how prices respond to demand shocks. An econometrician estimating demand and supply jointly can leverage this information to resolve price endogeneity without instruments from the supply-side model. In

[1]The international trade literature provides identification results for the special case where markups do not respond to demand shocks (e.g., Feenstra, 1994). We discuss this literature in more detail later. For applications in industrial organization that impose supply-side assumptions, see Thomadsen (2005), Cho et al. (2018), and Li et al. (2018). Thomadsen (2005) assumes no unobserved demand shocks, and Cho et al. (2018) assume no unobserved cost shocks; both are special cases of the covariance restriction approach.

[2]We consider instruments to be empirical objects that are inputs to an estimation procedure (e.g., two-stage least squares). Instruments may be constructed directly from data or estimated in a separate step.

fact, efficient estimates can be obtained from the output of ordinary least squares regression.

One of the largest obstacles to empirical work is the lack of valid supply shifters as instruments. Even if such instruments exist in theory, the data to operationalize them may not be available in practice. Our alternative approaches may allow empirical researchers to push ahead along new frontiers. For example, it would be very difficult to find the requisite amount of valid instruments for an empirical study of products across many categories. The covariance restriction approach may be appealing in such settings, as it provides a theory-based path to recovering causal parameters without requiring supplemental data.

We describe the three approaches to identification for our baseline model in Section 2. Our model makes two common assumptions about demand and supply, allowing for imperfect competition. In Section 3, we evaluate the empirical content of supply-side assumptions when combined with the demand model. We establish a fundamental link between the price coefficient, the covariance of unobservable cost and demand shocks, and the empirical variation present in the data. Somewhat surprisingly, we show that the supply-side assumptions do not improve efficiency when a consistent estimate of demand is obtained with a cost shifter. Our results do indicate how supplemental moments may be paired with supply-side assumptions to improve efficiency and obtain identification. Thus, we provide an explanation for the "folklore" around the Berry et al. (1995) results (Conlon and Gortmaker, 2019).

Based on this analysis, we develop an identification strategy using covariance restrictions alone. We show how, even in the absence of supply shifters or demand shifters, a restriction on the correlation of unobservable shocks can achieve point identification. We develop an efficient estimator that can be constructed from the output of ordinary least squares regression. In Section 4, we provide numerical simulations to compare the three approaches to identification and develop intuition around our estimator. In small samples, the covariance restriction approach performs well, even when an instrument-based approach suffers from the weak instruments problem. Section 5 provides extensions, showing that our result is not specific to the baseline assumptions about demand and supply. We make use of these extensions in Section 6, in which we demonstrate the utility of the approach with three empirical applications.

Our research builds on several strands of literature in economics. Early research at the Cowles Foundation (Koopmans et al., 1950) examines the identifying power of covariance restrictions in linear systems of equations, and a number of articles pursued this agenda in subsequent years (e.g., Fisher, 1963, 1965; Wegge, 1965). Hausman and Taylor (1983) show that identification of these systems can be interpreted in terms of instrumental variables: a demand-side instrument is used to recover the (initially unobserved) supply-side shock, which is then a valid instrument in demand estimation under the covariance restriction.[3] Matzkin (2016) and Chiappori et al. (2017) provide extensions to semi-parametric models. Estimators along these lines are consistent given sufficient variation in an excluded variable and the supply-side shock.

[3]The earliest known application of instrumental variables (Wright, 1928) uses precisely this approach.

By contrast, our covariance restriction approach does not require either source of variation, yet it extends to imperfect competition nonetheless. In a more general sense, our approach is conceptually related to control function estimation (e.g., Heckman, 1979).

A parallel literature explores the identification of supply and demand in models of international trade, building on the insight that a covariance restriction is sufficient to bound either the slope of supply or the slope of demand in linear models of perfect competition (Leamer, 1981). Feenstra (1994) considers monopolistic competition with constant elasticity demand. Identification is achieved if the econometrician observes at least two samples (e.g., from different countries) with equal slopes of demand and supply but unequal variances of the shocks. This approach is applied in Broda and Weinstein (2006, 2010), Soderbery (2015), and other articles. With constant elasticity demand, firms do not adjust markups in response to demand shocks, and therefore these trade models do not allow for a primary source of endogeneity bias that arises in a more general model of firm behavior. Hottman et al. (2016) and Feenstra and Weinstein (2017) estimate models that allow for some variation in markups, but identification for these extensions is not established. Thus, to our knowledge, we are the first to prove that a covariance restriction can identify models where firms respond strategically to demand shocks.[4]

Another approach is to estimate supply and demand via maximum likelihood, under the assumption that the distributions of demand and cost shocks are known to the econometrician and independent. At least one seminal article in industrial organization, Bresnahan (1987), pursues this approach. In the marketing literature, Yang et al. (2003) estimate an oligopoly model of supply and demand using Bayesian techniques to lessen the computation burden. For discussions and extensions of this approach, see Rossi et al. (2005), Dotson and Allenby (2010), and Otter et al. (2011). Published comments on Yang et al. (2003) point out that a coherent likelihood function may not exist due to multiple equilibria (Bajari, 2003; Berry, 2003). By contrast, our approach does not require a likelihood function and provides consistent estimates in the presence of multiple equilibria. Further, it does not require distributional assumptions. These advantages may make our approach relatively more palatable for oligopoly models.

Finally, price endogeneity has been a major focus of modern empirical and econometric research in industrial organization. Typically, the challenge is cast as a problem of finding valid supply-side instruments. Many possibilities have been developed, including the attributes of competing products (Berry et al., 1995; Gandhi and Houde, 2015), the prices of the same good in other markets (e.g., Hausman, 1996; Nevo, 2001; Crawford and Yurukoglu, 2012), or shifts in the equilibrium concept (e.g., Porter, 1983; Miller and Weinberg, 2017).[5] Our research highlights that, when demand and supply are estimated jointly, exogenous variation

[4]This trade literature has interesting historical antecedents. Leamer attributes an early version of his results to Schultz (1928). The identification argument of Feenstra (1994) is also proposed in Leontief (1929). Frisch (1933) provides an econometric critique of this argument.

[5]Byrne et al. (2016) proposes a novel set of instruments that leverage the structure of a discrete choice demand model with differentiated-products price competition. Nevo and Wolfram (2002) explore whether covariance restrictions can bound parameters (see footnote 41 of that article).

in a demand shifter also identifies the model. If valid instruments are available, then covariance restrictions may be used to construct overidentifying restrictions and test the model.

2 Model

We introduce the model and describe three approaches to identification. We focus initially on differentiated-products Bertrand competition with semi-linear demand, which allows a base set of results to be developed with minimal notation. To illustrate the generality of our approach, we establish that the results extend to alternative supply-side assumptions (Section 5) and alternative demand-side assumptions (Appendix B).

2.1 Data-Generating Process

Let there be $j = 1, 2, \ldots, J$ products in each of $t = 1, 2, \ldots, T$ markets, subject to downward-sloping demands. The econometrician observes vectors of prices, $p_t = [p_{1t}, p_{2t}, \ldots, p_{Jt}]'$, and quantities, $q_t = [q_{1t}, q_{2t}, \ldots, p_{Jt}]'$, corresponding to each market t, as well as a full rank matrix of covariates $X_t = [x_{1t}\ x_{2t}\ \ldots\ x_{Jt}]$. The covariates are orthogonal to a pair of demand and marginal cost shocks (i.e., $E[X\xi] = E[X\eta] = 0$) that are common knowledge among firms but unobserved by the econometrician.[6] We make the following assumptions:

Assumption 1 (Demand): *The demand schedule for each product is determined by the following semi-linear form:*

$$h_{jt} \equiv h(q_{jt}, w_{jt}; \sigma) = \beta p_{jt} + x'_{jt}\alpha + \xi_{jt} \tag{1}$$

where (i) $\frac{\partial h_{jt}}{\partial q_{jt}} > 0$, (ii) w_{jt} is a vector of observables and σ is a parameter vector, and (iii) the total derivatives of $h(\cdot)$ with respect to q exist as functions of the data and σ.

The demand system can be expressed as the inverse demand equation

$$p_{jt} = \frac{1}{\beta} h_{jt} - \frac{1}{\beta} x'_{jt}\alpha - \frac{1}{\beta}\xi_{jt} \tag{2}$$

which will prove convenient when we discuss identification.

Assumption 2 (Supply): *Each firm sells a single product and sets its price to maximize profit in each market. The firm takes the prices of other firms as given, knows the demand schedule in equation* (1)*, and has a linear constant marginal cost schedule given by*

$$c_{jt} = x'_{jt}\gamma + \eta_{jt}. \tag{3}$$

[6] We make the usual assumption that prices and covariates are linearly independent to allow for OLS estimation.

The demand assumption restricts attention to systems for which, after a transformation of quantities using observables (w_{jt}) and nonlinear parameters (σ), there is additive separability in prices, covariates, and the demand shock. The vector w_{jt} can be conceptualized as including the price and non-price characteristics of products, in particular those of other products that affect the demand of product j. Often, only a few observables are necessary to construct the transformation. For example, with the logit demand system, $h(q_{jt}; w_{jt}, \sigma) \equiv \ln(s_{jt}/w_{jt})$, where quantities are in shares ($q_{jt} = s_{jt}$), w_{jt} is the share of the outside good (s_{0t}), and there are no additional parameters in σ. Among the demand systems consistent with Assumption 1 are linear demand, nested logit demand, and random coefficients logit demand (e.g., Berry et al., 1995). We derive these connections in some detail in Appendix B.

The supply-side assumption restricts attention to Bertrand equilibria with constant marginal costs. In subsequent sections, we provide the additional notation necessary for models with multi-product firms, non-constant marginal costs, and Cournot competition. Assumptions 1 and 2 provide a mapping from the data and parameters to the structural error terms (ξ, η). Note that supply and demand may depend on different covariates; this is captured when non-identical components of α and γ are equal to zero.

We further assume the existence of a Nash equilibrium in pure strategies, such that prices satisfy the first-order condition

$$p_{jt} = c_{jt} - \frac{1}{\beta}\frac{dh_{jt}}{dq_{jt}} q_{jt}. \tag{4}$$

To obtain equation (4), we take the total derivative of h with respect to q_{jt}, re-arrange to obtain $\frac{dp_{jt}}{dq_{jt}} = \frac{1}{\beta}\frac{dh_{jt}}{dq_{jt}}$, and substitute into the more standard formulation of the first-order condition: $p = c - \frac{dp}{dq}q$. Markups are fully determined by β, the (assumed) structure of the model, and observables.[7] As equilibrium prices respond to demand shocks through markup adjustments, we are able to model the resulting correlation between prices and demand shocks. This allows for a broader set of identifying restrictions than previously recognized.

[7] We define markups as $p - c$. In particular, markups are proportional to the reciprocal of the price parameter due to the semi-linear demand system. The semi-linear structure is not strictly necessary. In practice, one could start with a known first-order condition and show that it takes the form $p_{jt} = c_{jt} - \frac{1}{\beta} f_{jt}$ for some function of the data f_{jt}. Our results can be extended to demand systems that admit multiplicative markups (Appendix B).

First-order conditions that admit multiple equilibria are unproblematic if the econometrician knows which equilibrium produces the data. It must be possible recover (ξ, η) from the data and parameters, but the mapping to prices from the parameters, exogenous covariates, and structural error terms need not be unique.

2.2 Three Approaches to Identification

By rearranging equation (4), the Nash equilibrium of the oligopoly model can be cast as the solution to the following two equations:

$$
\begin{aligned}
h_{jt} &= \beta p_{jt} + x'_{jt}\alpha + \xi_{jt} && \text{(Demand)} \\
\frac{dh_{jt}}{dq_{jt}} q_{jt} &= -\beta p_{jt} + \beta x'_{jt}\gamma + \beta\eta_{jt} && \text{(Supply)}
\end{aligned}
\tag{5}
$$

where again $h_{jt} \equiv h(q_{jt}, w_{jt}; \sigma)$. This formulation shows that consistent estimation of either supply or demand would be sufficient to identify the endogenous coefficient, β. With an estimate of β in hand, the other relation is identified and could be estimated using OLS. Thus, either approach allows for the identification of the entire system. There is also a class of assumptions that result in simultaneous identification of the two equations. Thus, we classify identification of these models in three approaches:

1. **Supply shifters**: A variable that shifts the supply curve but is uncorrelated with ξ may be used as an instrument for price to estimate the demand curve. Price is determined by marginal costs and markups, corresponding to the two right-hand side terms in equation (4). Valid instruments may shift either component. A marginal cost shifter corresponds to a variable $x^{(k)}$ for which $\alpha^{(k)} = 0$ and $\gamma^{(k)} \neq 0$, satisfying both the exclusion restriction (in demand) and the relevance condition (in supply). Alternatively, a markup shifter provides exogenous changes in the markup term $\frac{dh_{jt}}{dq_{jt}} q_{jt}$. Markup shifters may be obtained from the observable variables in w_{jt}; the characteristics of other products used in Berry et al. (1995) are one such example.

2. **Demand shifters**: Likewise, a demand shifter that is uncorrelated with η may be used to estimate supply. The inverse demand relation in equation (2) points to two possibilities for instruments. A variable $x^{(k)}$ may be used as an instrument for price if it satisfies the converse conditions to those of a supply shifter (i.e., $\alpha^{(k)} \neq 0$ and $\gamma^{(k)} = 0$). The second possibility is to employ an exogenous shifter of h as the instrument.[8] In contrast to the supply-shifter approach, knowledge of the other equation must be used in consistent estimation, as the structure of demand is used to construct $\frac{dh_{jt}}{dq_{jt}} q_{jt}$.

3. **Covariance restrictions**: The third approach to identification is to provide restrictions over ξ and η that simultaneously identify both equations. This approach can be used to identify the model even if no $x^{(k)}$ exists that satisfies either set of conditions above and the only exogenous variation in h and markups comes from the unobserved shocks.

The assumptions underlying the three approaches are not nested. Conceptually, each approach relies on untestable assumptions about the covariance structure of unobservables. In

[8] Thus, instruments constructed from w_{jt} may shift both markups and h. These could be used for consistent estimation of either the demand curve or the supply curve, or both jointly.

the instrument-based approaches, this takes the form of the exclusion restriction between an observable (or constructed) variable and either ξ or η. In the covariance restriction approach, the assumption is made over ξ and η jointly. The demand-shifter and supply-shifter approach have an additional relevance condition, which must be satisfied both theoretically and empirically. This points to an advantage of covariance restrictions, in that there is no "first-stage" empirical requirement. However, both the covariance restriction approach and the demand-shifter approach require formal supply-side specifications. Thus, an appealing feature of the supply-shifter approach compared to the demand-shifter approach is that there is no need to specify both supply and demand to recover the price parameter.

We do not view the requirement of a formal supply-side model as overly costly. Empirical papers are rarely concerned with identifying price coefficients alone. More often, the price coefficient is an input to a counterfactual analysis to measure the effect of a change in the market or the magnitude of an economic mechanism. For these counterfactual analyses, a formal supply-side model is required. Some articles have invoked these supply-side moments to improve efficiency in demand estimation (e.g., Berry et al., 1995). For robustness, the econometrician may consider alternative supply-side models, both in estimation and in simulating counterfactuals. We highlight that a formal treatment of supply provides two additional paths to identification, allowing the econometrician to either (i) exploit observed exogenous demand-side variation or (ii) correct for the relation between unobserved shocks.

One can also make assumptions about the correlation structure of unobservables to construct valid instruments. A well-known example of this approach are the prices of the good in other markets, which, in effect, are used to construct a proxy variable for unobserved costs. In practice, instruments are often constructed without a formal connection to the underlying model. In such cases, instruments for the demand equation that do not affect markups and are not in x, are, in effect, proxies for η and may interpreted as such.

As estimation using instrumental variables is generally well-understood, our focus for the rest of the paper is on developing results when both supply and demand are used jointly in estimation. Though the demand-shifter approach is less common,[9] previous papers have combined demand shifters with a covariance restriction to achieve identification (e.g., Hausman and Taylor, 1983; Matzkin, 2016). With both of these assumptions, the supply-side structural error term can be employed as an "unobserved instrument" for the demand equation. We show how covariance restrictions alone, i.e., in the absence of a demand shifter, may be used to jointly identify supply and demand.

[9] Igami and Sugaya (2018) may come closest to implementing this latter strategy. The authors estimate a Cournot supply relation using OLS, under the assumption that η is measurement error.

3 The Empirical Content of Supply-Side Assumptions

3.1 Expressing β in Terms of Data and Primitives

In this section we provide a constructive identification result for β. We show how the (biased) estimates obtained with OLS regression can be used to express the causal parameter in terms of model primitives. We take as given that the econometrician seeks to estimate $\theta = (\beta, \alpha, \gamma)$, with knowledge of σ.[10] Among the parameters in θ, we focus on β because the linear non-price parameters (α, γ) can be recovered trivially given β and σ,

An OLS regression of $h(\cdot)$ on p and x obtains an estimate of the price parameter with the probability limit:

$$\beta^{OLS} \equiv \frac{Cov(p^*, h)}{Var(p^*)} = \beta + \frac{Cov(p^*, \xi)}{Var(p^*)} \tag{6}$$

where $p^* = [I - x(x'x)^{-1}x']p$ is a vector of residuals from a regression of p on x. The conventional wisdom holds that $Cov(p^*, \xi) > 0$ in most applications (e.g., Nevo, 2001).

Intuitively, the maintained supply-side assumptions should inform how prices respond to demand shocks and thus the magnitude of OLS bias. To make progress formally, substitute for price on the right-hand-side of equation (6) using the first-order conditions:

$$\beta^{OLS} = \beta - \frac{1}{\beta} \frac{Cov\left(\frac{dh}{dq} q, \xi\right)}{Var(p^*)} + \frac{Cov(\eta, \xi)}{Var(p^*)}. \tag{7}$$

Our first general result obtains after a few additional lines of algebra, in which the unobserved demand shock ξ is expressed in terms of the OLS residuals and parameters. In particular, β solves a quadratic equation in which the coefficients are determined by (i) data, (ii) β^{OLS} and ξ^{OLS}, which are estimated consistently with OLS, and (iii) $Cov(\xi, \eta)$. Thus, the connection between OLS and the population parameters may be tighter than previously recognized.

Proposition 1. *Under assumptions 1 and 2, the probability limit of the OLS estimate can be written as a function of the true price parameter, the residuals from the OLS regression, the covariance between demand and supply shocks, prices, and quantities:*

$$\beta^{OLS} = \beta - \frac{1}{\beta + \frac{Cov\left(p^*, \frac{dh}{dq} q\right)}{Var(p^*)}} \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq} q\right)}{Var(p^*)} + \beta \frac{1}{\beta + \frac{Cov\left(p^*, \frac{dh}{dq} q\right)}{Var(p^*)}} \frac{Cov(\xi, \eta)}{Var(p^*)}. \tag{8}$$

[10]Alternatively, the econometrician may be considering a candidate σ and wishes to obtain corresponding estimates of (β, α, γ), as in the nested fixed-point estimation routine of Berry et al. (1995).

Therefore, the price parameter β solves the following quadratic equation:

$$
\begin{aligned}
0 = \; & \beta^2 \\
& + \left(\frac{Cov\left(p^*, \frac{dh}{dq} q\right)}{Var(p^*)} - \beta^{OLS} + \frac{Cov(\xi, \eta)}{Var(p^*)} \right) \beta \\
& + \left(-\beta^{OLS} \frac{Cov\left(p^*, \frac{dh}{dq} q\right)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq} q\right)}{Var(p^*)} \right).
\end{aligned} \tag{9}
$$

Proof. See appendix.

The proposition establishes a fundamental link between the price coefficient, the covariance of unobservable shocks, and the empirical variation present in the data. With the exceptions of β and $Cov(\xi, \eta)$, all of the terms in equation (9) can be constructed from data. Therefore, the quadratic admits at most two solutions for β for a given value of $Cov(\xi, \eta)$. It follows that, with prior knowledge of $Cov(\xi, \eta)$, the price parameter β is set identified with a maximum of two elements (points). Indeed, as we show below, conditions exist that guarantee point identification, allowing a consistent estimator to be constructed with the quadratic formula.

3.2 Invoking Supply-Side Assumptions in Estimation

We now consider the empirical content of supply-side assumptions when the econometrician does not have prior knowledge of $Cov(\xi, \eta)$. We assume that a consistent estimate of β has been obtained with the supply-shifter approach to estimation, and we wish to show whether supply-side assumptions improve the precision of the estimate. Our result is that the supply-side model provides *no additional information* about the parameter. In special cases, however, the supply-side allows for a specification check and efficiency improvements.

To understand why the supply side does not assist with identification, observe that equation (9) provides a function linking the endogenous parameter β to the covariance structure of unobservables. We can invert this function to obtain the implied value of $Cov(\xi, \eta)$ for any candidate value of β:

Corollary 1. ***(Implied Covariance)*** *There is a function mapping β to $Cov(\xi, \eta)$. The function is given by*

$$
\begin{aligned}
Cov(\xi, \eta) = \; & Var(p^*)\beta^{OLS} - Cov\left(p^*, \frac{dh}{dq} q\right) - \beta Var(p^*) \\
& + \frac{1}{\beta}\left(\beta^{OLS} Cov\left(p^*, \frac{dh}{dq} q\right) + Cov\left(\xi^{OLS}, \frac{dh}{dq} q\right)\right).
\end{aligned} \tag{10}
$$

Thus, $Cov(\xi, \eta)$ acts as a free parameter that rationalizes an estimate of β, conditional on the data, the structure of demand and supply, and the other parameters.

Nonetheless, supply-side assumptions can be used to check for misspecification. Under the demand and supply assumptions alone, it is possible to obtain a lower bound on $Cov(\xi, \eta)$. If the implied covariance from an estimate of β falls below this lower bound, then the estimate of the price parameter is incompatible with the data and the model. The lower bound is obtained by finding the value of β that minimizes the right-hand side of equation (10). This results in the following bound:

Proposition 2. ***(Covariance Bound)*** *Under assumptions 1 and 2, the model and data may bound* $Cov(\xi, \eta)$ *from below. The bound is given by*

$$Cov(\xi,\eta) \geq Var(p^*)\beta^{OLS} - Cov\left(p^*, \frac{dh}{dq}q\right) + 2Var(p^*)\sqrt{-\beta^{OLS}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}}. \tag{11}$$

The bound exists if and only if the term inside the radical is non-negative.

Proof. See appendix.

For an empirical application in which these covariance bounds prove useful, see the analysis of the airlines industry in Section 6.2.

It also can be possible to improve efficiency by imposing additional restrictions in conjunction with the supply-side assumptions. Berry et al. (1995) estimates both the demand and supply relations from equation (5) via instrumental variables. That is, they combine the first two approaches described in Section 2.2, using the same instruments as the supply shifters for demand and demand shifters for supply. The efficiency gain from joint estimation may be attributed to the additional assumption that the instruments are uncorrelated with respect to the cost shock, η, rather than from the supply-side structure alone. Critically, the Berry et al. (1995) instruments, which are constructed from the characteristics of other products, could be used to obtain a consistent estimate of β from either the demand equation or the supply equation.[11] If, instead, a marginal cost shifter were used to identify β, then imposing the supply-side assumptions would not yield efficiency improvements. This is because cost shifters are not able to separately identify the supply equation.

Additional information may also be obtained by restrictions on the covariance structure on unobservables. For example, applications that use the "optimal instruments" method (see, e.g., Reynaert and Verboven, 2014; Conlon and Gortmaker, 2019) make the assumption of homoskedasticity in the unobserved demand and cost shocks, which may provide modest efficiency gains. We now turn to examine the identifying power of one such restriction.

[11]Interestingly, this implies that a single instrument constructed from these characteristics *overidentifies* the price parameter, as could any (valid) instrument constructed from the broader set of w_{jt}.

3.3 Point Identification with Covariance Restrictions

Given Proposition 1, a restriction on the covariance of the unobserved structural error terms is a natural source of identification. This approach does not require the econometrician to observe variation in the exogenous covariates; rather the endogenous variation in prices and quantities is interpreted through the model. Here we focus on the conditions for point identification and provide a consistent estimator. Later we develop intuition using linear demand (Section 4) and discuss the how covariance restrictions can be evaluated (Section 3.4).

Assumption 3': *The econometrician has prior knowledge of $Cov(\xi, \eta)$.*

From equation (9), we know that Assumption 3 is sufficient to set identify β with at most two points. We now provide the additional conditions for point identification.

Proposition 3. ***(Point Identification)*** *Under assumptions 1 and 2, the price parameter β is the lower root of equation* (9) *if the following condition holds:*

$$0 \leq \beta^{OLS} \frac{Cov\left(p^*, \frac{dh}{dq} q\right)}{Var(p^*)} + \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq} q\right)}{Var(p^*)} \tag{12}$$

and, furthermore, β is the lower root of equation (9) *if and only if the following condition holds:*

$$-\frac{1}{\beta} \frac{Cov(\xi, \eta)}{Var(p^*)} \leq \frac{Cov\left(p^*, -\frac{1}{\beta}\xi\right)}{Var(p^*)} + \frac{Cov\left(p^*, \eta\right)}{Var(p^*)} \tag{13}$$

Therefore, under assumptions 1, 2 and 3', β is point identified if either of these conditions holds.

Proof. See appendix.

The first (sufficient) condition is derived as a simple application of the quadratic formula: if the constant term in the quadratic of equation (9) is negative then the upper root of the quadratic is positive and β must be the lower root. It can be evaluated empirically using the data and assumptions 1 and 2. The second (necessary and sufficient) condition is more nuanced because it contains elements that are not observed by the econometrician. However, provided that $Cov(\xi, \eta)$ is not too positive, the condition holds when prices increase in response to positive demand shocks. To see this in equation (13), note that the right-hand side depends on the covariance between prices and shocks to inverse demand ($-\frac{1}{\beta}\xi$) as well as the covariance between prices and marginal costs. We assume that this typical intuition holds in our empirical applications.[12]

We develop the estimator under the uncorrelatedness assumption, $Cov(\xi, \eta) = 0$. Let ($\hat{\beta}^{OLS}, \hat{\xi}^{OLS}$) be the OLS estimates of (β, ξ). Then we have:

[12]This condition can be verified analytically for some demand systems. When point identification may not be possible, Andrews and Soares (2010) offer an approach to construct valid confidence sets for inference.

Assumption 3 (Uncorrelatedness): $Cov(\xi, \eta) = 0$.

Corollary 2. ***(Three-Stage Estimator)*** *Under assumptions 1, 2, and 3, a consistent estimate of the price parameter β is given by*

$$\hat{\beta}^{\text{3-Stage}} = \frac{1}{2}\left(\hat{\beta}^{OLS} - \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - \sqrt{\left(\hat{\beta}^{OLS} + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\hat{\xi}^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}}\right) \quad (14)$$

if either condition (12) *or condition* (13) *holds.*

It is easily verified that the estimator is consistent for the lower root of equation (9). It can be calculated in three stages: (i) regress $h(q)$ on p and x with OLS, (ii) regress p on x with OLS and obtain the residuals p^*, and (iii) construct the estimator as shown. The computational burden is trivial, which may be beneficial if nested inside a search for the σ parameters.[13]

The three-stage estimator rests on the moment condition $E[\xi \cdot \eta] = 0$. Thus, an alternative approach to estimation is to search numerically for a $\tilde{\beta}$ that satisfies the corresponding empirical moment, yielding

$$\hat{\beta}^{MM} = \arg\min_{\tilde{\beta}<0}\left[\frac{1}{T}\sum_t \frac{1}{|J_t|}\sum_{j \in J_t} \xi(\tilde{\beta}; w_{jt}, \sigma, X_{jt}) \cdot \eta(\tilde{\beta}; w_{jt}, \sigma, X_{jt})\right]^2 \quad (15)$$

where $\xi(\tilde{\beta}; w, \sigma, X)$ and $\eta(\tilde{\beta}; w, \sigma, X)$ are the estimated residuals given the data and the candidate parameter, and the firms present in each market t are indexed by the set J_t. We see two main situations in which the numerical approach may be preferred despite its greater computational burden. First, additional moments can be incorporated in estimation, allowing for efficiency improvements and specification tests (e.g., Hausman, 1978; Hansen, 1982). Such moments could be constructed with valid instruments available in the data. Additional moments could also be constructed from further restrictions on unobservables. For example, the stronger assumption of mean independence ($E[\xi|\eta] = 0$) generates an infinite set of moments that can be employed using the techniques of the optimal instruments literature (Chamberlain, 1987; Berry et al., 1999; Reynaert and Verboven, 2014). Second, the three-stage estimator requires orthogonality between ξ and all regressors (i.e., $E[X\xi] = 0$), whereas the numerical approach can be pursued under a weaker assumption that allows for correlation between ξ and covariates that enter the cost function only.

[13] A more precise two-stage estimator is available for special cases in which the observed cost and demand shifters are uncorrelated (Appendix C).

3.4 Assessing Covariance Restrictions and Constructing Bounds

In many applications, econometricians may have detailed knowledge of the determinants of demand and marginal cost, even if these determinants are unobserved in the data. Such knowledge allows the econometrician to assess whether covariance restrictions along the lines of $Cov(\xi, \eta) = 0$ are reasonable. Covariance restrictions need not (and should not) be a "black box" that provides identification. The distinction between *understood* and *observed* is important, as the econometrician may have reasonable priors about the relationship between structural error terms even though they are (by definition) unobserved. For a constructive example, see the discussion about the cement industry in Section 6.2.

In many cases, institutional details suggest first-order correlations between unobserved determinants of demand and costs. Products with greater unobserved quality might be more expensive to produce, demand shocks could raise or lower marginal costs (e.g., due to capacity constraints), or firms might invest to lower the costs of their best-selling products. In these cases, the assumption of $Cov(\xi, \eta) = 0$ would not be reasonable, unless the confounding variation can be absorbed by control variables or fixed effects.

Econometricians with panel data may be able to incorporate fixed effects that absorb otherwise confounding correlations, such as those arising from product quality. To illustrate, consider the following generalized demand and cost functions:

$$\begin{aligned} h(q_{jt}, w_{jt}; \sigma) &= \beta p_{jt} + x'_{jt}\alpha + D_j + F_t + E_{jt} \\ c_{jt} &= x'_{jt}\gamma + U_j + V_t + W_{jt} \end{aligned} \tag{16}$$

with $Cov(E_{jt}, W_{jt}) = 0$. Let the unobserved shocks be $\xi_{jt} = D_j + F_t + E_{jt}$ and $\eta_{jt} = U_j + V_t + W_{jt}$, and let $h(\cdot)$ be known up to parameters. If products with higher quality have higher marginal costs then $Cov(U_j, D_j) > 0$. The econometrician can account for the relationship by estimating D_j for each firm; the residual $\xi^*_{jt} = \xi_{jt} - D_j$ is uncorrelated with U_j. Similarly, if costs are higher (or lower) in markets with high demand then market fixed effects can be incorporated. In this manner, panel data can account for product-specific and market-specific correlations in shocks. The econometrician can then assess whether a restriction on the correlation between product-specific deviations withing a market, E_{jt} and W_{jt}, is appropriate in the empirical setting.

Even without these controls, it may be possible to sign $Cov(\xi, \eta)$, allowing the econometrician to set identify parameters using bounds with priors. The demand and supply assumptions jointly imply two complementary sets of bounds, neither of which requires exact knowledge of $Cov(\xi, \eta)$. We start by developing what we refer to as *bounds with priors*. If the econometrician has a prior over the plausible range of $Cov(\xi, \eta)$, along the lines of $m \leq Cov(\xi, \eta) \leq n$, then a posterior set for β can be constructed from the quadratic of equation (9). Each plausible $Cov(\xi, \eta)$ maps into one or two valid (i.e., negative) roots. Further, a monotonicity result that we formalize below establishes that, under either condition (12) or (13), there is a one-to-one mapping between the value of $Cov(\xi, \eta)$ and the lower root:

Lemma 1. ***(Monotonicity)*** *Under assumptions 1 and 2, a valid lower root of equation* (9) ***(i.e., one that is negative)*** *is decreasing in $Cov(\xi, \eta)$. The range of the function is $(0, -\infty)$.*

Proof. See appendix.

It follows that a convex prior over $Cov(\xi, \eta)$ corresponds to convex posterior set. We suspect that, in practice, most priors will take the form $Cov(\xi, \eta) \geq 0$ or $Cov(\xi, \eta) \leq 0$. For example, an econometrician have reason to believe that higher quality products are more expensive to produce (yielding $Cov(\xi, \eta) \geq 0$) or that firms invest to lower the marginal costs of their best-selling products (yielding $Cov(\xi, \eta) \leq 0$). Priors of this firm generate one-sided bounds on β. Let $r(m)$ be the lower root of the quadratic evaluated at $Cov(\xi, \eta) = m$. Then under either condition (12) or (13), the prior $Cov(\xi, \eta) \geq m$ produces a posterior set of $(-\infty, r(m)]$, and the prior $Cov(\xi, \eta) \leq m$ produces a posterior set of $[r(m), 0)$.[14]

We also can construct a *prior-free bound*, even if the econometrician has no prior about $Cov(\xi, \eta)$. This result follows immediately from the monotonicity result when the covariance bound from equation (10) exists.

Corollary 3. ***(Prior-Free Bound)*** *When β is the lower root of equation* (9)*, evaluating this equation at the covariance bound provides an upper bound on β.*

For some intuition, it is helpful to represent the quadratic in equation (9) as az^2+bz+c, keeping in mind that one root is $z = \beta$ (< 0). Because $a = 1$, the quadratic forms a $\cup$-shaped parabola. If $c < 0$ then the existence of a negative root is guaranteed. However, if $c > 0$ then b must be positive and sufficiently large for a negative root to exist. By inspection of equation (9), this places restrictions on $Cov(\xi, \eta)$ and, in turn, β.

4 Small-Sample Performance

We provide numerical simulations for a stylized example to demonstrate the different approaches to identification and to provide intuition for the covariance restriction approach. We generate Monte Carlo results under different assumptions about the relative variance of unobserved demand and supply shocks.

4.1 Specification: Monopolist with Linear Demand

Consider the a specific version of the model in which a monopolist faces a downward-sloping linear demand schedule, $q_t = \beta p_t + x_t'\alpha + \xi_t$, and marginal cost is given by $c_t = x_t'\gamma + \eta_t$. The

[14]Nevo and Rosen (2012) develop similar bounds for estimation with imperfect instruments, defined as instruments that are less correlated with the structural error term than the endogenous regressor.

first order condition is $p_t = x_t'\gamma + \eta_t - \frac{1}{\beta}q_t$. The monopoly problem can be recast in terms of demand and supply:

$$\begin{aligned} q_t^d &= x_t'\alpha + \beta p_t + \xi_t && \text{(Demand)} \\ q_t^s &= \beta x_t'\gamma - \beta p_t + \beta\eta_t && \text{(Supply)} \end{aligned} \tag{17}$$

with the equilibrium condition $q_t^d = q_t^s$. These equations, which mirror the general formulation in equation (5), show that β can be consistently estimated with 2SLS from the demand equation using a supply shifter ($x^{(k)}$ s.t. $\alpha^{(k)} = 0$). Alternatively, demand shifters can be used to estimate the β from the supply equation ($x^{(k)}$ s.t. $\gamma^{(k)} = 0$).

The model is also identified under the covariance restriction $Cov(\xi, \eta) = 0$. The sufficient condition for point identification is simple to confirm analytically. Further, as $\frac{dh}{dq} = 1$ with linear demand, some terms in the three-stage estimator cancel and a simpler expression obtains:

$$\hat{\beta}^{\text{3-Stage}} = -\sqrt{\left(\hat{\beta}^{OLS}\right)^2 + \frac{Cov(\hat{\xi}^{OLS}, q)}{Var(p^*)}} \tag{18}$$

where p^* is the vector of residuals from a regression of p on x. We can express this three-stage estimator in a way that provides additional intuition about identification. Let q^* denote the vector of residuals from a regression of q on x. Because $\xi_t^{OLS} = q_t^* - \beta^{OLS}p_t^*$, we have $\frac{Cov(\xi^{OLS},q)}{Var(p^*)} = \frac{Var(q^*)}{Var(p^*)} - \beta^{OLS}\frac{Cov(p^*,q)}{Var(p^*)}$. Substituting obtains:

$$\hat{\beta}^{\text{3-Stage}} = -\sqrt{\frac{Var(q^*)}{Var(p^*)}}. \tag{19}$$

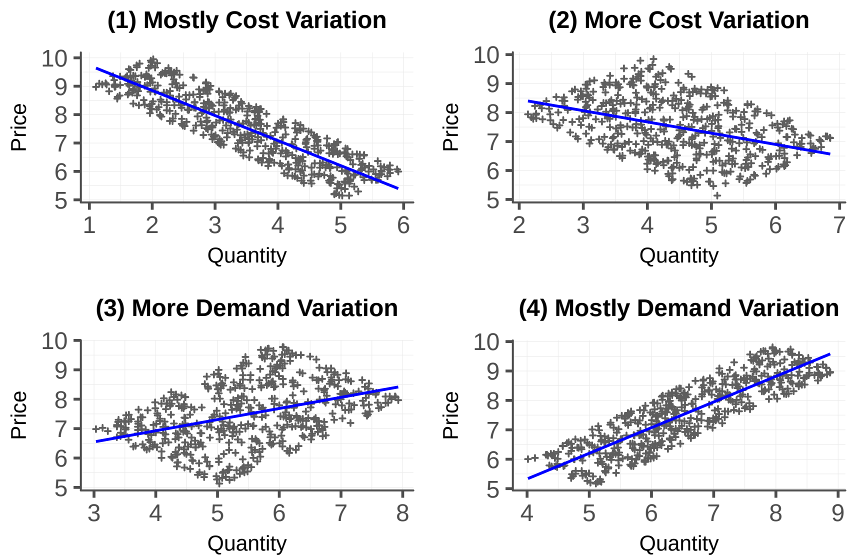
Thus, under a covariance restriction, the price parameter is identified from the relative variation in prices and quantities. In Appendix A.1, we connect these results to the Hayashi (2000) textbook treatment of bias with simultaneous equations, which relates the parameters to the relative variances of unobserved shocks.

4.2 Comparing Three Approaches

For our simulations, we parameterize demand as $q_t = 10 - p_t + \xi_t$ and marginal cost as $c_t = \eta_t$, so that $\beta = -1$. We let the demand and cost shocks have independent uniform distributions. We consider the three approaches for identification: supply shifters, demand shifters, and covariance restrictions. For the supply-shifter approach, we estimate demand with 2SLS using the cost shock η_t as the excluded instrument. Imposing the supply-side assumptions provide no benefit to this approach because our instrument is a cost shifter.[15] For the demand-shifter approach, we estimate the supply equation with 2SLS using ξ_t as the excluded instrument. Finally, we use the three-stage estimator for the covariance restriction approach.

[15] See Section 3.2. In addition, with the linear demand system, prior-free bounds do not exist and cannot be used as a specification check.

Figure 1: Price and Quantity in the Monopoly Model



Notes: Figure displays equilibrium prices and quantities under four different specifications for the distribution of unobserved shocks to demand and marginal costs. The line in each figure indicates the slope obtained by OLS regression.

We vary the support of η and ξ to compare environments in which empirical variation arises primarily from supply shocks or demand shocks. In case (1), $\xi \sim U(0,2)$ and $\eta \sim U(0,8)$. In case (2), $\xi \sim U(0,4)$ and $\eta \sim U(0,6)$. In case (3), $\xi \sim U(0,6)$ and $\eta \sim U(0,4)$. In case (4), $\xi \sim U(0,8)$ and $\eta \sim U(0,2)$. As is well known, if both cost and demand variation is present then equilibrium outcomes provide a "cloud" of data points that do not necessarily correspond to the demand curve. To illustrate this, we present one simulation of 500 observations from each specification in Figure 1, along with the fit of an OLS regression of quantity on price. Intuitively, the greater the demand-side variation, the larger the endogeneity bias. The probability limits of the OLS estimate in each scenario are -0.882, -0.385, 0.385, and 0.882.

To compare the small-sample performance of the approaches, we consider sample sizes of 25, 50, 100, and 500 observations. For each specification and sample size, we randomly draw 10,000 datasets. To account for large coefficients arising from the weak instrument problem, we bound the estimates of β on the range $[-100, 100]$. For specifications that suffer from weak instruments, this will bias the standard errors toward zero. This only affects specifications where the estimated standard error is greater than one, i.e., in 9 of our 48 specifications.

Table 1 summarizes the results. Panel (a) considers the supply-shifter approach. The left-most column shows that estimates that exploit variation in cost instruments are accurate when the variation in costs dominates the variation in demand shocks. Performance deteriorates as relatively more variation is caused by demand shocks, and this is exacerbated in smaller samples. The large bias in the upper right of panel (a) reflects a weak instrument problem,

Table 1: Small-Sample Properties: Mean Price Coefficients and Standard Errors

(a) Supply Shifters

Observations	(1) $Var(\eta) \gg Var(\xi)$		(2) $Var(\eta) > Var(\xi)$		(3) $Var(\eta) < Var(\xi)$		(4) $Var(\eta) \ll Var(\xi)$	
25	-1.004	(0.105)	-1.039	(0.303)	-1.310	(2.629)	-0.899	(13.921)
50	-1.001	(0.072)	-1.018	(0.201)	-1.113	(1.135)	-1.392	(10.890)
100	-1.001	(0.050)	-1.008	(0.138)	-1.048	(0.332)	-1.432	(5.570)
500	-1.000	(0.022)	-1.001	(0.060)	-1.009	(0.138)	-1.061	(0.411)

(b) Demand Shifters

Observations	(1) $Var(\eta) \gg Var(\xi)$		(2) $Var(\eta) > Var(\xi)$		(3) $Var(\eta) < Var(\xi)$		(4) $Var(\eta) \ll Var(\xi)$	
25	-0.881	(12.794)	-1.295	(3.087)	-1.040	(0.312)	-1.006	(0.106)
50	-1.448	(10.980)	-1.112	(0.596)	-1.016	(0.198)	-1.001	(0.073)
100	-1.597	(5.837)	-1.045	(0.333)	-1.009	(0.136)	-1.001	(0.050)
500	-1.070	(0.414)	-1.008	(0.137)	-1.002	(0.060)	-1.000	(0.022)

(c) Covariance Restrictions

Observations	(1) $Var(\eta) \gg Var(\xi)$		(2) $Var(\eta) > Var(\xi)$		(3) $Var(\eta) < Var(\xi)$		(4) $Var(\eta) \ll Var(\xi)$	
25	-1.004	(0.098)	-1.017	(0.201)	-1.018	(0.206)	-1.005	(0.099)
50	-1.001	(0.068)	-1.008	(0.136)	-1.007	(0.135)	-1.001	(0.068)
100	-1.001	(0.047)	-1.003	(0.094)	-1.004	(0.093)	-1.001	(0.047)
500	-1.000	(0.021)	-1.001	(0.041)	-1.001	(0.042)	-1.000	(0.021)

Notes: Results are based on 10,000 simulations of data for each specification and number of observations. The demand curve is $q_t = 10 - p_t + \xi_t$ and marginal costs are $c_t = \eta_t$, so that $(\beta_0, \beta, \gamma_0) = (10, -1, 0)$. The two-stage least squares estimates are calculated using marginal costs (η) as an instrument in the demand equation. In specification (1), $\xi \sim U(0,2)$ and $\eta \sim U(0,8)$. In specification (2), $\xi \sim U(0,4)$ and $\eta \sim U(0,6)$. In specification (3), $\xi \sim U(0,6)$ and $\eta \sim U(0,4)$. In specification (4), $\xi \sim U(0,8)$ and $\eta \sim U(0,2)$.

even though all the exogenous cost variation in observed. With specification (4), the mean F-statistics for the first-stage regression of p on η are 2.6, 4.2, 7.3, and 32.6 for markets with 25, 50, 100, and 500 observations, respectively. As this approach exploits only the price variation that can be attributed to costs, the success of the approach depends on the relative importance of cost to demand shocks in the data-generating process.[16]

Panel (b) considers the demand-shifter approach. The rightmost column shows that the demand-shifter approach performs well when the variation in demand shocks dominates the variation in supply shocks. The performance of this approach deteriorates as relatively more

[16]Additionally, specifications (3) and (4) generate positive OLS coefficients. In small samples, a positive OLS coefficient might suggest that an identification strategy relying on cost shifters or other supply-side instruments will not be fruitful, because much of the variation in prices is determined by demand. Of course, for any particular instrument, the first-stage F-statistic can be used to assess relevance.

variation in the model is attributable to costs. Where the supply-shifter approach does very well, the demand-shifter approach does poorly, and vice versa.

Finally, panel (c) considers the covariance restriction approach. The three-stage estimator is more efficient and has less bias than either of the instrument-based estimators, across all specifications and sample sizes considered. Because the estimator exploits all of the variation in the data, its performance does not depend on whether empirical variation arises more from variation in supply conditions or demand conditions. This can be seen by comparing specifications (1) and (2) (primarily cost variation) to specifications (3) and (4) (primarily demand variation). In particular, the standard errors are symmetric for specifications (1) and (4), as well as for specifications (2) and (3). In settings where instrumental variables perform poorly, a three-stage estimator may still provide a precise estimate.

Note that all three approaches rely on the orthogonality condition $Cov(\xi, \eta) = 0$. This is equivalent to the the exclusion restriction for the instrument-based approaches. Though the assumptions about the relations among unobservables overlap in this example, the instrument-based approaches also require the econometrician to observe either η_t or ξ_t. The covariance restriction approach is consistent when both are unobserved.

5 Extensions and Generalizations

The results developed thus far rely on some relatively strong (though common) restrictions on the form of demand and supply. In this section, we consider generalizations to non-constant marginal costs and alternative models of competition. First, we provide guidance on how additional restrictions may be imposed to estimate demand systems when σ is unknown. We provide additional extensions the appendices. In Appendix B, we discuss how to generalize the demand assumption to incorporate, for example, constant elasticity demand. In Appendix D, we provide the extension to multi-product firms.

5.1 Identification of Nonlinear Parameters

We now consider identification of the parameter vector, σ, which contains parameters that enter the demand system nonlinearly or load onto endogenous regressors other than price. Conditional on σ, knowledge of $Cov(\xi, \eta)$ can be sufficient to identify the linear parameters in the model, including β (Proposition 3). Thus, the single covariance restriction of Assumption 3 provides a function that maps σ to β and generates an identified set for (β, σ). To point identify (β, σ) when σ is unknown, supplemental moments must be employed. These moments may be generated from instruments or from covariance restrictions that extend the notion of uncorrelatedness. Covariance restrictions can be used to identify a large number of parameters without requiring the econometrician to isolate exogenous variation in each endogenous covariate.

Consider the relation between any demand shock and any marginal cost shock across products (j, k) and markets (t, s): (ξ_{jt}, η_{ks}). Assumption 3, which is sufficient to identify the price parameter, can be expressed as

$$E_{jt}[\xi_{jt} \cdot \eta_{jt}] = 0, \tag{20}$$

where the expectation is taken over products and markets. One could construct other first-order relations, such as

$$E_t[\xi_{jt} \cdot \eta_{kt}] = 0 \quad \forall j, k \tag{21}$$

where the expectation is taken over markets for each pair (j, k), providing $J \times J$ restrictions. We impose this set of moments in our first application, in which we estimate 12 nonlinear parameters in the random coefficients demand system of Nevo (2000). Other restrictions could be imposed on the time-series or cross-sectional correlation in shocks. We provide some intuition about the link between moments and parameters in each application. For a formal analysis of the sensitivity of parameters to estimation moments, see Andrews et al. (2017).

One could also construct restrictions on functions of these shocks, providing higher-order moments. In some cases, it may be reasonable to assume that the variance of the demand shock does not depend on the level of the cost shock, and vice versa, generating second-order moments of the form $E_{jt}[\xi_{jt}^2 \cdot \eta_{jt}]$ and $E_{jt}[\xi_{jt} \cdot \eta_{jt}^2]$. As another example, if product-level shocks are uncorrelated, it may be reasonable to assume that shocks are uncorrelated when aggregated by product group. These moments take the form

$$E_{gt}[\overline{\xi}_{gt} \cdot \overline{\eta}_{gt}] = 0, \tag{22}$$

where $\overline{\xi}_{gt} = \frac{1}{|g|} \sum_{j \in g} \xi_{jt}$ and $\overline{\eta}_{gt} = \frac{1}{|g|} \sum_{j \in g} \eta_{jt}$ are the mean demand and cost shocks within a group-market.[17] We impose related moment inequalities in our third application, in which we examine the demand system of Aguirregabiria and Ho (2012).

With these additional moments, the econometrician could pursue joint estimation via method-of-moments, searching over the parameter space for the pair (β,σ) that minimizes a weighted sum of the moments. Since the three-stage estimator is consistent for β conditional on σ, the econometrician could instead calculate $\hat{\beta}^{3\text{-}Stage}$ for each candidate parameter vector $\tilde{\sigma}$, selecting $(\hat{\beta}^{3\text{-}Stage}(\tilde{\sigma}), \tilde{\sigma})$ that minimize a weighted sum of the supplemental moments. The second approach can have an advantage in computational efficiency, especially if σ is low-dimensional.

5.2 Non-Constant Marginal Costs

If marginal costs are not constant in output, then unobserved demand shocks that change quantity also affect marginal cost. For example, consider a special case in which marginal costs

[17] Note that this assumption does not provide independent identifying power if (ξ, η) are jointly normal, because it would be implied by orthogonality.

take the form:

$$c_{jt} = x_{jt}'\gamma + g(q_{jt};\lambda) + \eta_{jt} \tag{23}$$

Here $g(q_{jt};\lambda)$ is some potentially nonlinear function that may (or may not) be known to the econometrician. Maintaining Bertrand competition and the baseline demand assumption, the first-order conditions of the firm are:

$$p_{jt} = \underbrace{x_{jt}'\gamma + g(q_{jt};\lambda) + \eta_{jt}}_{\text{Marginal Cost}} + \underbrace{\left(-\frac{1}{\beta}\frac{dh_{jt}}{dq_{jt}}q_{jt}\right)}_{\text{Markup}}. \tag{24}$$

Thus, provided $g'(\cdot;\lambda) \neq 0$, markup adjustments are no longer the only mechanism through with prices respond to demand shocks. This also can be seen from the OLS regression of $h(q_{jt}, w_{jt};\sigma)$ on p and x, which yields a price coefficient with the following probability limit:

$$plim(\hat{\beta}^{OLS}) = \beta - \frac{1}{\beta}\frac{Cov\left(\xi, \frac{dh}{dq}q\right)}{Var(p^*)} + \frac{Cov(\xi, g(q;\lambda))}{Var(p^*)} \tag{25}$$

The third term on the right-hand-side shows that bias depends on how demand shocks affect the non-constant potion of marginal costs. Unless prior knowledge of $g(q_{jt};\lambda)$ can be brought to bear, additional restrictions are necessary to extend the identification results of the preceding section.

There are two ways to make progress. First, if $g'(\cdot;\lambda)$ can be signed, then it is possible to bound the price parameter, β. A leading example is that of capacity constraints, for which it might be reasonable to assume that $Cov(\xi,\eta) = 0$ and $g'(\cdot;\lambda) \geq 0$. Thus, bounds with priors can be constructed from $Cov(\xi,\eta^*) \geq 0$ where $\eta^*_{jt} \equiv \eta_{jt} + g(q_{jt};\lambda)$ is a composite error term. Prior knowledge of $Cov(\xi,\eta)$ is sufficient to at least set identify β in such a situation. Second, the econometrician may be able to estimate $g(q_{jt};\lambda)$, either in advance or simultaneously with the price coefficient. Supplemental covariance restrictions, such as those discussed in the previous section, can be used to identify λ.

Proposition 4. *Under assumptions 1 and 3 and a modified assumption 2 in which marginal costs take the semi-linear form of equation* (23)*, the price parameter β solves the following quadratic* *equation:*

$$
\begin{aligned}
0 = \quad & \left(1 - \frac{Cov(p^*, g(q;\lambda))}{Var(p^*)}\right)\beta^2 & (26)\\
& + \left(\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)} - \hat{\beta}^{OLS} + \frac{Cov(p^*, g(q;\lambda))}{Var(p^*)}\hat{\beta}^{OLS} + \frac{Cov(\hat{\xi}^{OLS}, g(q;\lambda)}{Var(p^*)}\right)\beta \\
& + \left(-\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\hat{\beta}^{OLS} - \frac{Cov\left(\hat{\xi}^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}\right)
\end{aligned}
$$

where $\hat{\beta}^{OLS}$ *is the OLS estimate and* $\hat{\xi}^{OLS}$ *is a vector containing the OLS residuals.*

Proof. See appendix.

With the above quadratic in hand, the remaining results of Section 3 extend naturally. Although the estimation of $g(q_{jt};\lambda)$ is not our focus, we note that a three-stage estimator of β could be obtained for any candidate parameters in λ, thereby facilitating computational efficiency.

5.3 Alternative Models of Competition

Though our main results are presented under Bertrand competition in prices, our method applies to a broader set of competitive assumptions. Consider, for example, Nash competition among profit-maximizing firms that have a single choice variable, a, and constant marginal costs. The individual firm's objective function is:

$$\max_{a_j|a_i, i\neq j} (p_j(a) - c_j)q_j(a). \tag{27}$$

This generalized model of Nash competition nests Bertrand ($a = p$) and Cournot ($a = q$). The first-order condition, holding fixed the actions of the other firms, is given by:

$$p_j(a) = c_j - \frac{{p_j}'(a)}{{q_j}'(a)}q_j(a). \tag{28}$$

In equilibrium, we obtain the structural decomposition $p = c + \mu$, where μ incorporates the structure of demand and its parameters. This decomposition provides a restriction on how prices move with demand shocks, aiding identification. It can be obtained in other contexts, including consistent conjectures and competition in quantities with increasing marginal costs. We provide one such extension in the empirical application to the cement industry.

The approach can be extended to demand systems that generate an alternative structure for equilibrium prices. For example, consider a monopolist facing a constant elasticity demand schedule. The optimal price is $p = c\frac{\epsilon}{1+\epsilon}$ where $\epsilon < 0$ is the elasticity of demand. In this model, the monopolist has a multiplicative markup that does not respond to demand shocks. As we show in Appendix B, it is straightforward to extend our identification results to this model and other demand systems that admit a more general class of multiplicative markups, which result when demand is semi-linear in log prices.[18]

6 Empirical Applications

6.1 Ready-to-Eat Cereal

In our first application, we examine the pseudo-real cereals data of Nevo (2000).[19] The model features random coefficients logit demand and Bertrand competition among multi-product firms. We use the application to demonstrate that covariance restrictions can identify the price parameter, β, as well as the nonlinear parameters, σ. In the data, there is no variation in the product choice sets across markets, so the instruments defined in Berry et al. (1995) and Gandhi and Houde (2015) are unavailable. The instruments provided in the data set and employed in Nevo (2000) are constructed from the prices of the same product in other markets. We compare the estimates from a specification with these instruments to a specification with covariance restrictions.

The indirect utility that consumer i receives from product j in market m and period t is given by

$$u_{ijmt} \;\; = \;\; x_j \alpha_i^* + \beta_i^* p_{jmt} + \zeta_j + \xi_{jmt} + \epsilon_{ijmt} \tag{29}$$

where ζ_j is a product fixed effect and ϵ_{ijmt} is a logit error term. The indirect utility provided by the outside good, $j = 0$, is given by $u_{i0mt} = \epsilon_{i0mt}$. The consumer-specific coefficients take the form

$$\begin{bmatrix} \alpha_i^* \\ \beta_i^* \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \Pi D_i + \Sigma \nu_i \tag{30}$$

where D_i is a vector of observed demographics and ν_i is vector of unobserved demographics that have independent standard normal distributions. Within the notation of Section 2, the nonlinear parameters σ include the elements of Π and Σ, w_{jmt} is composed of the demographics, and $h(q_{jmt}, w_{jmt}; \tilde{\sigma})$ can be recovered using the contraction mapping of Berry et al. (1995) for any candidate parameters $\tilde{\sigma}$. The supply side of the model is the multiproduct version of our Assumption 2.

The data are a panel of 24 brands, 47 markets, and two quarters. We estimate the demand

[18] Our identification result relies on separability between marginal costs and markups, which may be obtained after a transformation of prices.

[19] See also Dubé et al. (2012), Knittel and Metaxoglou (2014), and Conlon and Gortmaker (2019). We focus on the "restricted" specification of Conlon and Gortmaker (2019), which addresses a multicollinearity problem by imposing that the nonlinear parameter on Price$\times$Income^2 takes a value of zero.

Table 2: Point Estimates for Ready-to-Eat Cereal

(a) Available Instruments

Variable	Means	Standard Deviations	Interactions with Demographics: Income	Age	Child
Price	-32.019	1.803	4.187	–	11.755
	(2.304)	(0.920)	(4.638)		(5.198)
Constant	–	0.120	3.101	1.198	–
		(0.163)	(1.105)	(1.048)	
Sugar	–	0.004	-0.190	0.028	–
		(0.012)	(0.035)	(0.032)	
Mushy	–	0.086	1.495	-1.539	–
		(0.193)	(0.648)	(1.107)	

(b) Covariance Restrictions

Variable	Means	Standard Deviations	Interactions with Demographics: Income	Age	Child
Price	-36.230	1.098	14.345	–	26.906
	(1.122)	(1.067)	(1.677)		(1.384)
Constant	–	0.051	-0.156	1.072	–
		(0.230)	(0.286)	(0.240)	
Sugar	–	0.003	-0.084	-0.004	–
		(0.014)	(0.018)	(0.010)	
Mushy	–	0.130	0.301	-0.845	–
		(0.162)	(0.196)	(0.103)	

Notes: Table reports point estimates for the random-coefficients logit demand system estimated using the Nevo (2000) dataset. Panel A employs the available instruments and replicates Conlon and Gortmaker (2019). Panel B employs covariance restrictions described by the text.

parameters, $\theta = (\beta, \alpha, \Pi, \Sigma)$, using the covariance restrictions $Cov(\xi_j, \eta_k) = 0$ for all j, k, as proposed in Section 5.1. The identifying assumption is that the demand shock of each product is orthogonal to its own marginal cost shock and those of all other products. The $J \times J$ (= 576) covariance restrictions are more than sufficient to identify the 12 nonlinear parameters in the Conlon and Gortmaker (2019) specification. Roughly, cross-product covariance restrictions allow the empirical relationship between the shares of product j and the prices of product k to be interpreted as arising from model parameters (σ), rather than systematic correlation between demand shocks to one product and marginal cost shocks to its substitute.[20]

Table 2 summarizes the results of estimation based on the instruments (panel (a)) and covariance restrictions (panel (b)). Both identification strategies yield similar mean own-price demand elasticities: -3.70 with instruments and -3.61 with covariance restrictions. Overall, the different approaches produce similar patterns for the coefficients. Most of the point esti-

[20]The restrictions allow for correlation in demand shocks across products and cost shocks across products.

mates under covariance restrictions fall in the 95 percent confidence intervals implied by the specification with instruments, including that of the mean price parameter. Only one of the interaction terms that is statistically significant (Constant×Income) changes sign. Excluding the standard deviation parameters, the standard errors are noticeably smaller with covariance restrictions, which likely reflects the greater number of identifying restrictions. Finally, covariance restrictions fit the data reasonably well. At the estimated parameters, the correlation between own demand shocks and own marginal cost shocks is 0.0015 and the mean absolute value across the 576 sample moments is 0.1145.

6.2 The Portland Cement Industry

Our second empirical application uses the setting and data of Fowlie et al. (2016) ["FRR"], which examines market power in the cement industry and its effects on the efficacy of environmental regulation. The model features Cournot competition among undifferentiated cement plants facing capacity constraints.[21] We use the application to demonstrate that knowledge of institutional details can help evaluate the uncorrelatedness assumption.

We begin by extending our results to Cournot competition with non-constant marginal costs. Let $j = 1, \ldots, J$ firms produce a homogeneous product demanded by consumers according to $h(Q; w) = \beta p + x'\gamma + \xi$, where $Q = \sum_j q_j$, and p represents a price common to all firms in the market. Marginal costs are semi-linear, as in equation (23), possibly reflecting capacity constraints. Working with aggregated first-order conditions, it is possible to show that the OLS regression of $h(Q; w_{jt})$ on price and covariates yields:

$$plim\left(\hat{\beta}^{OLS}\right) = \beta - \frac{1}{\beta}\frac{1}{J}\frac{Cov\left(\xi, \frac{dh}{dq}Q\right)}{Var(p^*)} + \frac{Cov(\xi, \overline{g})}{Var(p^*)} \tag{31}$$

where J is the number of firms in the market and $\overline{g} = \frac{1}{J}\sum_{j=1}^{J} g(q_j; \lambda)$ is the average contribution of $g(q, \lambda)$ to marginal costs. Bias arises due to markup adjustments and the correlation between unobserved demand and marginal costs generated through $g(q; \lambda)$.[22] The identification result provided in Section 5.2 for models with non-constant marginal costs extends.

[21] A published report of the Environment Protection Agency (EPA) states that "consumers are likely to view cement produced by different firms as very good substitutes.... there is little or no brand loyalty that allows firms to differentiate their product" EPA (2009).

[22] Bias due to markup adjustments dissipates as the number of firms grows large. Thus, if marginal costs are constant then the OLS estimate is likely to be close to the population parameter in competitive markets. In Monte Carlo experiments, we have found similar results for Bertrand competition and logit demand.

Corollary 4. *In the Cournot model, the price parameter β solves the following quadratic equation:*

$$
\begin{aligned}
0 = \; & \left(1 - \frac{Cov(p^*, \overline{g})}{Var(p^*)}\right)\beta^2 \qquad (32)\\
& + \left(\frac{1}{J}\frac{Cov\left(p^*, \frac{dh}{dq}Q\right)}{Var(p^*)} + \frac{Cov(\xi, \overline{\eta})}{Var(p^*)} - \hat{\beta}^{OLS} + \frac{Cov(p^*, \overline{g})}{Var(p^*)}\hat{\beta}^{OLS} + \frac{Cov(\hat{\xi}^{OLS}, \overline{g})}{Var(p^*)}\right)\beta \\
& + \left(-\frac{1}{J}\frac{Cov\left(p^*, \frac{dh}{dq}Q\right)}{Var(p^*)}\hat{\beta}^{OLS} - \frac{1}{J}\frac{Cov\left(\hat{\xi}^{OLS}, \frac{dh}{dq}Q\right)}{Var(p^*)}\right)
\end{aligned}
$$

The derivation tracks exactly the proof of Proposition 4. For the purposes of the empirical exercise, we compute the three-stage estimator as the empirical analog to the lower root.

Turning to the application, FRR examine 20 distinct geographic markets in the United States annually over 1984-2009. Let the demand curve in market m and year t have a logit form:

$$h(Q_{mt}; w) \equiv \ln(Q_{mt}) - \ln(M_m - Q_{mt}) = \alpha_r + \beta p_{mt} + \xi_{mt} \qquad (33)$$

where M_m is the "market size" of the market. We assume $M_r = 2 \times \max_t\{Q_{mt}\}$ for simplicity.[23] Further, let marginal costs take the "hockey stick" form of FRR:

$$
\begin{aligned}
c_{jmt} &= \gamma + g(q_{jmt}) + \eta_{jmt} \qquad (34)\\
g(q_{jmt}) &= 2\lambda_2 1\{q_{jmt}/k_{jm} > \lambda_1\}(q_{jmt}/k_{jm} - \lambda_1)
\end{aligned}
$$

where k_{jm} and q_{jmt}/k_{jm} are capacity and utilization, respectively. Marginal costs are constant if utilization is less than the threshold $\lambda_1 \in [0, 1]$, and increasing linearly at rate determined by $\lambda_2 \geq 0$ otherwise. The two unobservables, (ξ, η), capture demand shifts and shifts in the constant portion of marginal costs.

The institutional details of the industry suggest that uncorrelatedness may be reasonable. Demand is procyclical because cement is used in construction projects; given the demand specification this cyclicality enters through the unobserved demand shock. On the supply side, the two largest cost components are "materials, parts, and packaging" and "fuels and electricity" (EPA, 2009). Both depend on the price of coal. With regard to "fuels and electricity," most cement plants during the sample period rely on coal as their primary fuel, and electricity prices are known to correlate with coal prices. With regard to "material, parts, and packaging," the main input in cement manufacture is limestone, which requires significant amounts of electricity to extract (National Stone Council, 2008). Thus, an assessment of uncorrelatedness hinges largely on the relationship between construction activity and coal prices.

[23] We use logit demand rather than the constant elasticity demand of FRR because it fits easily into our framework. The 2SLS results are unaffected by the choice. Similarly, the 3-Stage estimator with logit obtains virtually identical results as a method-of-moments estimator with constant elasticity demand that imposes uncorrelatedness.

Table 3: Point Estimates for Cement

Estimator:	3-Stage	2SLS	OLS
Elasticity	-1.15	-1.07	-0.47
of Demand	(0.18)	(0.19)	(0.14)

Notes: The sample includes 520 market-year observations over 1984-2009. Bootstrapped standard errors are based on 200 random samples constructed by drawing markets with replacement.

In this context, there is a theoretical basis for orthogonality: if coal suppliers have limited market power and roughly constant (realized) marginal costs, then coal prices should not respond much to demand. Indeed, this is precisely the identification argument of FRR, as both coal and electricity prices are included in the set of excluded instruments.

Table 3 summarizes the results of demand estimation. The 3-Stage estimator is implemented taking as given the nonlinear cost parameters obtained in FRR: $\lambda_1 = 0.869$ and $\lambda_2 = 803.65$. In principle, these could be estimated simultaneously via the method of moments, provided some demand shifters can be excluded from marginal costs, but estimation of these parameters is not our focus. As shown, the mean price elasticity of demand obtained with the 3-Stage estimator under uncorrelatedness is -1.15. This is statistically indistinguishable from the 2SLS elasticity estimate of -1.07, which is obtained using the FRR instruments: coal prices, natural gas prices, electricity prices, and wage rates. The closeness of the 3-Stage and 2SLS is not coincidental and instead reflects that the identifying assumptions are quite similar. Indeed, the main difference is whether the cost shifters are treated as observed (FRR) or unobserved (3-Stage).

If the econometrician does not know (and cannot identify) the nonlinear parameters in the cost function, then consistent estimates cannot be obtained with our methodology. Further, prior-free bounds are unavailable as the empirical upper root of the quadratic in Corollary 4 is positive. Nonetheless, some progress can be made using posterior bounds. Define the composite marginal cost shock, $\eta^*_{jmt} = g(q_{jmt}) + \eta_{jmt}$, as inclusive of the capacity effects. Given the upward-sloping marginal costs, we have $Cov(\xi, \overline{\eta}^*) \geq 0$ if $Cov(\xi, \overline{\eta}) = 0$. This restriction generates an upper bound on the demand elasticity of -0.69, ruling out the OLS point estimate.

6.3 The Airline Industry

In our third empirical application, we examine demand for airline travel using the setting and data of Aguirregabiria and Ho (2012) ["AH"].[24] AH explores why airlines form hub-and-spoke networks; here, we focus on demand estimation only. The model features differentiated-products Bertrand competition among multi-product firms facing a nested logit demand system.

[24]We thank Victor Aguirregabiria for providing the data. Replication is not exact because the sample differs somewhat from what is used in the AH publication and because we employ a different set of fixed effects in estimation.

We use the application to demonstrate how bounds can narrow the identified set for (β, σ).

The nested logit demand system can be expressed as

$$h(s_{jmt}, w_{jmt}; \sigma) \equiv \ln s_{jmt} - \ln s_{0mt} - \sigma \ln \bar{s}_{jmt|g} = \beta p_{jmt} + x'_{jmt}\alpha + \xi_{jmt} \qquad (35)$$

where s_{jmt} is the market share of product j in market m in period t. The conditional market share, $\bar{s}_{j|g} = s_j / \sum_{k \in g} s_k$, is the the choice probability of product j given that its "group" of products, g, is selected. The outside good is indexed as $j = 0$. Higher values of σ increase within-group consumer substitution relative to across-group substitution. In contrast to the typical expression for the demand system, we place $\sigma \ln \bar{s}_{jmt|g}$ on the left-hand side so that the right-hand side contains a single endogenous regressor: price.

The data are drawn from the *Airline Origin and Destination Survey* (DB1B) survey, a ten percent sample of airline itineraries, for the four quarters of 2004. Markets are defined as directional round trips between origin and destination cities. Consumers within a market choose among airlines and whether to take a nonstop or one-stop itinerary. Thus, each airline offers zero, one, or two products per market. The nesting parameter, σ, governs consumer substitution within each product group: nonstop flights, one-stop flights, and the outside good. The supply side of the model is the multiproduct version of Assumption 2.[25]

The institutional details of the airline industry suggest that the covariance between demand shocks and marginal cost shocks is positive. The marginal production cost to carry an additional passenger is small and roughly constant because each additional passenger has little impact on the inputs needed to fly the plane from one airport to another. However, the airline bears an opportunity cost for each sold seat, as they can no longer sell the seat at a higher price to another passenger. When the airline expects the flight to be at capacity, this opportunity cost may become large (Williams, 2017). Thus, because positive shocks to demand produce more full flights, it is reasonable to assume $Cov(\xi, \eta) \geq 0$.

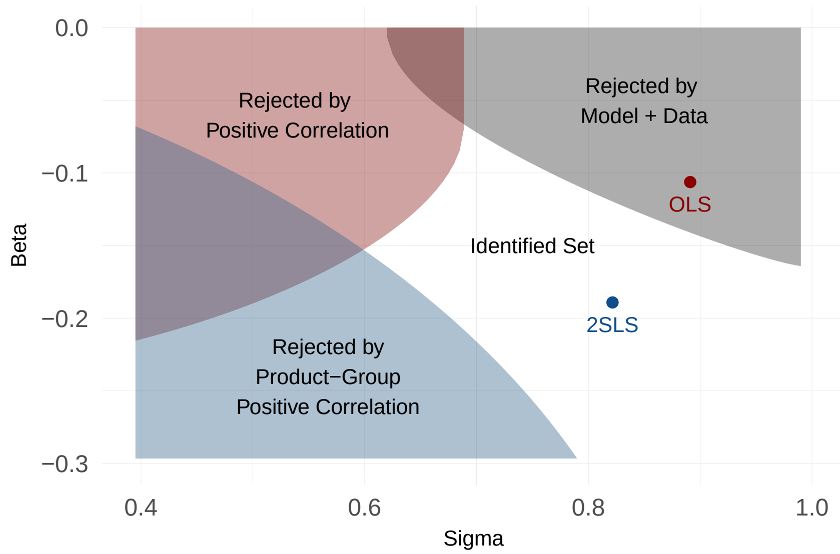
Under that assumption, it is possible to reject values (β, σ) that produce negative correlation in product-specific shocks. The econometrician can combine this prior with the prior-free bounds developed earlier. Finally, one can consider reasonable extensions of the priors over the correlation between demand and supply shocks. Following the logic from Section 5.1, we impose the group-level moment

$$E_{gmt}[\bar{\xi}_{gmt} \cdot \bar{\eta}_{gmt}] \geq 0, \qquad (36)$$

where $\bar{\xi}_{gmt} = \frac{1}{|g|} \sum_{j \in g} \xi_{jmt}$ and $\bar{\eta}_{gmt} = \frac{1}{|g|} \sum_{j \in g} \eta_{jmt}$ are the mean demand and cost shocks within a group-market-period. If the correlation in product-level shocks is weakly positive, it

[25]The covariates include an indicator for nonstop itineraries, the distance between the origin and destination cities, and a measure of the airline's "hub sizes" at the origin and destination cities. We also include airline fixed effects and route×quarter fixed effects. The latter expands on the city×quarter fixed effects described by AH. Market size, which determines the market share of the outside good, is equal to the total population in the origin and destination cities.

Figure 2: Identified Parameter Set Under Priors



Notes: Figure displays candidate parameter values for (σ, β). The gray region indicates the set of parameters that cannot generate the observed data from the assumptions of the model. The red region indicates the set of parameters that generate $Cov(\xi, \eta) < 0$, and the blue region indicates parameters that generate $Cov(\overline{\xi}, \overline{\eta}) < 0$. The identified set is obtained by rejecting values in the above regions under the assumption of (weakly) positive correlation. For context, the OLS and the 2SLS estimates are plotted, along with 3-Stage estimates I-IV. The parameter σ can only take values on $[0, 1)$.

is reasonable to assume that the group-level shocks are also weakly positive, through a similar deduction. By rejecting parameter values that fail to generate the data or that deliver negative correlation, the econometrician can narrow the identified set.

Figure 2 displays the rejected regions based on the model and above priors on unobserved shocks. The gray region corresponds to the parameter values rejected by the prior-free bounds. Based on these bounds only, some values of β can be rejected if $\sigma \geq 0.62$. As σ becomes larger, a more negative β is required to rationalize the data within the context of the model. If $\sigma = 0.80$ then it must be that $\beta \leq -0.11$. The dark red region corresponds to parameter values that generate negative correlation between demand and supply shocks. These can be rejected under the prior that $Cov(\xi, \eta) \geq 0$. The dark blue region provides the corresponding set for the prior $Cov(\overline{\xi}, \overline{\eta}) \geq 0$.

The three regions overlap, but no region is a strict subset of the other. The remaining non-rejected values provide the identified set. In our application, we are able to rule out values of σ less than 0.599 for any value of β, as these lower values cannot generate positive correlation in both product-level and product-group-level shocks. Similarly, we obtain an upper bound on β of -0.067 across all values of σ. For context, we plot the OLS and the 2SLS estimates in Figure 2. The OLS estimate falls in a rejected region and can be ruled out by the structure of the model alone. The 2SLS estimate, in contrast, falls within the identified set. This result is not mechanical, as these point estimates are generated with non-nested assumptions.

7 Conclusion

Our objective has been to evaluate the identifying power of supply-side assumptions in models of imperfect competition. Invoking the supply model in estimation expands the set of restrictions that obtain identification. We show that price endogeneity can be resolved by interpreting a reduced-form (OLS) estimate through the lens of a theoretical model. With a covariance restriction, the demand system is point identified, and weaker assumptions generate bounds on the structural parameters. Thus, causal demand parameters can be recovered without the use of instrumental variables. Though we focus our results on specific, widely-used assumptions about demand and supply, we view our method as not particular to these assumptions. Rather, the main insight is that information about supply-side behavior can be modeled to adjust the observed relationships between quantity and price. Our method provides a direct way to correct for endogeneity arising from the adjustment of markups in response to demand shocks. We hope that the methods developed help facilitate research in areas for which strong instruments are unavailable or difficult to find.

References

Aguirregabiria, V. and C.-Y. Ho (2012). A dynamic oligopoly game of the US airline industry: Estimation and policy experiments. *Journal of Econometrics 168*(1), 156–173.

Andrews, D. W. K. and G. Soares (2010). Inference for Parameters Defined by Moment Inequalities Using Generalized Moment Selection. *Econometrica 78*(1), 119–157.

Andrews, I., M. Gentzkow, and J. M. Shapiro (2017). Measuring the Sensitivity of Parameter Estimates to Estimation Moments. *The Quarterly Journal of Economics 132*(4), 1553–1592.

Bajari, P. (2003). Comment on "Bayesian analysis of simultaneous demand and supply". *Quantitative Marketing and Economics 1*, 277–283.

Berry, S. (2003). Comment on "Bayesian analysis of simultaneous demand and supply". *Quantitative Marketing and Economics 1*, 285–291.

Berry, S. and P. A. Haile (2014). Identification in differentiated products markets using market level data. *Econometrica 82*(5), 1749–1797.

Berry, S., J. Levinsohn, and A. Pakes (1995). Automobile Prices in Market Equilibrium. *Econometrica 63*(4), 841.

Berry, S. T. (1994). Estimating discrete-choice models of product differentiation. *RAND Journal of Economics 25*(2), 242–262.

Berry, S. T., J. Levinsohn, and A. Pakes (1999). Voluntary export restraints on automobiles: Evaluating a trade barrier. *American Economic Review 89*(3), 400–430.

Bresnahan, T. (1987). Competition and collusion in the american automobile oligopoly: The 1955 price war. *Journal of Industrial Economics 35*(4), 457–482.

Broda, C. and D. E. Weinstein (2006). Globalization and the gains from variety. *The Quarterly journal of economics 121*(2), 541–585.

Broda, C. and D. E. Weinstein (2010). Product creation and destruction: Evidence and price implications. *American Economic Review 100*, 691–723.

Byrne, D. P., S. Imai, N. Jain, V. Sarafidis, and M. Hirukawa (2016). Identification and estimation of differentiated products models using market size and cost data.

Cardell, S. N. (1997). Variance components structures for the extreme-value and logistic distributions with applications to models of heterogeneity. *Econometric Theory 13*, 185–213.

Chamberlain, G. (1987). Asymptotic efficiency in estimation with conditional moment restrictions. *Journal of Econometrics 34*(3), 305–334.

Chiappori, P.-A., I. Komunjer, and D. Kristensen (2017). Nonparametric identification in nonlinear simultaneous equations models: The case of covariance restrictions. *Working Paper*.

Cho, S., G. Lee, J. Rust, and M. Yu (2018). Optimal dynamic hotel pricing.

Conlon, C. and J. Gortmaker (2019). Best practices for differentiated products demand estimation with pyblp.

Crawford, G. S. and A. Yurukoglu (2012). The welfare effects of bundling in multichannel television markets. *American Economic Review 102*(2), 643–685.

De Loecker, J. (2011). Product differentiation, multiproduct firms, and estimating the impact of trade liberalization on productivity. *Econometrica 79*(3), 1407–1451.

Deaton, A. and J. Muellbauer (1980). An almost ideal demand system. *The American Economic Review 70*(3), pp. 312–326.

Dixit, A. K. and J. E. Stiglitz (1977). Monopolistic competition and optimum product diversity. *American Economic Review 67*(3), 297–308.

Doraszelski, U. and J. Jaumandreeu (2013). R&D and productivity: Estimating endogenous productivity. *Review of Economic Studies 80*, 1338–1383.

Dotson, J. P. and G. M. Allenby (2010). Investigating the strategic influence of customer and employee satisfaction on firm financial performance. *Marketing Science 29*(5), 783–961.

Dubé, J.-P., , J. T. Fox, and C.-L. Su (2012). Improving the numerical performance of static and dynamic aggregate discrete choice random coefficients demand estimation. *Econometrica 80*(5), 2231–2267.

EPA (2009). Regulatory Impact Analysis: National Emission Standards for Hazardous Air Pollutants from the Portland Cement Manufacturing Industry.

Feenstra, R. C. (1994). New product varieties and the measurement of international prices. *The American Economic Review*, 157–177.

Feenstra, R. C. and D. E. Weinstein (2017). Globalization, Markups, and U.S. Welfare. *Journal of Political Economy 125*(4), 1040–1074.

Fisher, F. (1963). Uncorrelated disturbances and identifiability criteria. *International Economic Review 4*, 134–152.

Fisher, F. (1965). Near-identifiability and the variances of the disturbance terms. *Econometrica 33*, 409–419.

Fowlie, M., M. Reguant, and S. P. Ryan (2016). Market-based emissions regulation and industry dynamics. *Journal of Political Economy 124*(1), 249–302.

Frisch, R. (1933). *Pitfalls in the Statistical Construction of Demand and Supply Curves.* Leipzig: Hans Buske, Verlag.

Gandhi, A. and J.-F. Houde (2015). Measuring substitution pattern in differentiated products industries.

Hansen, L. P. (1982). Large sample properties of generalized method of moments estimators. *Econometrica 50*, 1029–1054.

Hausman, J. (1996). Valuation of new goods under perfect and imperfect competition. In *The Economics of New Goods, Studies in Income and Wealth* Vol 58, ed. by T. Bresnahan and R. Gordon. Chicago: National Bureau of Economic Research.

Hausman, J. A. (1978). Specification tests in econometrics. *Econometrica 46*(6), 1251–1271.

Hausman, J. A. and W. E. Taylor (1983). Identification in linear simultaneous equations models with covariance restrictions: An instrumental variables interpretation. *Econometrica 51*(5), 1527–1549.

Hayashi, F. (2000). *Econometrics*, Princeton Universtiy Press.

Heckman, J. J. (1979). Sample Selection Bias as a Specification Error. *Econometrica 47*(1), 153.

Hottman, C. J., S. J. Redding, and D. E. Weinstein (2016). Quantifying the sources of firm heterogeneity. *Quarterly Journal of Economics 131*(3), 1291–1364.

Igami, M. and T. Sugaya (2018). Measuring the Incentive to Collude: The Vitamin Cartels, 1990-1999. Available at SSRN: `https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2889837`.

Knittel, C. R. and K. Metaxoglou (2014). Estimation of random-coefficients demand models: Two empiricists' perspective. *The Review of Economics and Statistics 96*(1), 34–59.

Koopmans, T. C., H. Rubin, and R. B. Leipnik (1950). *Statistical Inference in Dynamic Economic Models (Cowles Commission Monograph 10)*, Chapter Measuring the Equation Systems of Dynamic Economics. New York: John Wiley & Sons.

Leamer, E. E. (1981). Is it a demand curve, or is it a supply curve? Partial identification through inequality constraints. *The Review of Economics and Statistics 63*(3), 319–327.

Leontief, W. (1929). Ein versuch zur statistichen analyse von angebot und nachfrage. *Weltvirtschaftliches Archiv 30*(1), 1–53.

Li, S., J. Mazur, Y. Park, J. Roberts, A. Sweeting, and J. Zhang (2018). Endogenous and selective service choices after airline mergers.

Matzkin, R. L. (2016). On independence conditions in nonseparable models: Observable and unobservable instruments. *Journal of Econometrics 191*, 302–311.

Miller, N. H. and M. Weinberg (2017). Understanding the prices effects of the MillerCoors joint venture. *Econometrica 85*(6), 1763–1791.

National Stone Council (2008). *Limestone Quarrying and Processing: A Life-Cycle Inventory*. Report Prepared by University of Tennessee Center for Clean Products.

Nevo, A. (1998). Identification of the oligopoly solution concept in a differentiated-products industry. *Economics Letters 59*, 391–395.

Nevo, A. (2000). A practitioner's guide to estimation of random-coefficients logit models of demand. *Journal of Economics & Management Strategy 9*(4), 513–548.

Nevo, A. (2001). Measuring market power in the ready-to-eat cereal industry. *Econometrica 69*(2), 307–342.

Nevo, A. and A. Rosen (2012). Identification with imperfect instruments. *Review of Economics and Statistics 94*(3), 659–671.

Nevo, A. and C. Wolfram (2002). Why do manufacturers issue coupons? An empirical analysis of breakfast cereals. *RAND Journal of Economics 33*(2), 319–339.

Otter, T., T. J. Gilbride, and G. M. Allenby (2011). Testing models of strategic behavior characterized by conditional likelihoods. *Marketing Science 30*, 686–701.

Porter, R. H. (1983). A Study of Cartel Stability: The Joint Executive Committee, 1880-1886. *The Bell Journal of Economics 14*(2), 301.

Reynaert, M. and F. Verboven (2014). Improving the performance of random coefficients demand models: The role of optimal instruments. *Journal of Econometrics 179*(1), 83–98.

Romeo, C. (2010). Filling out the instrument set in mixed logit demand systems for aggregate data. Available at SSRN: https://ssrn.com/abstract=1593590 or http://dx.doi.org/10.2139/ssrn.1593590.

Rossi, P., G. Allenby, and R. McColloch (2005). *Bayesian Statistics and Marketing*. Wiley.

Schultz, H. (1928). *Statistical Laws of Demand and Supply*. The University of Chicago Press.

Soderbery, A. (2015). Estimating import supply and demand elasticities: Analysis and implications. *Journal of International Economics 96*(1), 1–17.

Thomadsen, R. (2005). The effect of ownership structure on prices in geographically differentiated industries. *RAND Journal of Economics 36*, 908–929.

Waldfogel, J. (2003). Preference externalities: An empirical study of who benefits whom in differentiated-product markets. *RAND Journal of Economics 34*(3), 557–568.

Wegge, L. (1965). Identifiability criteria for a system of equations as a whole. *The Australian Journal of Statistics 7*, 67–77.

Williams, K. R. (2017). Dynamic airline pricing and seat availability.

Working, E. J. (1927). What Do Statistical "Demand Curves" Show? *The Quarterly Journal of Economics 41*(2), 212.

Wright, P. G. (1928). *The Tariff on Animal and Vegetable Oils*. New York: Macmillan.

Yang, S., Y. Chen, and G. M. Allenby (2003). Bayesian analysis of simultaneous demand and supply. *Quantitative Marketing and Economics 1*, 251–275.

A Linear Models of Supply and Demand

In this appendix we recast the monopoly model of Section 4 in terms of supply and demand. We first provide an alternative representation of the OLS estimator that builds explicitly on the canonical textbook treatment of simultaneous equation bias in chapter 3 of Hayashi (2000). We then provide a numerical example to illustrate how a covariance restriction can identify the demand curve from the "cloud" of equilibrium price-quantity points. Finally, we compare to the monopoly model to a model of perfect competition that has been a focus of previous articles addressing demand estimation using covariance restrictions (e.g., Koopmans et al., 1950; Hausman and Taylor, 1983; Matzkin, 2016). There are many similarities and one critical difference.

A.1 Intuition from Simultaneous Equations: A Link to Hayashi

To start, given the first-order conditions of the monopolist, $p_t + (\frac{dq}{dp})^{-1} q_t = \gamma + \eta_t$ for $\frac{dq}{dp} = \beta$, equilibrium in the model can be characterized as follows:

$$\begin{aligned} q_t^d &= \alpha + \beta p_t + \xi_t && \text{(Demand)} \\ q_t^s &= \beta\gamma - \beta p_t + \nu_t && \text{(Supply)} \\ q_t^d &= q_t^s && \text{(Equilibrium)} \end{aligned} \tag{A.1}$$

where $\nu_t \equiv \beta\eta_t$. The only distinction between this model and that of Hayashi is that slope of the supply schedule is determined (solely) by the price parameter of the demand equation, rather than by the increasing marginal cost schedules of perfect competitors.[26]

If market power is the reason that the supply schedule slopes upwards, as it is with our monopoly example, then uncorrelatedness suffices for identification because the model fully pins down how firms adjust prices with demand shocks. Repeating the steps of Hayashi, we have:

$$\beta^{OLS} \equiv plim\left(\hat{\beta}^{OLS}\right) = \beta\left(\frac{Var(\nu) - Var(\xi)}{Var(\nu) + Var(\xi)}\right). \tag{A.2}$$

If variation in the data arises solely due to cost shocks (i.e., $Var(\xi) = 0$) then the OLS estimator is consistent for β. If instead variation arises solely due to demand shocks (i.e., $Var(\nu) = 0$) then the OLS estimator is consistent for $-\beta$. A third special case arises if the demand and cost shocks have equal variance (i.e., $Var(\nu) = Var(\xi)$). Then $\beta^{OLS} = 0$, exactly halfway between the demand slope (β) and the supply slope ($-\beta$). Thus the adjustment required to bring the OLS coefficient in line with either the demand or supply slope is maximized, in terms of absolute value.

It is when variation in the data arises due both cost and demand shocks that the OLS estimate is difficult to interpret. With uncorrelatedness, however, the OLS residuals provide the information required to correct bias. A few lines of algebra obtain:

[26] A implication of equation (A.1) is that it can be possible to estimate demand parameters by estimating the *supply side* of the model, taking as given the demand system and the nature of competition. We are aware of precisely one article that employs such a method: Thomadsen (2005) estimates a model of price competition among spatially-differentiated duopolists with (importantly) constant marginal costs.

Lemma A.1. *Under uncorrelatedness, we have*

$$\beta^2 = \left(\beta^{OLS}\right)^2 + \frac{Cov(q, \xi^{OLS})}{Var(p)}. \tag{A.3}$$

and

$$Cov(q, \xi^{OLS}) = \frac{Var(\nu)Var(\xi)}{Var(\nu) + Var(\xi)}. \tag{A.4}$$

Proof: See appendix E.

The first equation is a restatement of equation (18). The second equation expresses the correction term as function of $Var(\nu)$ and $Var(\xi)$. Notice that the correction term equals zero if variation in the data arises solely due to either cost or demand shocks—precisely the cases for which OLS estimator obtains β and $-\beta$, respectively. Further, the correction term is maximized if $Var(\nu) = Var(\xi)$ which, as developed above, is when the largest adjustment is required because $\beta^{OLS} = 0$.

A.2 Numerical Illustration

We demonstrate the covariance restriction approach with the following numerical example. Let demand be given by $q_t = 10 - p_t + \xi_t$ and let marginal cost be $c_t = \eta_t$, so that (α, β, γ) = $(10, -1, 0)$. Let the demand and cost shocks have independent uniform distributions. The monopolist sets price to maximize profit. As is well known, if both cost and demand variation is present then equilibrium outcomes provide a "cloud" of data points that do not necessarily correspond to the demand curve. To illustrate this, we consider four cases with varying degrees of cost and demand variation. In case (1), $\xi \sim U(0, 2)$ and $\eta \sim U(0, 8)$. In case (2), $\xi \sim U(0, 4)$ and $\eta \sim U(0, 6)$. In case (3), $\xi \sim U(0, 6)$ and $\eta \sim U(0, 4)$. In case (4), $\xi \sim U(0, 8)$ and $\eta \sim U(0, 2)$. We randomly take 500 draws for each case and calculate the equilibrium prices and quantities.

The data are plotted the main text in Figure 1 along with OLS fits. The experiment represents the classic identification problem of demand estimation: the empirical relationship between equilibrium prices and quantities can be quite misleading about the slope of the demand function. However, Proposition 2 and Corollary 19 state that the structure of the model together with the OLS estimates allow for consistent estimates of the price parameter. Table A.1 provides the required empirical objects. The OLS estimates, $\hat{\beta}^{OLS}$, are negative when there is greater variance in costs than demand shocks and positive when there is relatively more variation in demand shocks, as is also revealed in the scatter plots. By contrast, the adjustment term $\frac{Cov(\hat{\xi}^{OLS}, q)}{Var(p)}$ is larger if the cost and demand shocks have more similar support. Equation (18) yields estimates, $\hat{\beta}^{\textit{3-Stage}}$, that are nearly equal to the population value of -1.00. The variance of price and quantity are similar in each column, consistent with Corollary 19 and the data-generating process.[27]

[27] Inspection of Figure 1 further suggests that there is a connection between the magnitude of bias adjustment and the goodness-of-fit from the OLS regression of price on quantity. Starting with equation (18), a few lines of

Table A.1: Numerical Illustration for the Monopoly Model

	(1) $Var(\eta) \gg Var(\xi)$	(2) $Var(\eta) > Var(\xi)$	(3) $Var(\eta) < Var(\xi)$	(4) $Var(\eta) \ll Var(\xi)$
$\hat{\beta}^{OLS}$	-0.87	-0.42	0.42	0.88
$Var(q)$	1.42	1.11	1.20	1.36
$Var(p)$	1.44	1.01	1.09	1.35
$Cov(\hat{\xi}^{OLS}, q)$	0.33	0.92	1.01	0.32
$\frac{Cov(\hat{\xi}^{OLS},q)}{Var(p)}$	0.23	0.91	0.93	0.24
$\hat{\beta}^{\text{3-Stage}}$	-0.995	-1.045	-1.051	-1.003

Notes: Based on numerically generated data that conform to the motivating example of monopoly pricing. The demand curve is $q_t = 10 - p_t + \xi_t$ and marginal costs are $c_t = \eta_t$, so that $(\beta_0, \beta, \gamma_0) = (10, -1, 0)$. In column (1), $\xi \sim U(0, 2)$ and $\eta \sim U(0, 8)$. In column (2), $\xi \sim U(0, 4)$ and $\eta \sim U(0, 6)$. In column (3), $\xi \sim U(0, 6)$ and $\eta \sim U(0, 4)$. In column (4), $\xi \sim U(0, 8)$ and $\eta \sim U(0, 2)$. Thus, the support of the cost shocks are largest (smallest) relative to the support of the demand shocks in the left-most (right-most) column.

A.3 Perfect Competition

As a point of comparison, consider perfect competition with linear demand and supply curves. Let marginal costs be given by $c = x'\gamma + \lambda q + \eta$. Firms are price-takers and each has a first-order condition given by $p = x'\gamma + \lambda q + \eta$. The firm-specific supply curve is $q^s = -\frac{1}{\lambda}x'\gamma + \frac{1}{\lambda}p - \frac{\eta}{\lambda}$. Aggregating across firms and assuming with linearity in demand, we have the following market-level system of equations:

$$\begin{aligned} Q^D &= \beta p + x'\alpha + \xi && \text{(Demand)} \\ Q^S &= \frac{J}{\lambda}p - \frac{J}{\lambda}x'\gamma - \frac{J}{\lambda}\eta && \text{(Supply)} \\ Q^D &= Q^S && \text{(Equilibrium)} \end{aligned} \tag{A.5}$$

where Q^D and Q^S represent market quantity demanded and supplied, respectively. Though similar in structure to the monopoly problem described by (A.1), the supply slope in this case depends on the number of firms and the slope of marginal costs. In the monopoly problem above, the supply slope is fully determined by the demand parameter.

In this setting, uncorrelatedness allows for the consistent estimation of the price coefficient, but only if the supply slope $\frac{J}{\lambda}$ can be pinned down with additional moments. Hausman and Taylor (1983) propose the following methodology: (i) under the exclusion restriction $\gamma_k = 0$ for some k, estimate the supply-schedule using x_k as an instrument for p; (ii) recover estimates of the supply-side shock; (iii) use these estimated supply-side errors as instruments in demand estimation. Under uncorrelatedness, these supply-side errors are orthogonal to demand shocks. Matzkin (2016) proposes a similar procedure but relaxes the assumption of linearity.

additional algebra obtains

$$\beta = -\frac{|\beta^{OLS}|}{\sqrt{R^2}}$$

where R^2 is from the OLS regression. This reformulation can fail if $R^2 = 0$ but numerical results indicate the formula is robust for values of R^2 that are approximately zero. We thank Peter Hull for this suggestion.

B Demand System Applications and Extensions

The demand system of equation (1) is sufficiently flexible to nest monopolistic competition with linear demands (e.g., as in the motivating example) and the discrete choice demand models that support much of the empirical research in industrial organization. We illustrate with some typical examples:

1. *Nested logit demand:* Following the exposition of Cardell (1997), let the firms be grouped into $g = 0, 1, \ldots, G$ mutually exclusive and exhaustive sets, and denote the set of firms in group g as $\mathscr{J}_g$. An outside good, indexed by $j = 0$, is the only member of group 0. Then the left-hand-side of equation (1) takes the form

$$h(s_j, w_j; \sigma) \equiv \ln(s_j) - \ln(s_0) - \sigma \ln(\bar{s}_{j|g})$$

where $\bar{s}_{j|g} = \sum_{j \in \mathscr{J}_g} \frac{s_j}{\sum_{j \in \mathscr{J}_g} s_j}$ is the market share of firm j within its group. The parameter $\sigma \in [0, 1)$ determines the extent to which consumers substitute disproportionately among firms within the same group. If $\sigma = 0$ then the logit model obtains. We can construct the markup by calculating the total derivative of h with respect to s. At the Bertrand-Nash equilibrium,

$$\frac{dh_j}{ds_j} = \frac{1}{s_j \left(\frac{1}{1-\sigma} - s_j + \frac{\sigma}{1-\sigma} \bar{s}_{j|g} \right)}.$$

Thus, we verify that the derivatives can be expressed as a function of data and the nonlinear parameters, allowing for three-stage estimation. In our third application, we use the nested logit model to estimate bounds on the structural parameters (Section 6.3).

2. *Random coefficients logit demand:* Modifying slightly the notation of Berry (1994), let the indirect utility that consumer $i = 1, \ldots, I$ receives from product j be

$$u_{ij} = \beta p_j + x_j' \alpha + \xi_j + \left[\sum_k x_{jk} \sigma_k \zeta_{ik} \right] + \epsilon_{ij}$$

where x_{jk} is the kth element of x_j, ζ_{ik} is a mean-zero consumer-specific demographic characteristic, and ϵ_{ij} is a logit error. We have suppressed market subscripts for notational simplicity. Decomposing the right-hand side of the indirect utility equation into $\delta_j = \beta p_j + x_j' \alpha + \xi_j$ and $\mu_{ij} = \sum_k x_{jk} \sigma_k \zeta_{ik}$, the probability that consumer i selects product j is given by the standard logit formula

$$s_{ij} = \frac{\exp(\delta_j + \mu_{ij})}{\sum_k \exp(\delta_k + \mu_{ik})}.$$

Integrating yields the market shares: $s_j = \frac{1}{I} \sum_i s_{ij}$. Berry et al. (1995) prove that a contraction mapping recovers, for any candidate parameter vector $\tilde{\sigma}$, the vector $\delta(s, \tilde{\sigma})$ that equates these market shares to those observed in the data. This "mean valuation" is $h(s_j, w_j; \tilde{\sigma})$ in our notation. The three-stage estimator can be applied to recover the price coefficient, again taking some $\tilde{\sigma}$ as given. At the Bertrand-Nash equilibrium, dh_j / ds_j takes the form

$$\frac{dh_j}{ds_j} = \frac{1}{\frac{1}{I}\sum_i s_{ij}(1-s_{ij})}.$$

Thus, with the uncorrelatedness assumption the linear parameters can be recovered given the candidate parameter vector $\tilde{\sigma}$. The identification of σ is a distinct issue that has received a great deal of attention from theoretical and applied research (e.g., Waldfogel, 2003; Romeo, 2010; Berry and Haile, 2014; Gandhi and Houde, 2015; Miller and Weinberg, 2017). We demonstrate how to estimate these parameters using additional covariance restrictions in our first application (Section 6.1).

The semi-linear demand assumption (Assumption 1) can be modified to allow for semi-linearity in a transformation of prices, $f(p_{jt})$:

$$h_{jt} \equiv h(q_{jt}, w_{jt}; \sigma) = \beta f(p_{jt}) + x'_{jt}\alpha + \xi_{jt} \tag{B.1}$$

Under this modification assumptions, it is possible to employ a method-of-moments approach to estimate the structural parameters. When $f(p_{jt}) = \ln p_{jt}$, the three-stage estimator and identification results are applicable, under the modified assumptions that ξ is orthogonal to $\ln X$ and that $\ln \eta$ and ξ are uncorrelated.

The optimal price for these demand systems takes the form $p_{jt} = \mu_{jt} c_{jt}$, where μ_{jt} is a markup that reflects demand parameters and (in general) demand shocks. It follows that the probability limit of an OLS regression of h on $\ln p$ is given by:

$$\beta^{OLS} = \beta - \frac{1}{\beta}\frac{Cov(\ln \mu, \xi)}{Var(\ln p^*)} + \frac{Cov(\ln \eta, \xi)}{Var(\ln p^*)}. \tag{B.2}$$

Therefore, the results developed in this paper are extend in a straightforward manner. We opt to focus on semi-linear demand throughout this paper for clarity.

A special case that is often estimated in empirical work is when h and $f(p)$ are logarithms:

Constant elasticity demand: With the modified demand assumption of equation (B.1), the constant elasticity of substitution (CES) demand model of Dixit and Stiglitz (1977) can be incorporated:

$$\ln(q_{jt}/q_t) = \alpha + \beta \ln\left(\frac{p_{jt}}{\Pi_t}\right) + \xi_{jt}$$

where q_t is an observed demand shifter, Π_t is a price index, and β provides the constant elasticity of demand. This model is often used in empirical research on international trade and firm productivity (e.g., De Loecker, 2011; Doraszelski and Jaumandreeu, 2013). Due to the constant elasticity, profit-maximization and uncorrelatedness imply $Cov(p, \xi) = 0$, and OLS produces unbiased estimates of the demand parameters.[28] Indeed, this is an excellent illustration of our basic argument: so long as the data-generating process is sufficiently well understood, it is possible to characterize the bias of OLS estimates.

[28]The international trade literature following Feenstra (1994) consider non-constant marginal costs, which requires an additional restriction. See section 5.2 for an extension of our methodology to non-constant marginal costs.

The demand assumption in Equation (1) accommodates many rich demand systems. Consider the linear demand system, $q_{jt} = \alpha_j + \sum_k \beta_{jk} p_k + \xi_{jt}$, which sometimes appears in identification proofs (e.g., Nevo, 1998) but is seldom applied empirically due to the large number of price coefficients. In principle, the system could be formulated such that $h(q_{jt}, w_{jt}; \sigma) \equiv q_{jt} - \sum_{k \neq j} \beta_{jk} p_k$. In addition to the own-product uncorrelatedness restrictions that could identify β_{jj}, one could impose cross-product covariance restrictions to identify β_{jk} $(j \neq k)$. We discuss these cross-product covariance restrictions in Section 5.1. A similar approach could be used with the almost ideal demand system of Deaton and Muellbauer (1980).

C Two-Stage Estimation

In the presence of an additional restriction, we can produce a more precise estimator that can be calculated with one fewer stage. When the observed cost and demand shifters are uncorrelated, there is no need to project the price on demand covariates when constructing a consistent estimate, and one can proceed immediately using the OLS regression. We formalize the additional restriction and the estimator below.

Assumption 5: *Let the parameters $\alpha^{(k)}$ and $\gamma^{(k)}$ correspond to the demand and supply coefficients for covariate k in X. For any two covariates k and l, $Cov(\alpha^{(k)} x^{(k)}, \gamma^{(l)} x^{(l)}) = 0$.*

Proposition C.1. *Under assumptions 1-3 and 5, a consistent estimate of the price parameter β is given by*

$$\hat{\beta}^{\text{2-Stage}} = \frac{1}{2}\left(\hat{\beta}^{OLS} - \frac{\hat{Cov}\left(p, \frac{dh}{dq}q\right)}{\hat{Var}(p)} - \sqrt{\left(\hat{\beta}^{OLS} + \frac{\hat{Cov}\left(p, \frac{dh}{dq}q\right)}{\hat{Var}(p)}\right)^2 + 4\frac{\hat{Cov}\left(\hat{\xi}^{OLS}, \frac{dh}{dq}q\right)}{\hat{Var}(p)}}\right) \tag{C.1}$$

when the auxiliary condition, $\beta < \frac{Cov(p^*,\xi)}{Var(p^*)}\frac{Var(p)}{Var(p^*)} - \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}$, holds.

The estimator can be expressed entirely in terms of the data, the OLS coefficient, and the OLS residuals. The first stage is an OLS regression of $h(q; \cdot)$ on p and x, and the second stage is the construction of the estimator as in equation (C.1). Thus, we eliminate the step of projecting p on x. This estimator is consistent under the assumption that any covariate affecting demand does not covary with marginal cost. The auxiliary condition parallels that of the three-stage estimator, and we expect that it holds in typical cases.

D Multi-Product Firms

We now provide the notation necessary to extend our results to the case of multi-product firms under our maintained assumptions. Let K^m denote the set of products owned by multi-product firm m. When the firm sets prices on each of its products to maximize joint profits, there are $|K^m|$ first-order conditions, which can be expressed as

$$\sum_{k \in K^m} (p_k - c_k)\frac{\partial q_k}{\partial p_j} = -q_j \;\; \forall j \in K^m.$$

The market subscript, t, is omitted to simplify notation. For demand systems satisfying Assumption 1,

$$\frac{\partial q_k}{\partial p_j} = \beta \frac{1}{\frac{dh_j}{dq_k}}.$$

where the derivative $\frac{dh_j}{dq_k} = \frac{\partial h_j}{\partial q_j}\frac{dq_j}{dq_k} + \frac{\partial h_j}{\partial w_j}\frac{dw_j}{dq_k}$ is calculated holding the prices of other products fixed. Therefore, the set of first-order conditions can be written as

$$\sum_{k \in K^m} (p_k - c_k) \frac{1}{dh_j/dq_k} = -\frac{1}{\beta} q_j \;\; \forall j \in K^m.$$

For each firm, stack the first-order conditions, writing the left-hand side as the product of a matrix A^m of loading components and a vector of markups, $(p_j - c_j)$, for products owned by the firm. The loading components are given by $A^m_{i(j),i(k)} = \frac{1}{dh_j/dq_k}$, where $i(\cdot)$ indexes products within a firm. Next, invert the loading matrix to solve for markups as function of the loading components and $-\frac{1}{\beta}\boldsymbol{q}^m$, where $\boldsymbol{q}^m$ is a vector of the multi-product firm's quantities. Equilibrium prices equal marginal costs plus a markup, where the markup is determined by the inverse of A^m $((A^m)^{-1} \equiv \Lambda^m)$, quantities, and the price parameter:

$$p_j = c_j - \frac{1}{\beta} \left(\Lambda^m \boldsymbol{q}^m\right)_{i(j)}. \tag{D.1}$$

Here, $\left(\Lambda^m \boldsymbol{q}^m\right)_{i(j)}$ provides the entry corresponding to product j in the vector $\Lambda^m \boldsymbol{q}^m$. As the matrix Λ^m is not a function of the price parameter after conditioning on observables, this form of the first-order condition allows us to solve for β using a quadratic three-stage solution analogous to that in equation (14).[29] Letting $\tilde{h} \equiv \left(\Lambda^m \boldsymbol{q}^m\right)_{i(j)}$ be the multi-product analog for $\frac{dh}{dq}q$, we obtain a quadratic in β, and the remaining results of Section 3 then obtain easily:

Corollary 5. *Under assumptions 1 and 3, along with a modified assumption 2 that allows for multi-product firms, the price parameter β solves the following quadratic equation:*

$$\begin{aligned} 0 &= \beta^2 \\ &+ \left(\frac{Cov(p^*, \tilde{h})}{Var(p^*)} + \frac{Cov(\xi, \eta)}{Var(p^*)} - \hat{\beta}^{OLS}\right)\beta \\ &+ \left(-\frac{Cov(p^*, \tilde{h})}{Var(p^*)}\hat{\beta}^{OLS} - \frac{Cov(\hat{\xi}^{OLS}, \tilde{h})}{Var(p^*)}\right). \end{aligned}$$

where $\tilde{h}$ is constructed from the first-order conditions of multi-product firms.

[29] At this point, the reader may be wondering where the prices of other firms are captured under the adjusted first-order conditions for multi-product ownership. As is the case with single product firms, we expect prices of other firm's products to be included in w_j, which is appropriate under Bertrand price competition.

E Proofs

Lemma: A Consistent and Unbiased Estimate for ξ

The following proof shows a consistent and unbiased estimate for the unobserved term in a linear regression when one of the covariates is endogenous. Though demonstrated in the context of semi-linear demand, the proof also applies for any endogenous covariate, including when (transformed) quantity depends on a known transformation of price, as no supply-side assumptions are required. For example, we may replace p with $\ln p$ everywhere and obtain the same results.

Lemma E.1. *A consistent and unbiased estimate of ξ is given by $\xi_1 = \xi^{OLS} + \left(\beta^{OLS} - \beta\right) p^*$*

We can construct both the true demand shock and the OLS residuals as:

$$\begin{aligned} \xi &= h(q) - \beta p - x'\alpha \\ \xi^{OLS} &= h(q) - \beta^{OLS} p - x'\alpha^{OLS} \end{aligned}$$

where this holds even in small samples. Without loss of generality, we assume $E[\xi] = 0$. The true demand shock is given by $\xi_0 = \xi^{OLS} + (\beta^{OLS} - \beta)p + x'(\alpha^{OLS} - \alpha)$. We desire to show that an alternative estimate of the demand shock, $\xi_1 = \xi^{OLS} + (\beta^{OLS} - \beta)p^*$, is consistent and unbiased. (This eliminates the need to estimate the true α parameters). It suffices to show that $(\beta^{OLS} - \beta)p^* \to (\beta^{OLS} - \beta)p + x'(\alpha^{OLS} - \alpha)$. Consider the projection matrices

$$\begin{aligned} Q &= I - P(P'P)^{-1}P' \\ M &= I - X(X'X)^{-1}X', \end{aligned}$$

where P is an $N \times 1$ matrix of prices and X is the $N \times k$ matrix of covariates x. Denote $Y \equiv h(q) = P\beta + X\alpha + \xi$. Our OLS estimators can be constructed by a residualized regression

$$\begin{aligned} \alpha^{OLS} &= \left((XQ)'QX\right)^{-1}(XQ)'Y \\ \beta^{OLS} &= \left((PM)'MP\right)^{-1}(PM)'Y. \end{aligned}$$

Therefore

$$\begin{aligned} \alpha^{OLS} = &\left(X'QX\right)^{-1}\left(X'QP\beta + X'QX\alpha + X'Q\xi\right) \\ = &\alpha + \left(X'QX\right)^{-1} X'Q\xi. \end{aligned}$$

Similarly,

$$\begin{aligned} \beta^{OLS} = &\left(P'MP\right)^{-1}\left(P'MP\beta + P'MX\alpha + P'M\xi\right) \\ = &\beta + \left(P'MP\right)^{-1} P'M\xi. \end{aligned}$$

We desire to show

$$P^*(\beta^{OLS} - \beta) \to P(\beta^{OLS} - \beta) + X(\alpha^{OLS} - \alpha).$$

Note that $P^* = MP$. Then

$$
\begin{aligned}
P^*(\beta^{OLS} - \beta) \to & P(\beta^{OLS} - \beta) + X(\alpha^{OLS} - \alpha) \\
MP\left(P'MP\right)^{-1} P'M\xi \to & P\left(P'MP\right)^{-1} P'M\xi + X\left(X'QX\right)^{-1} X'Q\xi \\
-X(X'X)^{-1}X'P\left(P'MP\right)^{-1} P'M\xi \to & X\left(X'QX\right)^{-1} X'Q\xi \\
-X(X'X)^{-1}X'P\left(P'MP\right)^{-1} P'\left[I - X(X'X)^{-1}X'\right]\xi \to & X\left(X'QX\right)^{-1} X'\left[I - P(P'P)^{-1}P'\right]\xi \\
-X(X'X)^{-1}X'P\left(P'MP\right)^{-1} P'\xi \to & X\left(X'QX\right)^{-1} X'\xi \\
+X(X'X)^{-1}X'P\left(P'MP\right)^{-1} P'X(X'X)^{-1}X'\xi \quad & - X\left(X'QX\right)^{-1} X'P(P'P)^{-1}P'\xi.
\end{aligned}
$$

We will show that the following two relations hold, which proves consistency and completes the proof.

1. $X(X'X)^{-1}X'P\left(P'MP\right)^{-1} P'\xi = X\left(X'QX\right)^{-1} X'P(P'P)^{-1}P'\xi$
2. $X(X'X)^{-1}X'P\left(P'MP\right)^{-1} P'X(XX\left(X'QX\right)^{-1} X'X)^{-1}X'\xi \to X\left(X'QX\right)^{-1} X'\xi$

Part 1: Equivalence

It suffices to show that $X(X'X)^{-1}X'P\left(P'MP\right)^{-1} = X\left(X'QX\right)^{-1} X'P(P'P)^{-1}$.

$$
\begin{aligned}
X(X'X)^{-1}X'P\left(P'MP\right)^{-1} =& X\left(X'QX\right)^{-1} X'P(P'P)^{-1} \\
X(X'X)^{-1}X'P =& X\left(X'QX\right)^{-1} X'P(P'P)^{-1}\left(P'MP\right) \\
X(X'X)^{-1}X'P =& X\left(X'QX\right)^{-1} X'P(P'P)^{-1}\left(P'P\right) \\
& - X\left(X'QX\right)^{-1} X'P(P'P)^{-1}\left(P'X(X'X)^{-1}X'P\right) \\
X(X'X)^{-1}X'P =& X\left(X'QX\right)^{-1} X'P \\
& - X\left(X'QX\right)^{-1} X'\left[I - Q\right]X(X'X)^{-1}X'P \\
X(X'X)^{-1}X'P =& X\left(X'QX\right)^{-1} X'P \\
& - X\left(X'QX\right)^{-1} X'X(X'X)^{-1}X'P \\
& + X\left(X'QX\right)^{-1} X'QX(X'X)^{-1}X'P \\
X(X'X)^{-1}X'P =& X(X'X)^{-1}X'P
\end{aligned}
$$

QED.

Part 2: Consistency (and Unbiasedness)

Because $X(X'X)^{-1}X'P = X\left(X'QX\right)^{-1}X'P(P'P)^{-1}(P'MP)$, as shown above:

$$\begin{aligned}
X(X'X)^{-1}X'P\left(P'MP\right)^{-1}P'X(X'X)^{-1}X'\xi &\to X\left(X'QX\right)^{-1}X'\xi \\
X\left(X'QX\right)^{-1}X'P(P'P)^{-1}P'X(X'X)^{-1}X'\xi &\to X\left(X'QX\right)^{-1}X'\xi \\
X\left(X'QX\right)^{-1}X'\left[I-Q\right]X(X'X)^{-1}X'\xi &\to X\left(X'QX\right)^{-1}X'\xi \\
X\left(X'QX\right)^{-1}X'X(X'X)^{-1}X'\xi &\to X\left(X'QX\right)^{-1}X'\xi \\
-X(X'X)^{-1}X'\xi & \\
X\left(X'QX\right)^{-1}X'\xi - X(X'X)^{-1}X'\xi &\to X\left(X'QX\right)^{-1}X'\xi \\
X(X'X)^{-1}X'\xi &\to 0.
\end{aligned}$$

The last line, where the projection of ξ onto the exogenous covariates X converges to zero, holds by assumption. We say that two vectors converge if the mean absolute deviation goes to zero as the sample size gets large. Note that also $E[X(X'X)^{-1}X'\xi] = 0$, so ξ_1 is both a consistent and unbiased estimate for ξ_0. QED.

Proof of Proposition 1 (Set Identification)

From the text, we have $\hat{\beta}^{OLS} \xrightarrow{p} \beta + \frac{Cov(p^*,\xi)}{Var(p^*)}$. The general form for a firm's first-order condition is $p = c + \mu$, where c is the marginal cost and μ is the markup. We can write $p = p^* + \hat{p}$, where $\hat{p}$ is the projection of p onto the exogenous demand variables, X. By assumption, $c = X\gamma + \eta$. If we substitute the first-order condition $p^* = X\gamma + \eta + \mu - \hat{p}$ into the bias term from the OLS regression, we obtain

$$\begin{aligned}
\frac{Cov(p^*,\xi)}{Var(p^*)} &= \frac{Cov(\xi, X\gamma + \eta + \mu - \hat{p})}{Var(p^*)} \\
&= \frac{Cov(\xi,\eta)}{Var(p^*)} + \frac{Cov(\xi,\mu)}{Var(p^*)}
\end{aligned}$$

where the second line follows from the exogeneity assumption ($E[X\xi] = 0$). Under our demand assumption, the unobserved demand shock may be written as $\xi = h(q) - x\alpha - \beta p$. At the probability limit of the OLS estimator, we can construct a consistent estimate of the unobserved demand shock as $\xi = \xi^{OLS} + \left(\beta^{OLS} - \beta\right)p^*$ (see Lemma E.1 above). From the prior step in this proof, $\beta^{OLS} - \beta = \frac{Cov(\xi,\eta)}{Var(p^*)} + \frac{Cov(\xi,\mu)}{Var(p^*)}$. Therefore, $\xi = \xi^{OLS} + \left(\frac{Cov(\eta,\xi)}{Var(p^*)} + \frac{Cov(\mu,\xi)}{Var(p^*)}\right)p^*$. This implies

$$\begin{aligned}
\frac{Cov\left(\xi,\mu\right)}{Var(p^*)} &= \frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)} + \left(\frac{Cov(\xi,\eta)}{Var(p^*)} + \frac{Cov\left(\xi,\mu\right)}{Var(p^*)}\right)\frac{Cov(p^*,\mu)}{Var(p^*)} \\
\frac{Cov\left(\xi,\mu\right)}{Var(p^*)}\left(1 - \frac{Cov(p^*,\mu)}{Var(p^*)}\right) &= \frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\mu)}{Var(p^*)} \\
\frac{Cov\left(\xi,\mu\right)}{Var(p^*)} &= \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)} + \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\mu)}{Var(p^*)}
\end{aligned}$$

When we substitute this expression in for β^{OLS}, we obtain

$$\beta^{OLS} = \beta + \frac{Cov(\xi,\eta)}{Var(p^*)} + \frac{1}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)} + \frac{\frac{Cov(p^*,\mu)}{Var(p^*)}}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}$$

$$\beta^{OLS} = \beta + \frac{1}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)} + \frac{1}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}$$

Thus, we obtain an expression for the OLS estimator in terms of the OLS residuals, the residualized prices, the markup, and the correlation between unobserved demand and cost shocks. If the markup can be parameterized in terms of observables and the correlation in unobserved shocks can be calibrated, we have a method to estimate β from the OLS regression. Under our supply and demand assumptions, $\mu = -\frac{1}{\beta}\frac{dh}{dq}q$, and plugging in obtains the first equation of the proposition:

$$\beta^{OLS} = \beta - \frac{1}{\beta + \frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\frac{dh}{dq}q\right)}{Var(p^*)} + \beta\frac{1}{\beta + \frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}.$$

The second equation in the proposition is obtained by rearranging terms. QED.

Proof of Proposition 2 (Covariance Bound)

The proof is again an application of the quadratic formula. Any generic quadratic, $ax^2 + bx + c$, with roots $\frac{1}{2}\left(-b \pm \sqrt{b^2-4ac}\right)$, admits a real solution if and only if $b^2 \geq 4ac$. Given the formulation of (9), real solutions satisfy the condition:

$$\left(\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)} - \beta^{OLS}\right)^2 \geq 4\left(-\beta^{OLS}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS},\frac{dh}{dq}q\right)}{Var(p^*)}\right).$$

As $a = 1$, a solution is always possible if $c < 0$. This is the sufficient condition for point identification from the text. If $c \geq 0$, it must be the case that $b \geq 0$; otherwise, both roots are positive. Therefore, a real solution is obtained if and only if $b \geq 2\sqrt{c}$, that is

$$\left(\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)} - \beta^{OLS}\right) \geq 2\sqrt{-\beta^{OLS}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS},\frac{dh}{dq}q\right)}{Var(p^*)}}.$$

Solving for $Cov(\xi,\eta)$, we obtain the prior-free bound,

$$Cov(\xi,\eta) \geq Var(p^*)\beta^{OLS} - Cov(p^*,\frac{dh}{dq}q) + 2Var(p^*)\sqrt{-\beta^{OLS}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS},\frac{dh}{dq}q\right)}{Var(p^*)}}.$$

This bound exists if the expression inside the radical is positive, which is the case if and only if the sufficient condition for point identification from Proposition 3 fails. QED.

Alternative Proof

Alternatively, one can calculate the minimum of equation (10), which is repeated here for convenience.

$$
\begin{aligned}
Cov(\xi,\eta) = \ & Var(p^*)\beta^{OLS} - Cov\left(p^*, \frac{dh}{dq}q\right) - \beta Var(p^*) \\
& + \frac{1}{\beta}\left(\beta^{OLS} Cov\left(p^*, \frac{dh}{dq}q\right) + Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)\right).
\end{aligned}
$$

The minimum is obtained at

$$
\underline{\beta} = \arg\min_{\beta} -\beta Var(p^*) + \frac{1}{\beta}\left(\beta^{OLS} Cov\left(p^*, \frac{dh}{dq}q\right) + Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)\right)
$$

which has the solution

$$
\underline{\beta} = -\sqrt{-\beta^{OLS}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}}.
$$

The sign of the solution is determined because β must be negative. Verifying the second-order condition for a minimum is straightforward. Therefore, the covariance bound may be expressed as:

$$
Cov(\xi,\eta) \geq Var(p^*)\beta^{OLS} - Cov\left(p^*, \frac{dh}{dq}q\right) - 2Var(p^*)\underline{\beta}.
$$

Proof of Proposition 3 (Point Identification)

Part (1). We first prove the sufficient condition, i.e., that under assumptions 1 and 2, β is the lower root of equation (9) if the following condition holds:

$$
0 \leq \beta^{OLS}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} + \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)} \tag{E.1}
$$

Consider a generic quadratic, ax^2+bx+c. The roots of the quadratic are $\frac{1}{2a}\left(-b \pm \sqrt{b^2-4ac}\right)$. Thus, if $4ac < 0$ and $a > 0$ then the upper root is positive and the lower root is negative. In equation (9), $a = 1$, and $4ac < 0$ if and only if equation (E.1) holds. Because the upper root is positive, $\beta < 0$ must be the lower root, and point identification is achieved given knowledge of $Cov(\xi,\eta)$. QED.

Part (2). In order to prove the necessary and sufficient condition for point identification, we first state and prove a lemma:

Lemma E.2. *The roots of equation* (9) *are* β *and* $\frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}$.

Proof of Lemma E.2. We first provide equation (9) for reference:

$$
\begin{aligned}
0 &= \beta^2 \\
&+ \left(\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)} - \beta^{OLS}\right)\beta \\
&+ \left(-\beta^{OLS}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}\right)
\end{aligned}
$$

To find the roots, begin by applying the quadratic formula

$$
\begin{aligned}
(r_1, r_2) &= \frac{1}{2}\left(-B \pm \sqrt{B^2 - 4AC}\right) \\
&= \frac{1}{2}\left(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right. \\
&\left. \pm\sqrt{\left(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 + 4\beta^{OLS}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} + 4\frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}}\right) \\
&= \frac{1}{2}\left[\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right. \\
&\pm\left(\left(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)\right. \\
&\left.\left. +4\beta^{OLS}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} + 4\frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}\right)^{\frac{1}{2}}\right] \\
&= \frac{1}{2}\left(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right. \qquad \text{(E.2)} \\
&\left. \pm\sqrt{\left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)} + \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)}\right)
\end{aligned}
$$

Looking inside the radical, consider the first part: $\left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}$

$$
\begin{aligned}
&\left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)} \\
&= \left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi - p^*(\beta^{OLS} - \beta), \frac{dh}{dq}q\right)}{Var(p^*)} \\
&= \left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi, \frac{dh}{dq}q\right)}{Var(p^*)} - 4\frac{Cov(p^*, \xi)}{Var(p^*)}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} \\
&= \left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi, \frac{dh}{dq}q\right)}{Var(p^*)} - 4\left(\frac{Cov(\xi, \eta)}{Var(p^*)} + \frac{Cov(\xi, -\frac{1}{\beta}\frac{dh}{dq}q)}{Var(p^*)}\right)\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} \\
&= \left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi, \frac{dh}{dq}q\right)}{Var(p^*)}\left(1 + \frac{1}{\beta}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right) - 4\frac{Cov(\xi, \eta)}{Var(p^*)}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} \qquad \text{(E.3)}
\end{aligned}
$$

To simplify this expression, it is helpful to use the general form for a firm's first-order condition, $p = c + \mu$, where c is the marginal cost and μ is the markup. We can write $p = p^* + \hat{p}$, where $\hat{p}$ is the projection of p onto the exogenous demand variables, X. By assumption, $c = X\gamma + \eta$. It follows that

$$
\begin{aligned}
p^* &= X\gamma + \eta + \mu - \hat{p} \\
&= X\gamma + \eta - \frac{1}{\beta}\frac{dh}{dq}q - \hat{p}
\end{aligned}
$$

Therefore

$$
Cov(p^*, \xi) = Cov(\xi, \eta) - \frac{1}{\beta}Cov(\xi, \frac{dh}{dq}q)
$$

and

$$
\begin{aligned}
Cov(\xi, \frac{dh}{dq}q) &= -\beta\left(Cov(p^*, \xi) - Cov(\xi, \eta)\right) \\
\frac{Cov(\xi, \frac{dh}{dq}q)}{Var(p^*)} &= -\beta\left(\frac{Cov(p^*, \xi)}{Var(p^*)} - \frac{Cov(\xi, \eta)}{Var(p^*)}\right) \qquad \text{(E.4)}
\end{aligned}
$$

Returning to equation (E.3), we can substitute using equation (E.4) and simplify:

$$\begin{aligned}
&\left(\beta^{OLS}+\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}\right)^2+4\frac{Cov\left(\xi,\frac{dh}{dq}q\right)}{Var(p^*)}\left(1+\frac{1}{\beta}\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}\right)-4\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}\\
=&\left(\beta^{OLS}\right)^2+\left(\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}\right)^2+2\beta^{OLS}\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}-4\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}\\
&+4\frac{Cov\left(\xi,\frac{dh}{dq}q\right)}{Var(p^*)}+4\frac{1}{\beta}\frac{Cov\left(\xi,\frac{dh}{dq}q\right)}{Var(p^*)}\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}\\
=&\left(\beta+\frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\right)^2+\left(\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}\right)^2+2\left(\beta+\frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\right)\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}-4\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}\\
&-4\beta\left(\frac{Cov(p^*,\xi)}{Var(p^*)}-\frac{Cov(\xi,\eta)}{Var(p^*)}\right)-4\left(\frac{Cov(p^*,\xi)}{Var(p^*)}-\frac{Cov(\xi,\eta)}{Var(p^*)}\right)\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}\\
=&\left(\beta+\frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\right)^2+\left(\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}\right)^2+2\left(\beta+\frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\right)\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}\\
&-4\beta\left(\frac{Cov(p^*,\xi)}{Var(p^*)}\right)-4\left(\frac{Cov(p^*,\xi)}{Var(p^*)}\right)\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}+4\beta\frac{Cov(\xi,\eta)}{Var(p^*)}\\
=&\beta^2+\left(\frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\right)^2+\left(\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}\right)^2+2\beta\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}\\
&-2\beta\frac{Cov\left(p^*,\xi\right)}{Var(p^*)}-2\frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}+4\beta\frac{Cov(\xi,\eta)}{Var(p^*)}\\
=&\left(\left(\beta+\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}\right)-\frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\right)^2+4\beta\frac{Cov(\xi,\eta)}{Var(p^*)}
\end{aligned}$$

Now, consider the second part inside of the radical in equation (E.2):

$$\begin{aligned}
&\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2-2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta^{OLS}-\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}\right)\\
=&\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2-2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta+\frac{Cov(\xi,\eta)}{Var(p^*)}-\frac{1}{\beta}\frac{Cov(\xi,\frac{dh}{dq}q)}{Var(p^*)}-\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}\right)\\
=&\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2-2\beta\frac{Cov(\xi,\eta)}{Var(p^*)}-2\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2+2\frac{1}{\beta}\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(\xi,\frac{dh}{dq}q)}{Var(p^*)}+2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}\\
=&-\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2-2\beta\frac{Cov(\xi,\eta)}{Var(p^*)}-2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\frac{Cov(p^*,\xi)}{Var(p^*)}-\frac{Cov(\xi,\eta)}{Var(p^*)}\right)+2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}\\
=&\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2-2\frac{Cov(\xi,\eta)}{Var(p^*)}\beta-2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\xi)}{Var(p^*)}+2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}
\end{aligned}$$

Combining yields a simpler expression for the terms inside the radical of equation (E.2):

$$\begin{aligned}
&\left(\left(\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right) - \frac{Cov\left(p^*, \xi\right)}{Var(p^*)}\right)^2 + 4\beta\frac{Cov(\xi,\eta)}{Var(p^*)} \\
&+ \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\beta - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\xi)}{Var(p^*)} + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} \\
=&\left(\left(\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right) - \frac{Cov\left(p^*, \xi\right)}{Var(p^*)}\right)^2 + \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 \\
&+ 2\beta\frac{Cov(\xi,\eta)}{Var(p^*)} - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\xi)}{Var(p^*)} + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} \\
=&\left(\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(p^*, \xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2
\end{aligned}$$

Plugging this back into equation (E.2), we have:

$$\begin{aligned}
(r_1, r_2) &= \frac{1}{2}\left(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right. \\
&\left. \pm\sqrt{\left(\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(p^*, \xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2}\right) \\
&= \frac{1}{2}\left(\beta + \frac{Cov\left(p^*, \xi\right)}{Var(p^*)} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right. \\
&\left. \pm\sqrt{\left(\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(p^*, \xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2}\right)
\end{aligned}$$

The roots are given by

$$\begin{aligned}
&\frac{1}{2}\left(\beta + \frac{Cov\left(p^*, \xi\right)}{Var(p^*)} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} + \beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(p^*, \xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right) \\
=&\beta
\end{aligned}$$

and

$$\frac{1}{2}\left(\beta+\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}-\frac{Cov(p^{*},\frac{dh}{dq}q)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}-\beta-\frac{Cov\left(p^{*},\frac{dh}{dq}q\right)}{Var(p^{*})}+\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}\right)$$
$$=\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}-\frac{Cov(p^{*},\frac{dh}{dq}q)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}$$

which completes the proof of the intermediate result. QED.

Part (3). Consider the roots of equation (9), β and $\frac{Cov(p^{*},\xi)}{Var(p^{*})}-\frac{Cov(p^{*},\frac{dh}{dq}q)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}$. The price parameter β may or may not be the lower root.[30] However, β is the lower root iff

$$\begin{aligned}
\beta &< \frac{Cov(p^{*},\xi)}{Var(p^{*})}-\frac{Cov(p^{*},\frac{dh}{dq}q)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}\\
\beta &< -\beta\frac{Cov(p^{*},-\frac{1}{\beta}\xi)}{Var(p^{*})}+\beta\frac{Cov(p^{*},-\frac{1}{\beta}\frac{dh}{dq}q)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}\\
\beta &< -\beta\frac{Cov(p^{*},-\frac{1}{\beta}\xi)}{Var(p^{*})}+\beta\frac{Cov(p^{*},p^{*}-c)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}\\
\beta &< \beta\frac{Var(p^{*})}{Var(p^{*})}-\beta\frac{Cov(p^{*},-\frac{1}{\beta}\xi)}{Var(p^{*})}-\beta\frac{Cov(p^{*},\eta)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}\\
0 &< -\beta\frac{Cov(p^{*},-\frac{1}{\beta}\xi)}{Var(p^{*})}-\beta\frac{Cov(p^{*},\eta)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}\\
0 &< \frac{Cov(p^{*},-\frac{1}{\beta}\xi)}{Var(p^{*})}+\frac{Cov(p^{*},\eta)}{Var(p^{*})}+\frac{1}{\beta}\frac{Cov(\xi,\eta)}{Var(p^{*})}
\end{aligned}$$

The third line relies on the expression for the markup, $p-c=-\frac{1}{\beta}\frac{dh}{dq}q$. The final line holds

[30] Consider that the first root is the upper root if

$$\beta+\frac{Cov\left(p^{*},\frac{dh}{dq}q\right)}{Var(p^{*})}-\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}+\frac{Cov(\xi,\eta)}{Var(p^{*})}>0$$

because, in that case,

$$\sqrt{\left(\beta+\frac{Cov\left(p^{*},\frac{dh}{dq}q\right)}{Var(p^{*})}-\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}+\frac{Cov(\xi,\eta)}{Var(p^{*})}\right)^{2}}=\beta+\frac{Cov\left(p^{*},\frac{dh}{dq}q\right)}{Var(p^{*})}-\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}+\frac{Cov(\xi,\eta)}{Var(p^{*})}$$

When $\beta+\frac{Cov\left(p^{*},\frac{dh}{dq}q\right)}{Var(p^{*})}-\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}+\frac{Cov(\xi,\eta)}{Var(p^{*})}<0$, then $\sqrt{\left(\beta+\frac{Cov\left(p^{*},\frac{dh}{dq}q\right)}{Var(p^{*})}-\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}+\frac{Cov(\xi,\eta)}{Var(p^{*})}\right)^{2}}=-\left(\beta+\frac{Cov\left(p^{*},\frac{dh}{dq}q\right)}{Var(p^{*})}-\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}+\frac{Cov(\xi,\eta)}{Var(p^{*})}\right)$, and the first root is then the lower root (i.e., minus the negative value).

because $\beta < 0$ so $-\beta > 0$. It follows that β is the lower root of (9) iff

$$-\frac{1}{\beta}\frac{Cov(\xi,\eta)}{Var(p^*)} \leq \frac{Cov\left(p^*, -\frac{1}{\beta}\xi\right)}{Var(p^*)} + \frac{Cov\left(p^*,\eta\right)}{Var(p^*)}$$

in which case β is point identified given knowledge of $Cov(\xi,\eta)$. QED.

Proof of Lemma 1 (Monotonicity in $Cov(\xi,\eta)$)

We return to the quadratic formula for the proof. The lower root of a quadratic ax^2+bx+c is $L \equiv \frac{1}{2}\left(-b-\sqrt{b^2-4ac}\right)$. In our case, $a=1$.

We wish to show that $\frac{\partial L}{\partial \gamma} < 0$, where $\gamma = Cov(\xi,\eta)$. We evaluate the derivative to obtain

$$\frac{\partial L}{\partial \gamma} = -\frac{1}{2}\left(1+\frac{b}{(b^2-4c)^{\frac{1}{2}}}\right)\frac{\partial b}{\partial \gamma}.$$

We observe that, in our setting, $\frac{\partial b}{\partial \gamma} = \frac{1}{Var(p^*)}$ is always positive. Therefore, it suffices to show that

$$1+\frac{b}{(b^2-4c)^{\frac{1}{2}}} > 0. \tag{E.5}$$

We have two cases. First, when $c<0$, we know that $\left|\frac{b}{(b^2-4c)^{\frac{1}{2}}}\right| < 1$, which satisfies (E.5). Second, when $c>0$, it must be the case that $b>0$ also. Otherwise, both roots are positive, invalidating the model. When $b>0$, it is evident that the left-hand side of (E.5) is positive. This demonstrates monotonicity.

Finally, we obtain the range of values for L by examining the limits as $\gamma \to \infty$ and $\gamma \to -\infty$. From the expression for L and the result that $\frac{\partial b}{\partial \gamma}$ is a constant, we obtain

$$\lim_{\gamma\to-\infty} L = 0$$
$$\lim_{\gamma\to\infty} L = -\infty$$

When $c<0$, the domain of the quadratic function is $(-\infty,\infty)$, which, along with monotonicity, implies the range for L of $(0,-\infty)$. When $c>0$, the domain is not defined on the interval $(-2\sqrt{c}, 2\sqrt{c})$, but L is equal in value at the boundaries of the domain. QED.

Additionally, we note that the upper root, $U \equiv \frac{1}{2}\left(-b+\sqrt{b^2-4ac}\right)$ is increasing in γ. When the upper root is a valid solution (i.e., negative), it must be the case that $c>0$ and $b>0$, and it is straightforward to follow the above arguments to show that $\frac{\partial U}{\partial \gamma} > 0$ and that the range of the upper root is $[-\frac{1}{2}b, 0)$.

Proof of Proposition 4 (Non-Constant Marginal Costs)

Under the semi-linear marginal cost schedule of equation (23), the plim of the OLS estimator is equal to

$$\text{plim}\hat{\beta}^{OLS} = \beta + \frac{Cov(\xi, g(q))}{Var(p^*)} - \frac{1}{\beta}\frac{Cov\left(\xi, \frac{dh}{dq}q\right)}{Var(p^*)}.$$

This is obtain directly by plugging in the first–order condition for p: $Cov(p^*, \xi) = Cov(g(q) + \eta - \frac{1}{\beta}\frac{dh}{dq}q - \hat{p}, \xi) = Cov(\xi, g(q)) - \frac{1}{\beta}Cov(\xi, \frac{dh}{dq}q)$under the assumptions. Next, we re-express the terms including the unobserved demand shocks in in terms of OLS residuals. The unobserved demand shock may be written as $\xi = h(q) - x\beta_x - \beta p$. The estimated residuals are given by $\xi^{OLS} = \xi + \left(\beta - \beta^{OLS}\right)p^*$. As $\beta - \beta^{OLS} = \frac{1}{\beta}\frac{Cov\left(\xi, \frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov(\xi,g(q))}{Var(p^*)}$, we obtain $\xi^{OLS} = \xi + \left(\frac{1}{\beta}\frac{Cov\left(\xi, \frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov(\xi,g(q))}{Var(p^*)}\right)p^*$. This implies

$$Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right) = \left(1 + \frac{1}{\beta}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)Cov(\xi, \frac{dh}{dq}q) - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}Cov(\xi, g(q))$$
$$Cov\left(\xi^{OLS}, g(q)\right) = \frac{1}{\beta}\frac{Cov(p^*, g(q))}{Var(p^*)}Cov\left(\xi, \frac{dh}{dq}q\right) + \left(1 - \frac{Cov(p^*, g(q))}{Var(p^*)}\right)Cov(\xi, g(q))$$

We write the system of equations in matrix form and invert to solve for the covariance terms that include the unobserved demand shock:

$$\begin{bmatrix} Cov(\xi, \frac{dh}{dq}q) \\ Cov(\xi, g(q)) \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{\beta}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} & -\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} \\ \frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1 - \frac{Cov(p^*,g(q))}{Var(p^*)} \end{bmatrix}^{-1} \begin{bmatrix} Cov(\xi^{OLS}, \frac{dh}{dq}q) \\ Cov(\xi^{OLS}, g(q)) \end{bmatrix}$$

where

$$\begin{bmatrix} 1 + \frac{1}{\beta}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} & -\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} \\ \frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1 - \frac{Cov(p^*,g(q))}{Var(p^*)} \end{bmatrix}^{-1} =$$
$$\frac{1}{1 + \frac{1}{\beta}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(p^*,g(q))}{Var(p^*)}}\begin{bmatrix} 1 - \frac{Cov(p^*,g(q))}{Var(p^*)} & \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} \\ -\frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1 + \frac{1}{\beta}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} \end{bmatrix}.$$

Therefore, we obtain the relations

$$Cov(\xi, \frac{dh}{dq}q) = \frac{\left(1 - \frac{Cov(p^*,g(q))}{Var(p^*)}\right)Cov(\xi^{OLS}, \frac{dh}{dq}q) + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}Cov(\xi^{OLS}, g(q))}{1 + \frac{1}{\beta}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(p^*,g(q))}{Var(p^*)}}$$
$$Cov(\xi, g(q)) = \frac{-\frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)}Cov(\xi^{OLS}, \frac{dh}{dq}q) + \left(1 + \frac{1}{\beta}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)Cov(\xi^{OLS}, g(q))}{1 + \frac{1}{\beta}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(p^*,g(q))}{Var(p^*)}}.$$

In terms of observables, we can substitute in for $Cov(\xi, g(q)) - \frac{1}{\beta}Cov\left(\xi, \frac{dh}{dq}q\right)$ in the plim of the OLS estimator and simplify:

$$
\begin{aligned}
&\left(1+\frac{1}{\beta}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(p^*, g(q))}{Var(p^*)}\right)\left(Cov(\xi, g(q)) - \frac{1}{\beta}Cov\left(\xi, \frac{dh}{dq}q\right)\right) \\
=& -\frac{1}{\beta}\frac{Cov(p^*, g(q))}{Var(p^*)}Cov(\xi^{OLS}, \frac{dh}{dq}q) + \left(1+\frac{1}{\beta}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)Cov(\xi^{OLS}, g(q)) \\
&-\frac{1}{\beta}\left(1-\frac{Cov(p^*, g(q))}{Var(p^*)}\right)Cov(\xi^{OLS}, \frac{dh}{dq}q) - \frac{1}{\beta}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}Cov(\xi^{OLS}, g(q)) \\
=& Cov(\xi^{OLS}, g(q)) - \frac{1}{\beta}Cov(\xi^{OLS}, \frac{dh}{dq}q).
\end{aligned}
$$

Thus, we obtain an expression for the probability limit of the OLS estimator,

$$
\text{plim}\hat{\beta}^{OLS} = \beta - \frac{\frac{Cov(\xi^{OLS}, \frac{dh}{dq}q)}{Var(p^*)} - \beta\frac{Cov(\xi^{OLS}, g(q))}{Var(p^*)}}{\beta + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \beta\frac{Cov(p^*, g(q))}{Var(p^*)}},
$$

and the following quadratic β.

$$
\begin{aligned}
0 =& \left(1-\frac{Cov(p^*, g(q))}{Var(p^*)}\right)\beta^2 \\
&+\left(\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \hat{\beta}^{OLS} + \frac{Cov(p^*, g(q))}{Var(p^*)}\hat{\beta}^{OLS} + \frac{Cov(\xi^{OLS}, g(q))}{Var(p^*)}\right)\beta \\
&+\left(-\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\hat{\beta}^{OLS} - \frac{Cov(\xi^{OLS}, \frac{dh}{dq}q)}{Var(p^*)}\right).
\end{aligned}
$$

QED.

Proof of Lemma A.1

The proof is by construction. Note that model has the solutions $p_t^* = \frac{1}{2}\left(-\frac{\alpha}{\beta} - \frac{\xi_t}{\beta} + \gamma + \frac{\nu_t}{\beta}\right)$ and $q_t^* = \frac{1}{2}(\alpha + \xi_t + \beta\gamma + \nu_t)$, where again $\nu_t \equiv \beta\eta_t$. The following objects are easily derived:

$$
\begin{aligned}
Cov(p, \xi) &= -\frac{1}{2\beta}Var(\xi) & Cov(p, \nu) &= \frac{1}{2\beta}Var(\nu)2 \\
Var(p) &= \frac{Var(\nu) + Var(\xi)}{(2\beta)^2} & Var(q) &= \frac{1}{4}(Var(\xi) + Var(\nu))
\end{aligned}
$$

Using the above, we have

$$\begin{aligned} Cov(p,q) &= Cov(p, \alpha + \beta p + \xi) = \beta Var(p) + Cov(p,\xi) = \beta \frac{Var(\nu) + Var(\xi)}{(2\beta)^2} - \frac{2\beta}{(2\beta)^2} Var(\xi) \\ &= \frac{\beta Var(\nu) + \beta Var(\xi) - 2\beta Var(\xi)}{(2\beta)^2} = \beta \frac{Var(\nu) - Var(\xi)}{(2\beta)^2} \end{aligned}$$

And that obtains equation (A.2):

$$plim\left(\hat{\beta}^{OLS}\right) \equiv \beta^{OLS} = \frac{Cov(p,q)}{Var(p)} = \beta \frac{Var(\nu) - Var(\xi)}{Var(\nu) + Var(\xi)}$$

Equation (A.4) requires an expression for $Cov(q, \xi^{OLS})$. Define

$$plim(\hat{\xi}^{OLS}) \equiv \xi^{OLS} = q - \alpha^{OLS} - \beta^{OLS} p$$

Then, plugging into $Cov(q, \xi^{OLS})$ using the objects derived above, we have

$$\begin{aligned} Cov(q,\xi^{OLS}) &= Cov(q, q - \beta^{OLS} p) \\ &= Var(q) - \beta^{OLS} Cov(p,q) \\ &= \frac{1}{4}(Var(\xi) + Var(\nu)) - \left(\beta \frac{Var(\nu) - Var(\xi)}{Var(\nu) + Var(\xi)}\right)\left(\beta \frac{(Var(\nu) - Var(\xi))}{(2\beta)^2}\right) \\ &= \frac{1}{4}\left(\frac{[Var(\xi) + Var(\nu)]^2 - [Var(\nu) - Var(\xi)]^2}{Var(\nu) + Var(\xi)}\right) \\ &= \frac{Var(\xi)Var(\nu)}{Var(\nu) + Var(\xi)} \end{aligned}$$

We turn now to equation (A.3). Based on the above, we have that

$$\frac{Cov(q,\xi^{OLS})}{Var(p)} = \left(\frac{Var(\xi)Var(\nu)}{Var(\nu) + Var(\xi)}\right) \frac{(2\beta)^2}{Var(\nu) + Var(\xi)} = (2\beta)^2 \frac{Var(\xi)Var(\nu)}{[Var(\nu) + Var(\xi)]^2}$$

and now only few more lines of algebra are required:

$$\begin{aligned} (\beta^{OLS})^2 + \frac{Cov(q,\xi^{OLS})}{Var(p)} &= \beta^2 \left[\frac{Var(\nu) - Var(\xi)}{Var(\nu) + Var(\xi)}\right]^2 + (2\beta)^2 \frac{Var(\xi)Var(\nu)}{[Var(\nu) + Var(\xi)]^2} \\ &= \frac{\beta^2[Var^2(\nu) + Var^2(\xi) - 2Var(\nu)Var(\xi)] + 4\beta^2 Var(\nu)Var(\xi)}{[Var(\nu) + Var(\xi)]^2} \\ &= \frac{\beta^2[Var^2(\nu) + Var^2(\xi) + 2Var(\nu)Var(\xi)]}{[Var(\nu) + Var(\xi)]^2} \\ &= \beta^2 \frac{[Var(\nu) + Var(\xi)]^2}{[Var(\nu) + Var(\xi)]^2} = \beta^2 \end{aligned}$$

QED.

Proof of Proposition C.1 (Two-Stage Estimator)

Suppose that, in addition to assumptions 1-3, that marginal costs are uncorrelated with the exogenous demand factors (Assumption 5). Then, the expression $\frac{1}{\beta+\frac{Cov\left(p,\frac{dh}{dq}q\right)}{Var(p)}}\frac{Cov\left(\xi^{OLS},\frac{dh}{dq}q\right)}{Var(p)}$ is equal to $\frac{1}{\beta+\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\frac{dh}{dq}q\right)}{Var(p^*)}$.

Assumption 4 implies $Cov(\hat{p},c)=0$, allowing us to obtain

$$
\begin{aligned}
Cov(\hat{p},\beta(\hat{p}+p^*-c)) &= \beta Var(\hat{p}) \\
Cov(p-p^*,\beta(\hat{p}+p^*-c)) &= \beta Var(p)-\beta Var(p^*) \\
Var(p)\beta+Cov\left(p,\frac{dh}{dq}q\right) &= Var(p^*)\beta+Cov\left(p^*,\frac{dh}{dq}q\right) \\
\left(\beta+\frac{Cov\left(p,\frac{dh}{dq}q\right)}{Var(p)}\right)\frac{1}{Var(p^*)} &= \left(\beta+\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}\right)\frac{1}{Var(p)} \\
\frac{1}{\beta+\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\frac{dh}{dq}q\right)}{Var(p^*)} &= \frac{1}{\beta+\frac{Cov\left(p,\frac{dh}{dq}q\right)}{Var(p)}}\frac{Cov\left(\xi^{OLS},\frac{dh}{dq}q\right)}{Var(p)}.
\end{aligned}
$$

Therefore, the probability limit of the OLS estimator can be written as:

$$
\text{plim}\hat{\beta}^{OLS}=\beta-\frac{1}{\beta+\frac{Cov\left(p,\frac{dh}{dq}q\right)}{Var(p)}}\frac{Cov\left(\xi^{OLS},\frac{dh}{dq}q\right)}{Var(p)}.
$$

The roots of the implied quadratic are:

$$
\frac{1}{2}\left(\beta^{OLS}-\frac{Cov\left(p,\frac{dh}{dq}q\right)}{Var(p)}\pm\sqrt{\left(\beta^{OLS}+\frac{Cov\left(p,\frac{dh}{dq}q\right)}{Var(p)}\right)^2+4\frac{Cov\left(\xi^{OLS},\frac{dh}{dq}q\right)}{Var(p)}}\right)
$$

which are equivalent to the pair $\left(\beta,\beta\left(1-\frac{Var(p^*)}{Var(p)}\right)+\frac{Cov(p^*,\xi)}{Var(p^*)}-\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p)}\right)$. Therefore, with the auxiliary condition $\beta<\frac{Cov(p^*,\xi)}{Var(p^*)}\frac{Var(p)}{Var(p^*)}-\frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)}$, the lower root is consistent for β. QED.