Incorporating the Customer's Perspective into the Newsvendor Problem

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Abstract

The newsvendor problem is a classic in management science partly because selecting an optimal inventory level in the face of uncertain demand is an important problem but also because the solution is so elegant and intuitive: the inventory should be selected so that the probability that the vendor stocks out should be set equal to the ratio of the item's unit cost to its unit price.

A number of attempts have been made to enrich the factors treated in the analysis. For example, if price can be set by the vendor, this will influence demand. Extensions in which demand is related to the price charged lose the closed form, and elegance, of the original solution. Another assumption in the problem is that the demand, while uncertain, is not affected by the chosen inventory level. It might be supposed that a better in-stock position will ultimately lead to higher demand.

In this paper we examine the case in which expected demand is related to the expected value anticipated by the customer. We define this value as the consumer surplus (willingness to pay less price charged) times the likelihood that the customer finds the item in stock. With this refinement we show that the optimal inventory is such that the probability of stocking out is equal to the ratio of the item's unit cost to the customer's willingness to pay.

We apply the method to cases in which there are multiple products and find that the simple solution is preserved.

1. Introduction

A newsvendor sells one particular newspaper. Each day he can order as many copies of the paper as he likes at 20¢ per copy. He sells them at the suggested price of 50¢ but demand is highly variable, and he frequently has copies left over which he can only throw away. Assuming he wishes to maximize his expected profit, how many newspapers should he order?

If he orders too many he is likely to waste money, if he orders too few he is likely to miss sales. This is the classic newsvendor problem. The solution, as most students learn early in their careers, is that the optimal choice of inventory is such that the chance of at least one lost sale equals 2/5 (20 cents divided by 50 cents), or in general, unit cost divided by unit price. The solution makes sense: if the cost is low, but the price is high, you want a lot of inventory on hand. If the cost is high but the profit margin is low, you'd like to be sure you can sell what you buy.

Let c be the cost per unit of the newspapers, p be the price charged and denote the uncertain demand by $D\tilde{z}$ where D, a constant, represents the expected (average) demand, and $\tilde{z}$ is a non-negative random variable with mean 1, cumulative distribution F and density function f. The optimal inventory I^* is defined, implicitly, by

$$1 - F(I^* / D) = c / p \ . \qquad (1.1)$$

The problem dates from Whitin (1955). An extensive analysis of variations and extensions may be found in Porteus (1990) or Federgruen and Heching (1999). A recent review is given by Petruzzi and Dada (1999).

It is easy to modify the story slightly while still preserving the style of the solution. For example, if we are selling umbrellas not newspapers then any unsold inventory may be carried forward to the next day (period). If h is the holding cost of the inventory for one period (expressed as a fraction of the unit cost) then the new inventory solution is

$$1 - F(I^* / D) = hc / (p - (1-h)c) \ . \qquad (1.2)$$

The "critical fractile" on the right hand side of (1.2) is the ratio of the marginal cost of having one unit too many ("overage") to the sum of that number and the lost marginal revenue of having one too few ("underage"). (See Porteus (1990) or Bell and Schleifer (1995).) The same logic is true in (1.1).

Another easy refinement is to consider that customer goodwill is lost if a customer finds you out of stock. If the customer has traveled many miles to come to a store for an item and it is out of stock, one can suppose that customer might be reluctant to make the trip again. If g is a dollar amount reflecting the lost goodwill that can be attributed to a customer finding you out of stock then analysis reveals that the critical fractile becomes $c/(p+g)$. But there has been little success in extending the basic model—while preserving its simple solution—much beyond this.

A desirable extension would be to make demand dependent on price, say $D(p)\tilde{z}$. As Petruzzi and Dada (1999) demonstrate, while such a formulation can be solved and some general properties of the solution described, the solution is not of a closed form, even implicitly.

In the next section we propose a refinement of the newsvendor problem which does include price as a decision variable that affects both profit and demand, and which includes endogenous feedback on demand of the deleterious effects on the customer of being out of stock.

2. Customer Added Value

The analysis we have reviewed treats the newsvendor as a rational economic agent carefully weighing costs of underage and overage in selecting inventory levels. But let us also consider the customer as an economic actor. Why is she buying this product? Presumably it is because she feels better off for doing so: the price charged is less than her willingness-to-pay. Her "consumer surplus" can be calculated as $a - p$ where a represents the maximum amount she is willing to pay for an item. (This amount may be thought of as a function, in part, of where else, and at what price, the item is available, or of the cost and value of substitutes.)

Once at the store, the customer's optimal strategy is, rather simply, to buy the item if $a > p$ and not to if $a < p$. In what follows we will assume that the customer also buys if $a = p$. In light of this we might suppose that the store should set the price at $p = a$. Indeed many retailers do try to extract the customer's willingness to pay especially when the customer has no alternatives (beer at the ballpark) or is otherwise committed (two hours into haggling over a new car). But most retailers set prices in advance and clearly label them, they must allow for different levels of willingness to pay among potential customers, and perhaps most importantly, most retailers rely on repeat business and so endeavor to give customers good value.

For the moment we will continue to focus on one store, selling one product, at a price p and cost c, to customers all of whom have a maximum willingness to pay a. The retailer must select an inventory level I in advance of observing the demand level. Excess inventory is worthless (though recall that if excess may be carried forward to the next period the algebraic modifications needed are routine).

Let us consider the decision problem faced by the customer. Should she travel to the store to buy the item or not? If she does, and the item is in stock, she benefits by a-p less her cost of travel. If the store is out of stock, we assume her trip is worthless. Let us denote the probability that the store has the item she seeks as q. Thus her expected value from a store visit may be written

$$\text{Customer's Expected Value } = (a - p)q \ .$$

She should make the trip if this expected value exceeds her travel cost, but not otherwise. With this argument in mind we will assume in our refinement of the newsvendor problem that expected demand is a function of this expected value, namely $D((a-p)q)$: the higher the expected value, the more people will be attracted to the store. We will assume that D is an increasing, twice differentiable function; for non-extreme solutions we also require that at the optimal solution, $DD'' < 2D'D'$. This is a very mild restriction; all concave functions satisfy it, as do all power functions. In Bell (1997) I suggest that $D((a-p)q) = k(a-p)^2 q^2$ is a plausible function; if travel costs are proportional to distance from the store, and if potential customers are distributed evenly around the store, then the number of potential customers whose travel cost is less than the expected value added will grow as the square of that value added.

Finally, our formulation assumes an equilibrium between the actions of the store and its customers. Customers know what price the store charges, and how often it is in stock (but not, of course, whether it is in stock on a particular occasion until arriving at the store). The store, based on experience, can estimate a demand function $D\tilde{z}$ where $\tilde{z}$ is an error distribution, with mean 1, independent of *D*.

Theorem 1 A newsvendor, facing an uncertain demand $D((a-p)q)\tilde{z}$, where *p* is the sale price of an item costing *c*, *q* is the probability a customer finds the item in stock, *a* is a constant and $\tilde{z}$ is an error distribution that is independent of *p* and *q*, should select an inventory level so that the probability that at least one customer finds the item to be out of stock equals *c/a*.

Proof The traditional newsvendor problem is a special case in which *D* is constant and *p* is fixed. If the newsvendor selects an inventory *I*, at a cost of *cI*, he will stock out (lose at least one sale) if $D\tilde{z} > I$. The theorem suggests that *I* should be chosen so that the probability of this is *c/a*. Note that this is a different probability from the quantity *1-q* which represents the *fraction of customers* who are disappointed. For example if demand is equally likely to be five or six and the vendor stocks five items, then the probability of a stockout is one half. However, the probability that any given customer is disappointed is only 1/11. Our assumption is that from the customer's point of view it is the 1/11 probability that matters not the one half. More generally *q* = Expected Sales/Expected Demand.

Instead of dealing with the quantity *I* we will express our calculations in terms of the proportional inventory *i* defined by *I=Di*. This will make the analysis which follows more transparent. For any given choice of inventory, the vendor's sales will vary, not only because of $\tilde{z}$ but also because $D((a-p)q)$ depends on *q* the probability that a customer is able to buy an item.

The vendor's expected profit is

$$
\begin{aligned}
& p \text{ Expected Sales } - cI \\
&= pq \text{ Expected Demand } - ci \text{ Expected Demand} \\
&= (pq - ci) \text{ Expected Demand} \\
&= (pq - ci) D((a-p)q) \ .
\end{aligned}
$$

Recall that *f* is the density function of $\tilde{z}$ so that

$$q = \int_0^i x\, f(x)dx + \int_i^\infty i\, f(x)dx \tag{2.0}$$

and

$$\frac{dq}{di} = \int_i^\infty f(x)dx = 1 - F(i) \ .$$

So *q* is the probability a customer is able to obtain the product but dq/di is the probability that the store stocks out.

We now maximize the vendor's expected profit by differentiating with respect to the two decision variables, *p* and *i*, and setting the differentials to zero. We obtain, assuming a non-extreme solution,

$$qD((a-p)q) - q(pq - ci)D'((a-p)q) = 0 \tag{2.1}$$

and

$$\left(p\frac{dq}{di} - c\right)D((a-p)q) + (a-p)\frac{dq}{di}(pq - ci)D'((a-p)q) = 0. \tag{2.2}$$

From the first we deduce that $D = (pq - ci)D'$ so that, from the second,

$$\left(p\frac{dq}{di} - c\right) + (a-p)\frac{dq}{di} = 0$$

or

$$\frac{dq}{di} = c/a \ .$$

Even if the optimal price is the extreme solution *a*, the derivative with respect to *i* becomes

$$\left(a\frac{dq}{di}-c\right)D(0)=0 \tag{2.3}$$

so that we still have $\frac{dq}{di}=c/a$. The second order conditions (for a maximum) are satisfied if and only if, at the solution to (2.1) and (2.2), we also have $DD''<2D'D'$. //

Because *D* is arbitrary (though well-behaved) we have ignored the question of solving for *p*. For some choices of *D*, for example if $D(x)=x^n\ (n>0)$, there are closed form solutions for *p*, but not in general.

It will help later if we note that the inventory solution $dq/di=c/a$ is unchanged if we generalize the objective function from $(pq-ci)D((a-p)q)$ to $(pq-ci+k)D((a-p)q)$ where *k* is any constant. For the purposes of the current formulation *k* could represent an expense for each customer that visits the store, whether they buy or not; for example it could represent the cost of sales assistance or wear and tear on the store. The constant *k* could also be positive, for example if visiting customers could be expected to spend money on other (unspecified) impulse items. It will also be useful to note the following result.

Theorem 2 If the vendor is to select, from a set of possible products with varying values of *c* and *a*, the most profitable product to sell then

(i) Among those with the same ratio *c/a* he will prefer the product with the highest value of *a – c*.

(ii) Among those with the same difference *a – c* he will prefer the product with the lowest value of *c/a*.

In particular, one product is more profitable than another if it has both a higher *a – c* and a lower *c/a*.

Proof Let *a – c* = *k* and *c/a* = λ. The profit equation $(pq-ci)D((a-p)q)$ may be rewritten as $\left(pq-\frac{\lambda k}{1-\lambda}i\right)D\left(\left(\frac{k}{1-\lambda}-p\right)q\right)$. The first order derivative with respect to *k* is

$$\frac{-\lambda}{1-\lambda} i\, D + \left(pq - \frac{\lambda k i}{1-\lambda} \right) \frac{q}{(1-\lambda)} D'$$

which, in light of (2.1) is equal to

$$(q - \lambda i) D / (1-\lambda) \ .$$

This is positive since, by (2.0), $q - \lambda i = \int_0^i x\, f(x) dx$. The first order derivative with respect to λ is

$$\frac{-ki}{(1-\lambda)^2} D + \frac{kq}{(1-\lambda)^2} \left(pq - \frac{\lambda k i}{1-\lambda} \right) D'$$

which, in light of (2.1) is equal to

$$(q-i) \frac{k}{(1-\lambda)^2} D \ .$$

This is negative since, by (2.0) $i - q = \int_0^i (i-x) f(x) dx$. //

In the remainder of the paper we describe some extensions of the model. In each case the inventory solution has the simple style of solution of the classic newsvendor problem.

3. Two Independent Products

Suppose the vendor (store) offers two products. Each customer desires either or both of the products so long as the price is below respective reservation values of a_1 and a_2. If the vendor sets prices of p_1 and p_2 and an inventory policy such that the customer finds the products in stock a fraction q_1 and q_2 of the time then the expected added value to the customer of a trip to the store is

$$\text{Customer's Expected Value} = (a_1 - p_1) q_1 + (a_2 - p_2) q_2 \ .$$

We assume therefore that the uncertain demand is $D((a_1 - p_1)q_1 + (a_2 - p_2)q_2)\tilde{z}$. The unit costs to the vendor are c_1 and c_2 respectively. If I_1 and I_2 are the vendor's chosen inventory levels then, as before, we define the "inventory policy" variables i_1 and i_2 by $I_1 = Di_1$, $I_2 = Di_2$. The vendor's goal is to maximize

$$(p_1q_1 - c_1i_1 + p_2q_2 - c_2i_2)D((a_1 - p_1)q_1 + (a_2 - p_2)q_2)$$

and the optimal inventory occurs when

$$\frac{dq_1}{di_1} = c_1/a_1 \quad \text{and} \quad \frac{dq_2}{di_2} = c_2/a_2$$

that is, the single product solution carries over to each product independently.

To see this, consider the vendor's problem in stocking product 1, with product 2's policy assumed fixed (perhaps optimally, perhaps not). The problem is to maximize $(p_1q_1 - c_1i_1 + k)D((a_1 - p_1)q_1 + b_2)$ where k is a constant (equal to $p_2q_2 - c_2i_2$) and b_2 is a constant (equal to $(a_2 - p_2)q_2$). We know the solution to this is $\frac{dq_1}{di_1} = c_1/a_1$. That is, no matter how we go about pricing, and stocking, product 2, the inventory policy for product 1 is the same (it is in this sense that the products are "independent").

The presence of product 2 increases overall demand for product 1 but does not change the inventory policy of product 1, though it might well influence the optimal price.

Our conclusion obviously generalizes to any number of products. It also may be adapted to reflect the real world concern that total inventory must be less than some budget constraint. In this case inventory policies conform to some ratio $\frac{\lambda c_i}{a_i}$ where λ is selected so that the budget is met.

4. Two Substitute Products

Again we assume that the store has two items but this time customers do not want both. If both are available they will prefer item 1 if $a_1 - p_1 \geq a_2 - p_2$, and buy it if also $a_1 - p_1 \geq 0$. For the moment however, we will assume that product 1 is automatically preferred over product 2 so long as $a_1 - p_1 \geq 0$. Let q_1 be the probability that a customer buys product 1 (which is the same as the probability that the customer finds it in stock) and let q_2 be the probability that the customer buys product 2 (which equals the probability that product 2 is available *and* product 1 is not).

The expected value to the customer is again $(a_1 - p_1)q_1 + (a_2 - p_2)q_2$ and the uncertain demand is $D((a_1 - p_1)q_1 + (a_2 - p_2)q_2)\tilde{z}$. Unit costs are as before. Note that we now have

$$q_1 = \int_0^{i_1} xf(x)dx + \int_{i_1}^{\infty} i_1 f(x)dx \qquad (4.1)$$

but

$$\begin{aligned} q_2 &= \int_{i_1}^{i_1+i_2} (x - i_1) f(x)dx + \int_{i_1+i_2}^{\infty} i_2 f(x)dx \\ &= \int_{i_1}^{i_1+i_2} xf(x)dx - i_1(F(i_1 + i_2) - F(i_1)) + i_2(1 - F(i_1 + i_2)) \ . \end{aligned} \qquad (4.2)$$

Again we wish to maximize $(p_1q_1 - c_1i_1 + p_2q_2 - c_2i_2)D$ but the problem is now more complex, for unlike the independent product case, we do *not* have $\frac{dq_2}{di_1} = 0$.

Differentiating the objective by p_1 and setting to zero we get

$$q_1D - q_1(p_1q_1 - c_1i_1 + p_2q_2 - c_2i_2)D' = 0$$

or

$$D = (p_1q_1 - c_1i_1 + p_2q_2 - c_2i_2)D' \ . \qquad (4.3)$$

The same equation results when differentiating by p_2. Differentiating by i_1 and setting to zero:

$$\left(p_1\frac{dq_1}{di_1} - c_1 + p_2\frac{dq_2}{di_1}\right)D + (p_1q_1 - c_1i_1 + p_2q_2 - c_2i_2)\left((a_1 - p_1)\frac{dq_1}{di_1} + (a_2 - p_2)\frac{dq_2}{di_1}\right)D' = 0 \ .$$

Substituting from (4.3) we deduce

$$p_1\frac{dq_1}{di_1} - c_1 + p_2\frac{dq_2}{di_1} + (a_1 - p_1)\frac{dq_1}{di_1} + (a_2 - p_2)\frac{dq_2}{di_1} = 0$$

or

$$a_1 \frac{dq_1}{di_1} + a_2 \frac{dq_2}{di_1} = c_1 \quad . \tag{4.4}$$

Differentiating with respect to i_2 and setting to zero:

$$\left(p_2 \frac{dq_2}{di_2} - c_2 \right) D + \left(p_1 q_1 - c_1 i_1 + p_2 q_2 - c_2 i_2 \right)(a_2 - p_2) \frac{dq_2}{di_2} D' = 0 \ .$$

Substituting from (4.3) we deduce

$$p_2 \frac{dq_2}{di_2} - c_2 + (a_2 - p_2) \frac{dq_2}{di_2} = 0$$

or

$$\frac{dq_2}{di_2} = {c_2}/{a_2} \quad . \tag{4.5}$$

From (4.1) and (4.2) we can calculate

$$\frac{dq_1}{di_1} = 1 - F(i_1)$$
$$\frac{dq_2}{di_2} = 1 - F(i_1 + i_2)$$
$$\frac{dq_2}{di_1} = F(i_1) - F(i_1 + i_2) \quad .$$

From (4.5) we conclude $1 - F(i_1 + i_2) = {c_2}/{a_2}$ and from (4.4) that

$$a_1(1 - F(i_1)) + a_2 \left(F(i_1) - F(i_1 + i_2) \right) = c_1$$

or

$$1 - F(i_1) = \frac{c_1 - c_2}{a_1 - a_2} \quad .$$

Evidently this solution requires $c_1 > c_2$ and $a_1 > a_2$ (or $c_1 < c_2$ and $a_1 < a_2$) and

$$\frac{c_1 - c_2}{a_1 - a_2} > \frac{c_2}{a_2}$$

since product 2 stocks out only if product 1 already has stocked out. Hence

$$\frac{c_1}{a_1} > \frac{c_2}{a_2}.$$

Theorem 3 Suppose the vendor may carry either or both of two products for which $a_1 - c_1 > a_2 - c_2 > 0$ and $c_1 / a_1 > c_2 / a_2 > 0$. If customers will buy one, but not both of the products, then the vendor should stock product 1 so that there is only a $(c_1 - c_2)/(a_1 - a_2)$ probability that at least one customer will find it stocked out, and should stock product 2 so that the probability that at least one customer will find both products stocked out is c_2 / a_2.

Proof The argument developed above shows that *conditional* on customers preferring product 1 over product 2 the vendor should act as the theorem requires. So it remains to show that if $a_1 - c_1 > a_2 - c_2 > 0$ and $c_1 / a_1 > c_2 / a_2 > 0$ then also $a_1 - p_1^* > a_2 - p_2^*$, that is, there exist optimal prices so that indeed customers prefer product 1 to product 2.

The optimal prices are derived from (4.3) which, with q_1, q_2, i_1 and i_2 known is a function of $p_1 q_1 + p_2 q_2$. Indeed if we write $p_1 q_1 + p_2 q_2 = p_0$ and $a_1 q_1 + a_2 q_2 = a_0$ then (4.3) becomes $(p_0 - c_1 i_1 - c_2 i_2) D'(a_0 - p_0) = D(a_0 - p_0)$. This equation has some solution p_0^*. Thus any set of prices that satisfy $(a_1 - p_1) q_1 + (a_2 - p_2) q_2 = a_0 - p_0^*$ is also optimal. In particular we may select prices so that $a_1 - p_1 > a_2 - p_2$. This makes product 1 more attractive to the customer, as required. //

Because product 1 is costly to keep in stock ($c_1 / a_1 > c_2 / a_2$), the presence of product 2 permits the vendor to be less aggressive in stocking product 1, knowing that a disappointed customer may be able to buy product 2 instead of incurring a wasted trip. Product 1 plays the role of the “preferred product” even though the vendor might prefer to carry only product 2 if he could only stock one or the other.

5. Discussion

The paper has concerned the methodological implications of endogenizing the customer as a rational actor into the newsvendor problem. Perhaps more important are the intuitive implications. Traditionally, overage (the cost of having stocked one too many) and underage (the opportunity cost of stocking one too few) have been measured from the vendor's (short term) perspective. In our formulation however, we see that these should more properly be considered relative to the combined vendor-customer value chain: overage remains c but underage is $a - c$. It may be that such a perspective will prove beneficial in more complex problems where the system solution is less obvious.

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