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Cadet-Branch Matching in a Kelso-Crawford Economy†

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Sönmez (2013) and Sönmez and Switzer (2013) used matching theory with unilaterally substitutable priorities to propose mechanisms to match cadets to military branches. This paper shows that, alternatively, the Sönmez and Sönmez–Switzer mechanisms can be constructed as descending salary adjustment processes in Kelso-Crawford (1982) economies in which cadets are (grossly) substitutable. The lengths of service contracts serve as (inverse) salaries. The underlying substitutability explains the unilateral substitutability of the priorities utilized by Sönmez and Sönmez-Switzer. (*JEL* C78, D82, D86, J31, J45)

In a pair of recent papers, Sönmez (2013) and Sönmez and Switzer (2013) brought the problem of cadet-branch matching to market design. Cadets graduating from the US Military Academy (USMA) and Reserve Officers' Training Corps (ROTC) are required to serve as officers in the US Army for three years (for ROTC non-scholarship graduates), four years (for ROTC scholarship graduates), or five years (for USMA graduates). Until a few years ago, cadets were ranked in an Order of Merit List (OML) based on performance evaluations and chose branches via serial dictatorship. In response to low retention of officers after the end of their obligatory service, the army instituted a *branch-of-choice* program whereby cadets are allowed to commit to three additional years of service in exchange for increased priority. The army attempted to assign cadets to branches using deferred acceptance, but the army's implementation caused multiple problems with cadets' incentives.[1]

Sönmez (2013) and Sönmez and Switzer (2013) proposed mechanisms based on deferred acceptance to match cadets to branches. The *bid for your career* (*BfYC*)

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[1] Sönmez (2013) and Sönmez and Switzer (2013) showed that the current ROTC and USMA mechanisms are not strategy-proof. Furthermore, these mechanisms can assign more desirable branches to weaker cadets, so that the current mechanisms do not *respect unambiguous improvements in priority*. As a result, the ROTC and USMA mechanisms incentivize cadets to fail their exams intentionally, which happens in practice.

priorities (Sönmez 2013), which are the branches' priorities in the proposed mechanisms,[2] sometimes favor long contracts but sometimes favor short contracts. This inconsistency prevents contracts from being interpreted as salaries and causes complementarities, in that gaining access to one contract can make a branch desire another contract more.[3] Complementarities usually preclude the existence of stable outcomes (Kelso and Crawford 1982, Hatfield and Kojima 2008, Hatfield and Kominers 2017), but the BfYC priorities satisfy Hatfield and Kojima's (2010) *unilateral substitutability* condition, which guarantees that deferred acceptance produces a stable outcome.

This paper shows that cadet-branch matching does not formally require matching theory with weakened substitutability conditions or many-to-many matching. I restore *substitutability* (in the sense of Kelso and Crawford 1982 and Hatfield and Milgrom 2005) by changing priorities to systematically favor long contracts. This change of priorities does not affect the deferred acceptance mechanism. Defining the "salary" corresponding to a contract to be any decreasing function of the service time, the substitutable priorities are generated by maximizing a quasi-linear utility function.[4] If cadets prefer short contracts, then the cadet-branch economy can be regarded as a job market in the Kelso-Crawford (1982) model, and the Sönmez (2013) and Sönmez-Switzer (2013) mechanisms correspond to the descending salary adjustment process. Thus, the Sönmez and Sönmez-Switzer mechanisms feature cadets bidding against each other in an ascending auction in service length.

The results of this paper complement the original constructions of the proposed cadet-branch matching mechanisms. The branches' priorities in the Sönmez (2013) and Sönmez-Switzer (2013) models are as faithful as possible to the exact priorities that are currently used in practice,[5] while this paper's approach to the construction of the proposed mechanisms using substitutable priorities relies on deviation of the branch priorities from the currently implemented priorities. Hence, the priorities considered by Sönmez (2013) and Sönmez and Switzer (2013) may be easier to implement in practice. However, the priorities considered in this paper make the deferred acceptance mechanism the interpretable as an ascending auction in service length. Because the discrepancy between the Sönmez and Sönmez-Switzer priorities and the ones considered in this paper does not affect the proposed matching mechanisms, the auction interpretation applies to the Sönmez and Sönmez-Switzer mechanisms as well.

Furthermore, this paper provides a conceptual explanation of why the priorities involved in cadet-branch matching are unilaterally substitutable. This phenomenon

[2] The USMA priorities (Sönmez and Switzer 2013) are special cases of the BfYC priorities.

[3] Substitutability plays a key role in interpreting contracts as salaries (Echenique 2012, Kominers 2012). Schlegel (2015) interpreted contracts as salaries under weakened substitutability conditions, but without maintaining the natural monotonicity properties of preferences with respect to salaries.

[4] Switzer (2011) pursued a similar approach, but using responsive priorities. Recovering the Sönmez and Sönmez-Switzer (2013) mechanisms requires the use of non-responsive priorities.

[5] Sönmez and Switzer (2013) showed that the currently implemented USMA priority structure is compatible with fairness, strategy-proofness, and respect for improvements, while Sönmez (2013) explained how to modify the currently implemented ROTC priority structure minimally in order to obtain such compatibility.

was called "remarkabl[e]" by Sönmez (2013) and Sönmez and Switzer (2013),[6] who used it to derive their stability and strategy-proofness results. Theorem 4 proves the unilateral substitutability of any priority whose corresponding deferred acceptance mechanism is a descending salary adjustment process in a Kelso-Crawford economy.[7] Combined with the fact that the Sönmez and Sönmez-Switzer mechanisms are descending salary adjustment processes, it follows that the branches' priorities in the Sönmez (2013) and Sönmez-Switzer (2013) models are unilaterally substitutable. Thus, the framework proposed in this paper provides a conceptual explanation of why the priorities proposed by Sönmez and Sönmez-Switzer satisfy this crucial unilateral substitutability condition, whereas only a technical justification was previously known (Sönmez 2013, Sönmez and Switzer 2013).

The isomorphism result proved in this paper sharpens general results that relate contracts and salaries. Echenique (2012) showed that any many-to-one matching market (in the sense of Kelso and Crawford 1982 and Hatfield and Milgrom 2005) can be *embedded* into a potentially non-quasi-linear *matching market with salaries* in which workers are grossly substitutable, and Schlegel (2015) extended the embedding result to the case when branches' priorities are unilaterally substitutable (in the sense of Hatfield and Kojima 2010) by allowing branches' priorities to be only weakly monotone in salary. This paper proves an isomorphism instead of merely an embedding and requires branches' priorities to be quasi-linear and strictly monotone in salary, so that the branches' priorities can be taken to be the choices of profit-maximizing firms that regard workers as (gross) substitutes. As discussed in detail in Section VC, quasi-linearity and strict monotonicity are not only conceptually appealing but also technically useful in proving that the deferred acceptance mechanism is stable and group strategy-proof.

The example of cadet-branch matching illustrates a general phenomenon in many-to-one matching with contracts: some priorities that are not substitutable are *effectively* substitutable from the perspective of matching mechanisms. More formally, I call a priority *DA-substitutable* if it induces the same deferred acceptance mechanism as a substitutable priority. Passing to simpler priorities without changing the deferred acceptance mechanism may offer insight on general mechanism design problems in many-to-one matching with contracts.

As Echenique (2012) suggested, it is natural to ask whether the full generality of matching with contracts is needed in any given application. In cadet-branch matching, cadets are employed by the military and compensated in terms of education. This paper shows that regarding the cadet-branch matching market as a job market and education as a salary offers simpler constructions of the Sönmez (2013) and Sönmez-Switzer (2013) mechanisms and explains the unilateral substitutability of branch priorities in the original models. In general, changing priority structures may yield more intuitive descriptions of mechanisms while capturing features of the

[6] On page 192, Sönmez (2013) wrote "Remarkably, although the substitutes condition fails in my framework, the unilateral substitutes condition is satisfied." On page 454, Sönmez and Switzer (2013) wrote, "Remarkably, although the substitutability condition fails in the context of cadet-branch matching, the unilateral substitutability condition is satisfied."

[7] Theorem 1 also shows that such priorities satisfy Hatfield and Milgrom's (2005) *law of aggregate demand*.

underlying design problems.[8] Moreover, such simplifications might clarify the role and interpretation of contracts and substitutability conditions.

The remainder of this paper is organized as follows. Section I explains some of the results of this paper through an example. Section II reviews the basic model. Section III presents necessary and sufficient conditions for two deferred acceptance mechanisms to coincide. Section IV presents Sönmez's (2013) model of cadet-branch matching and the substitutable branch choice functions. Section V proves that the cadet-branch economy with substitutable choice functions is isomorphic to a Kelso-Crawford economy. Section VI defines DA-substitutability and gives a conceptual proof that the branches' priorities in the Sönmez (2013) and Sönmez-Switzer (2013) models are unilaterally substitutable. Section VII presents an extension, and Section VIII is a conclusion. The Appendices contain the proofs that are omitted from the text, and the online Appendices presents additional examples.

I. An Illustrative Example

Suppose that the United States Military Academy (USMA) is seeking to assign three cadets i_1, i_2, i_3 to the aviation branch a and the medical specialist branch m. Contracts can last for five or eight years. Cadets i_1 and i_2 would like to serve as little as possible and prefer to serve in the aviation branch: therefore, their preferences are

$$i_1\colon\quad (i_1,a,5) \succ_{i_1} (i_1,m,5) \succ_{i_1} (i_1,a,8) \succ_{i_1} (i_1,m,8),$$

$$i_2\colon\quad (i_2,a,5) \succ_{i_2} (i_2,m,5) \succ_{i_2} (i_2,a,8) \succ_{i_2} (i_2,m,8).$$

However, cadet i_3 would like to serve in the aviation branch regardless of the length of his contract—therefore, his preference is

$$i_3\colon\quad (i_3,a,5) \succ_{i_3} (i_3,a,8) \succ_{i_3} (i_3,m,5) \succ_{i_3} (i_3,m,8).$$

Suppose that the order of merit list (OML), which ranks cadets by academic, military, and physical performance, is

$$i_1 \succ_{\mathrm{OML}} i_2 \succ_{\mathrm{OML}} i_3.$$

The medical specialist branch would like to hire one cadet for a term of five years and prefers cadets that are high in the order of merit. Following Sönmez and Switzer (2013), the medical specialist branch prioritizes the short contract over the long contract with each cadet. Therefore, the medical specialist branch's priority is

$$m\colon\quad (i_1,m,5) \succ_m (i_1,m,8) \succ_m (i_2,m,5) \succ_m (i_2,m,8) \succ_m (i_3,m,5) \succ_m (i_3,m,8).$$

[8] The application of substitutable completability to the design of the Israel Psychology Masters' Match by Hassidim, Romm, and Shorrer (2017) cannot be formulated in a substitutable Kelso-Crawford economy. Indeed, note that unilateral substitutability is not satisfied in this setting, and thus the contrapositive of Theorem 4 in Section VIA rules out embedding the market into a quasi-linear Kelso-Crawford economy in which students are substitutable.

The aviation branch would like to hire two cadets, one of whom is high in the order of merit and the other of whom is ideally willing to serve a long term. Following Sönmez and Switzer (2013), shorter contracts are given priority in the high-OML slot and all cadets are first considered for the high-OML slot. Therefore, the aviation branch is comprised of two slots with priorities

$$a^1\colon \quad (i_1,a,5) \succ_1^a (i_1,a,8) \succ_1^a (i_2,a,5) \succ_1^a (i_2,a,8) \succ_1^a (i_3,a,5) \succ_1^a (i_3,a,8),$$

$$a^2\colon \quad (i_1,a,8) \succ_2^a (i_2,a,8) \succ_2^a (i_3,a,8) \succ_2^a (i_1,a,5) \succ_2^a (i_2,a,5) \succ_2^a (i_3,a,5).$$

The aviation branch's priority $\succ_a$ is defined by filling the first slot with a contract with a cadet and then filling the second slot with a contract with a different cadet (see Kominers and Sönmez 2016).

A priority is *substitutable* (in the sense of Hatfield and Milgrom 2005) if having access to one contract does not make a branch want another contract more.[9] Note that the aviation branch's priority is not substitutable because there is complementarity between $(i_1,a,5)$ and $(i_3,a,8)$. Indeed, the aviation branch chooses $(i_3,a,5)$ over $(i_3,a,8)$ if no other contracts are available but sometimes chooses $(i_3,a,8)$ over $(i_3,a,5)$ when the contract $(i_1,a,5)$ is also available. If the aviation branch instead ranked long contracts above short contracts even in the first slot, its priority would be substitutable. Indeed, the priority $\hat{\succ}_a$ induced by slot priorities

$$a^1\colon \quad (i_1,a,8) \hat{\succ}_1^a (i_1,a,5) \hat{\succ}_1^a (i_2,a,8) \hat{\succ}_1^a (i_2,a,5) \hat{\succ}_1^a (i_3,a,8) \hat{\succ}_1^a (i_3,a,5),$$

$$a^2\colon \quad (i_1,a,8) \hat{\succ}_2^a (i_2,a,8) \hat{\succ}_2^a (i_3,a,8) \hat{\succ}_2^a (i_1,a,5) \hat{\succ}_2^a (i_2,a,5) \hat{\succ}_2^a (i_3,a,5),$$

is substitutable because the aviation branch always favors long contracts with every cadet (Proposition 1). For example, the aviation branch always chooses $(i_3,a,8)$ over $(i_3,a,5)$ under $\hat{\succ}_a$, regardless of whether $(i_1,a,5)$ is available. The medical specialist branch's priority is already substitutable but can be modified to encapsulate the intuition that the branch should give priority to long contracts:

$$m\colon \quad (i_1,m,8) \hat{\succ}_m (i_1,m,5) \hat{\succ}_m (i_2,m,8) \hat{\succ}_m (i_2,m,5) \hat{\succ}_m (i_3,m,8) \hat{\succ}_m (i_3,m,5).$$

The change of priorities from $(\succ_a, \succ_m)$ to $(\hat{\succ}_a, \hat{\succ}_m)$ does not affect the deferred acceptance mechanism (Theorem 2).[10] Indeed, the deferred acceptance algorithm proceeds as follows regardless of whether the aviation branch's priority is taken to be $\succ_a$ or $\hat{\succ}_a$ and whether the medical specialist branch's priority is taken to be $\succ_m$ or $\hat{\succ}_m$.

Step 1: Cadet i_j proposes contract $(i_j,a,5)$ to the aviation branch for all i. From the proposal set $\{(i_1,a,5),(i_2,a,5),(i_3,a,5)\}$, the aviation branch rejects contracts $(i_3,a,5)$ and holds contracts $(i_1,a,5)$ and $(i_2,a,5)$.

[9] See Section IIA for a formal definition of substitutability.
[10] See Section IIC for a formal description of deferred acceptance.

Step 2: Cadet i_3 proposes contract $(i_3, a, 8)$ to the aviation branch. From the proposal set $\{(i_1, a, 5), (i_2, a, 5), (i_3, a, 8)\}$, the aviation branch rejects contract $(i_2, a, 5)$ and holds contracts $(i_1, a, 5)$ and $(i_3, a, 8)$.

Step 3: Cadet i_2 proposes contract $(i_2, m, 5)$ to the medical specialist branch. The medical specialist branch holds this contract, and no further rejections occur.

The deferred acceptance mechanism cannot distinguish between the priorities $\succ_m$ and $\hat{\succ}_m$ because the medical specialist branch is only asked to compare contracts between distinct cadets during deferred acceptance. Indeed, note that a cadet only proposes a contract after its most recent proposal is rejected, so each cadet has at most one active proposal at each step of deferred acceptance. As the branches together have at most one active proposal from each cadet at a time, the set of contracts considered by any branch at any stage of deferred acceptance must contain at most one contract with each cadet. In the language of Section III, the priorities $\succ_m$ and $\hat{\succ}_m$ induce *DA-equivalent* choice functions. Similarly, deferred acceptance cannot distinguish between priorities $\succ_a$ and $\hat{\succ}_a$ because they make the same choices from *feasible* sets of contracts, which are sets of contracts that contain at most one contract with each cadet (Theorem 1). Indeed, the slot priorities that define $\succ_a$ and $\hat{\succ}_a$ only differ in the relative order of contracts with individual cadets, and the aviation branch only considers one contract with a given cadet at a time (Theorem 2).

Another appeal of the substitutable priorities $\hat{\succ}_a$ and $\hat{\succ}_m$ is that they have representations that are quasi-linear in the inverses of contract lengths (Proposition 2). Indeed, let the medical specialist branch value a set A of cadets by the valuation function

$$\gamma_m(A) = \begin{cases} 0 & \text{if } A = \varnothing \\ 5 - \min_{i_j \in A} j & \text{if } A \neq \varnothing. \end{cases}$$

That is, the medical specialist branch only values the smartest cadet assigned to it. The priority $\hat{\succ}_m$ is represented by the quasi-linear utility function induced by γ_m. More formally, let

$$\iota(Y) = \{ i_j \,|\, \text{there exists } b \in \{a, m\} \text{ and } t \in \{5, 8\} \text{ such that } (i_j, b, t) \in Y \}$$

be the set of cadets associated with contracts in Y. The choice function corresponding to $\hat{\succ}_m$ maximizes the utility function u_m that is defined by

$$u_m(Y) = \gamma_m(\iota(Y)) - \sum_{(i_j, m, t) \in Y} \frac{1}{t},$$

for all sets Y of contracts that involve the medical specialist branch. Similarly, let the aviation branch value a set A of cadets by the valuation function

$$\gamma_a(A) = \begin{cases} 0 & \text{if } A = \varnothing \\ 5 - j & \text{if } A = \{i_j\} \\ 6 - j + \frac{1}{100k} & \text{if } A = \{i_j, i_k\} \text{ with } j < k \\ 5 + \frac{1}{200} & \text{if } |A| = 3. \end{cases}$$

That is, the aviation branch only values the smartest two cadets assigned to it and only particularly cares about the OML rank of the highest-merit cadet assigned to it. The valuation γ_a induces a quasi-linear utility function that represents the substitutable aviation branch priority $\hat{\succ}_a$.

Thus, the cadet-branch economy with branch priority profile $(\hat{\succ}_a, \hat{\succ}_m)$ can be regarded as a Kelso-Crawford labor market (Theorem 3). The cadet-proposing deferred acceptance algorithm corresponds to the descending salary adjustment process under this isomorphism. The "salary" corresponding to a contract (c_i, h, t) can be taken $1/t$ and the branches' utility functions can be taken to be quasi-linear (Proposition 2). A different choice of γ_a and γ_n would allow the salary to be taken to be $g(t)$ for any decreasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$.

II. Model: Matching with Contracts

I work with a model of many-to-one matching with contracts (Crawford and Knoer 1981, Kelso and Crawford 1982, Roth 1984a, Hatfield and Milgrom 2005, Hatfield and Kominers 2017). Let I be a set of cadets i and let B be a set of branches b,[11] so that $F = B \cup I$ is the set of agents f.[12] There is a fixed set of contracts X, and each contract $x \in X$ is between a cadet $\iota(x)$ and a branch $\beta(x)$. For all agents $f \in F$ and all sets of contracts $Y \subseteq X$, let

$$Y_f = \{x \in Y \mid \iota(x) = f \text{ or } \beta(x) = f\}$$

denote the set of contracts in Y that involve f. For ease of notation, I will not distinguish between singleton sets and their unique elements. A set of contracts $A \subseteq X$ is *unfeasible* if there exists a cadet i such that $|A_i| > 1$, and *feasible* otherwise.

Each cadet i has a strict preference order $\succ_i$ over $X_i \cup \{\varnothing\}$. Let $\succ = (\succ_i)_{i \in I}$ denote the cadets' preference profile. Given a set $Y \subseteq X$, let

$$C^i(Y) = \max_{\succ_i}\{Y \cup \{\varnothing\}\}.$$

Note that cadets can only choose at most one contract.

Each branch b has a choice function $C^b : \mathcal{P}(X_b) \to \mathcal{P}(X_b)$ satisfying $C^b(Y) \subseteq Y$ for all $Y \subseteq X$. Abusing notation, I extend C^b to $\mathcal{P}(X)$ by letting $C^b(Y) = C^b(Y_b)$. Let $C = (C^b)_{b \in B}$ denote the branches' priority profile. I allow branches to accept more than one contract with each cadet: if a branches' choice function only returns feasible sets, then I say that the branch's choice function is *feasible*.[13] In the applications to cadet-branch matching, the branches always have feasible choice functions.

[11] There are alternative terminologies "workers" and "firms" (Kelso and Crawford 1982, Roth 1984a) or "doctors" and "hospitals" (Roth 1984b, Hatfield and Milgrom 2005).

[12] The sets B and I are assumed to be disjoint.

[13] In contrast, Hatfield and Milgrom (2005) required that all branches have feasible choice functions. Hatfield and Kominers (2019) allowed branches to choose unfeasible sets in *completed* choice functions, and Hatfield and Kominers (2017) always allowed unfeasible choice functions.

A. *Conditions on Choice Functions*

A choice function C^b is *substitutable* (Kelso and Crawford 1982, Hatfield and Milgrom 2005) if $x \notin C^b(Y \cup \{x,y\})$ whenever $x \notin C^b(Y \cup \{x\})$. Substitutability requires that access to an additional contract y does not make b want a contract x more. A choice function C^b is *unilaterally substitutable* (Hatfield and Kojima 2010) if $x \notin C^b(Y \cup \{x,y\})$ whenever $x \notin C^b(Y \cup \{x\})$ and $\iota(x) \notin \iota(Y)$.

A choice function C^b satisfies the *law of aggregate demand* (Hatfield and Milgrom 2005) if $|C^b(Y)| \leq |C^b(Y')|$ whenever $Y \subseteq Y' \subseteq X$. The law of aggregate demand requires that b chooses weakly more contracts as the set of available contracts expands. A choice function C^b satisfies the *irrelevance of rejected contracts condition* (Aygün and Sönmez 2012, 2013) if $C^b(A) = C^b(A')$ whenever $C^b(A') \subseteq A \subseteq A'$. The irrelevance of rejected contracts condition requires that rejected contracts not affect b's choice set.

B. *Stability*

An *allocation* is a set of contracts $A \subseteq X$. An allocation $A \subseteq X$ is *individually rational* if $C^f(A) = A_f$ for all agents $f \in F$. A nonempty set $Z \subseteq X$ *blocks* an individually rational allocation $A \subseteq X$ if $Z \cap A = \varnothing$ and $Z_f \subseteq C^f(A_f \cup Z_f)$ for all agents $f \in F$. An allocation is *stable* if it is individually rational and unblocked.

C. *Mechanisms*

A *mechanism* is a function from the set of cadets' preference profiles to the set of feasible allocations, for fixed branches' choice functions. A mechanism $\mathcal{M}$ is *stable* if it returns stable outcomes. A mechanism $\mathcal{M}$ is *group strategy-proof* (*for cadets*) if for all $I' \subseteq I$ and all preference profiles $\hat{\succ} = (\hat{\succ}_i)_{i \in I'}$, there exists $i \in I'$ such that

$$\mathcal{M}(\succ)_i \succeq_i \mathcal{M}(\hat{\succ}, \succ_{I \setminus I'})_i.$$

The mechanism that I consider is the *deferred acceptance mechanism*, which returns the outcome of the deferred acceptance algorithm. I use a simultaneous-proposal deferred acceptance algorithm, following Gale and Shapley (1962), Crawford and Knoer (1981), Kelso and Crawford (1982), and Roth (1984a); and I always assume that cadets propose.[14] The algorithm proceeds iteratively as follows.[15]

Step 1: Each cadet proposes his most preferred contract to the corresponding branch. If no contracts are proposed, then terminate the algorithm.

Otherwise, each branch *holds* the set of contracts that it chooses from the proposed contracts. Each branch then rejects any proposed contract that is not held.

[14] Note that Crawford and Knoer (1981) and Kelso and Crawford (1982) considered branch-proposing deferred acceptance algorithms.

[15] For a formal definition of deferred acceptance, see Appendix A.

Step $t > 1$**:** Each cadet with whom no branch is holding a contract proposes his most preferred unrejected contract to the corresponding branch. If no contracts are proposed, then branches accept the contracts that they are holding and the algorithm is terminated.

Otherwise, each branch holds the set of contracts that it chooses from the proposed contracts and the previously held contracts. Each branch then rejects any proposed or previously held contracts that is not held.

Denote the deferred acceptance mechanism with respect to branch priority profile C by $\mathcal{DA}_C$.

III. Choice Functions That Induce the Equivalent Deferred Acceptance Mechanisms

This section derives a necessary and sufficient condition for two branch priority profiles to induce the same deferred acceptance mechanisms. In Section IV, I use this condition to compare deferred acceptance mechanisms for two families of branch priority profiles in cadet-branch matching.

Call a contract *available* at a step of deferred acceptance if it has been proposed but not rejected. Note that the set of available contracts is feasible at every step of deferred acceptance, because only cadets that are rejected are allowed to propose new contracts. Indeed, cadets only propose contracts after their previous proposals are rejected, so that each cadet has at most one active proposal at a time. Thus, the deferred acceptance algorithm only ever queries C^b on feasible sets of contracts. I say that two choice functions (or priority profiles) are DA-equivalent if they agree on all feasible sets of contracts, formally defined below.

DEFINITION 1: *Let b be a branch. A choice function $\hat{C}^b$ is DA-equivalent to C^b if $\hat{C}^b(Y) = C^b(Y)$ for all feasible sets $Y \subseteq X$. A branch priority profile $\hat{C}$ is DA-equivalent to C if $\hat{C}^b$ is DA-equivalent to C^b for all branches b.*

Deferred acceptance cannot distinguish between priority profiles that agree on all feasible sets of contracts, because the set of available contracts is feasible at every step of the deferred acceptance algorithm. As any feasible set of contracts can be the set of available contracts at Step 1 of the deferred acceptance algorithm, deferred acceptance can distinguish between any pair of priority profiles that are not DA-equivalent.

THEOREM 1: *A priority profile $\hat{C}$ is DA-equivalent to C if and only if $\mathcal{DA}_C = \mathcal{DA}_{\hat{C}}$.*

IV. Choice Functions for Cadet-Branch Matching

As an application of Theorem 1, this section explicitly realizes the main mechanisms proposed by Sönmez (2013) and Sönmez and Switzer (2013) as deferred acceptance mechanisms in matching markets with feasible, substitutable branch choice functions that satisfy the law of aggregate demand. Section IVA reviews Sönmez's (2013) priorities, and Section IVB presents the substitutable branch priorities. Section IVC discusses the practical advantages and disadvantages of the two approaches.

A. *Sönmez's* (2013) *Model of Cadet-Branch Matching*

In applications of matching with contracts to cadet-branch matching, additional structure is present in the branches' priorities and in the set of contracts. I consider a model of cadet-branch matching based on that of Sönmez (2013).

Each branch b has a strict priority order $\succ^b_{\mathrm{OML}}$ over $D \cup \{\varnothing\}$, called the *order of merit*. A cadet is *acceptable for b* if it is preferred to $\varnothing$ under $\succ^b_{\mathrm{OML}}$.[16] Each branch allows $\mathcal{K} \geq 1$ different contract lengths $0 < \mathcal{T}_1 < \cdots < \mathcal{T}_{\mathcal{K}}$,[17] and the set of contracts is

$$X = I \times B \times \{\mathcal{T}_1, \ldots, \mathcal{T}_{\mathcal{K}}\}.$$

An element of X is a contract (i,b,t) for cadet i to serve in branch b for t years. The functions ι and β are given by the projections onto the first and second factors, respectively.

Motivated by the Reserve Officers' Training Corps' (ROTC) existing matching mechanism, Sönmez (2013) defined bid for your career (BfYC) choice functions for cadet-branch matching. The USMA choice functions described by Sönmez and Switzer (2013) are the special cases of BfYC choice functions when $\mathcal{K} = 2$.

Each branch b has a capacity vector $(q^1_b, q^2_b) \in \mathbb{Z}^2_{\geq 0}$, where q^1_b is the number of contracts with high order-of-merit that b wants to hire and q^2_b represents the number of long contracts that b wants.[18] Fix a branch b and a set of contracts $Y \subseteq X$. The BfYC choice set is defined by selecting the shortest available contracts with the q^1_b cadets with contracts in Y that are most preferred under $\succ^b_{\mathrm{OML}}$; and the longest q^2_b contracts in Y with other cadets, where ties are broken (in the second step) by ordering the cadets according to $\succ^b_{\mathrm{OML}}$.

More precisely, the BfYC choice set is defined as follows. A contract x is *available* at some stage of the choice procedure if x has neither been *chosen* nor *removed from consideration* yet. Run the following iterative procedure to compute $C^b_{\mathrm{BfYC}}(Y)$.

Step 1: For $j = 1, 2, \ldots, q^1_b$:

- Step 1.j: If there are no available contracts in Y with acceptable cadets, then terminate the process. Otherwise, let i be the cadet with the highest priority under $\succ^b_{\mathrm{OML}}$ who has an available contract in Y. Choose the shortest available contract in Y with i and remove from consideration all other contracts with i.

[16] Sönmez (2013) and Sönmez and Switzer (2013) assumed that every cadet is acceptable to every branch.
[17] In Section I, I take $\mathcal{K} = 2$, $\mathcal{T}_1 = 5$, and $\mathcal{T}_2 = 8$.
[18] In Section I, I take $q^1_a = q^2_a = q^1_m = 1$ and $q^2_m = 0$.

Step 2: For $j = 1, 2, \ldots, q_b^2$:

- Step 2.j: If there are no available contracts in Y with acceptable cadets, then terminate the process. Otherwise, let t be the length of the longest available contract in Y. Let i be the cadet with the highest priority under $\succ_{\mathrm{OML}}^{b}$ who has an available contract in Y of length t. Choose contract (i, b, t) and remove from consideration all other contracts with i.

Note that the choice function C_{BfYC}^{b} is feasible by construction and let $\mathcal{DA}_{\mathrm{BfYC}} = \mathcal{DA}_{C_{\mathrm{BfYC}}}$ denote the corresponding deferred acceptance mechanism.

Unfortunately, the choice functions C_{BfYC}^{b} are not substitutable, as Sönmez (2013) and Sönmez and Switzer (2013) showed (see also Section I). However, the choice functions C_{BfYC}^{b} are unilaterally substitutable and satisfy the law of aggregate demand and the irrelevance of rejected contracts condition, as shown by Lemma 1 in Sönmez (2013). By Theorem 7 in Hatfield and Kojima (2010), it follows that the mechanism $\mathcal{DA}_{\mathrm{BfYC}}$ is group strategy-proof (Sönmez 2013, Sönmez and Switzer 2013).

B. *Substitutable BfYC Choice Functions*

This section defines new branch choice functions that are designed to select the longest available contract with a cadet at each step but be otherwise identical to the BfYC choice functions. More formally, I define *substitutable BfYC choice functions* for cadet-branch matching as follows. Fix a branch b and a set of contracts $Y \subseteq X_b$ and run the following iterative procedure to compute $C_{\mathrm{sBfYC}}^{b}(Y)$.

Step 1: For $j = 1, 2, \ldots, q_b^1$:

- Step 1.j: If there are no available contracts in Y with acceptable cadets, then terminate the process. Otherwise, let i be the cadet with the highest priority under $\succ_{\mathrm{OML}}^{b}$ who has an available contract in Y. Choose the *longest* available contract in Y with i and remove from consideration all other contracts with i.

Step 2: Run Step 2 of the process defining C_{BfYC}.

Note that the choice function C_{sBfYC}^{b} is feasible by construction and let $\mathcal{DA}_{\mathrm{sBfYC}} = \mathcal{DA}_{C_{\mathrm{sBfYC}}}$ denote the corresponding deferred acceptance mechanism.

Intuitively, the choice functions C_{BfYC}^{b} and C_{sBfYC}^{b} only differ in the relative priorities of contracts with individual cadets in the first step, and are identical in the second step. Thus, C_{BfYC}^{b} and C_{sBfYC}^{b} make the same trade-offs in sets of contracts between different cadets. More formally, C_{BfYC}^{b} and C_{sBfYC}^{b} are DA-equivalent, and therefore Theorem 1 guarantees that they induce the same deferred acceptance mechanism.

THEOREM 2: *The priority profiles C_{BfYC} and C_{sBfYC} are DA-equivalent, and thus we have that $\mathcal{DA}_{\mathrm{BfYC}} = \mathcal{DA}_{\mathrm{sBfYC}}$.*

The choice functions C^b_{sBfYC} give relatively more priority to long contracts than the choice functions C^b_{BfYC} do. Theorem 2 shows that the use of the cadet-proposing deferred acceptance mechanism prevents the branches from forcing longer contracts on cadets by giving higher priority to longer contracts.

One advantage of the choice functions C^b_{sBfYC} is that they are substitutable.

PROPOSITION 1: *The choice functions C^b_{sBfYC} are substitutable and satisfy the law of aggregate demand. In particular, $\mathcal{DA}_{\text{sBfYC}}$ is group strategy-proof.*

Intuitively, the substitutable BfYC choice functions consistently choose among the longest available contracts with each cadet. This consistency condition is precisely *Pareto separability* (in the sense of Hatfield and Kojima 2010), which implies substitutability when taken in conjunction with unilateral substitutability (Hatfield and Kojima 2010). To prove Proposition 1, I follow a more classical approach using a quasi-linear utility representation (Proposition 2) and properties of the choice function of a profit-maximizing firm that regards workers as substitutes (Hatfield and Milgrom 2005)—see Section V.[19]

Theorem 2 and Proposition 1 show that the main mechanisms proposed by Sönmez (2013) and Sönmez and Switzer (2013) are deferred acceptance mechanisms with respect to a profile of branch choice functions that are feasible, substitutable, and satisfy the law of aggregate demand. That is, matching theory with unilaterally substitutable choice functions (Hatfield and Kojima 2010) or unfeasible choice functions (Hatfield and Kominers 2017, 2019) is not needed for cadet-branch matching.[20] It also follows that the Sönmez mechanism $\mathcal{DA}_{\text{BfYC}}$ is group strategy-proof.[21] Indeed, Theorem 2 shows that the Sönmez mechanism $\mathcal{DA}_{\text{BfYC}}$ coincides with $\mathcal{DA}_{\text{sBfYC}}$, which is in turn group strategy-proof by Proposition 1.

COROLLARY 1 (Sönmez 2013, Sönmez and Switzer 2013): *$\mathcal{DA}_{\text{BfYC}}$ is group strategy-proof.*

C. *Practical Issues*

The BfYC priorities have the practical advantage of being as faithful to the currently implemented priorities as possible.[22] Specifically, the substitutable BfYC priorities differ from the BfYC priorities and the currently implemented priorities in how a branch treats the contract lengths for the first q^1_b cadets that it selects.

[19] It is possible to prove Proposition 1 directly using the properties of choice functions induced by lexicographic priorities. I prove Proposition 1 using a quasi-linear utility representation to illustrate the connection of cadet-branch matching with the job matching model of Kelso and Crawford (1982).

[20] Results on matching with weakened substitutability conditions (Hatfield and Kojima 2010, Hatfield and Kominers 2019) or unfeasible choice functions (Hatfield and Kominers 2012, 2017) are not even needed to show that $\mathcal{DA}_{\text{sBfYC}}$ is group strategy-proof—group strategy-proofness in Proposition 1 follows directly from the main result of Hatfield and Kojima (2009).

[21] Theorem 5 in Hatfield and Kojima (2010) and the unilateral substitutability of C^b_{BfYC} for all b (Sönmez 2013, Sönmez and Switzer 2013) ensure that the $\mathcal{DA}_{\text{BfYC}}$ coincides with the *cumulative offer mechanism* with respect to C_{BfYC}, which was the exact mechanism proposed by Sönmez (2013).

[22] In the case of the USMA, the currently implemented priorities are actually the special case of the BfYC priorities with two different contract lengths (Sönmez and Switzer 2013).

The substitutable BfYC priorities select the *longest* available contracts with the selected cadets, while the BfYC priorities and the currently implemented priorities select the *shortest* available contracts with the selected cadets. Theorem 2 implies that this difference in priorities does not affect the deferred acceptance mechanism. However, it might be difficult to persuade policymakers to change the treatment of contract length for the first few cadets to be selected at each branch, despite the fact that the choice does not affect the outcome of deferred acceptance.[23]

However, the substitutable BfYC priorities have the advantage of making deferred acceptance become interpretable as a descending salary adjustment process, as I show formally in Section V. This property may make deferred acceptance easier to explain to cadets if a mechanism based on deferred acceptance is adopted for cadet-branch matching. In light of Theorem 2, the salary adjustment process interpretation applies to deferred acceptance with BfYC priorities as well, although the use of the substitutable BfYC priorities makes the connection more transparent.

V. Contracts versus Salaries in Cadet-Branch Matching

In this section, I first show that the substitutable BfYC choice functions can be taken to be the choices of profit-maximizing firms. I then show that the cadet-branch economy with substitutable branch priorities is isomorphic to a Kelso-Crawford economy and that deferred acceptance corresponds to a descending salary adjustment process.

A. *Quasi-linearity of the Substitutable Branch Choice Functions*

This section shows that the substitutable BfYC choice functions are quasi-linear in "salary," where branches value cadets according to *assignment valuations* (Shapley 1962). I use assignment valuations to capture the fact that the branches have several different slots for cadets and the slots place different values on ranking in the order of merit.

Let branch b have $q_b^1 + q_b^2$ slots and let $\alpha^b \in \mathbb{R}^{I\times\{1,\ldots,q_b^1+q_b^2\}}$ be a matrix of *assignment values*, so that $\alpha_{i,j}^b$ is the value that branch b receives if cadet i is assigned to slot j in b. Following Shapley (1962), define valuation $\gamma_b : \mathcal{P}(I) \rightarrow \mathbb{R}$ to value (sets of) cadets by assigning cadets to slots in b in a way that maximizes total value. More formally, let

$$\gamma_b(E) = \max_{\substack{\{d_{i_1},\ldots,d_{i_k}\}\subseteq E \\ 1\leq j_1<\cdots<j_k\leq q_b^1+q_b^2}} \sum_{\ell=1}^{k} \alpha_{d_{i_\ell},j_\ell}^b.$$

[23] In other applications, it has proven difficult to get policymakers to change the priorities that are used in deferred acceptance. For example, it required substantial effort to eliminate "walk zones" in Boston despite their counterintuitive (and sometimes counterproductive) behavior (Dur et al. 2018).

Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a strictly decreasing function, which converts contract lengths to salaries. The requirement that g be decreasing ensures that short contracts correspond to high salaries.[24] Define utility function $u_b : \mathcal{P}(X) \rightarrow \mathbb{R}$ by

$$u_b(Y) = \gamma_b(\iota(Y_b)) - \sum_{(i,b,t) \in Y} g(t).$$

Thus, u_b values cadets according to γ_b and is quasi-linear in $g(\text{contract length})$.

Consider an assignment value matrix for b described as follows. Let any difference in values of doctors to the first q_b^1 slots dominate any difference in salaries. For the last q_b^2 slots, let any difference in salaries dominate any difference between values of cadets.[25] Intuitively, maximizing utility among subsets of a given set of contracts then selects up to q_b^1 contracts with preferred cadets and then up to q_b^2 more contracts that are as long as possible. Thus, maximizing u_b coincides with C^b_{BfYC} for suitably chosen assigment values.

PROPOSITION 2: *For all branches b and all strictly decreasing $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, there exists a matrix $\alpha^b \in \mathbb{R}^{I \times \{1, \ldots, q_b^1 + q_b^2\}}$ such that*

$$\left\{C^b_{\text{sBfYC}}(Y)\right\} = \arg\max_{Z \subseteq Y} u_b(Z)$$

for all sets of contracts $Y \subseteq X$.

Proposition 2 implies that the substitutable BfYC choice functions can be interpreted as the choice functions of profit-maximizing firms. In contrast, the original branch choice functions C^b_{BfYC} are not the choice functions of profit-maximizing firms. Intuitively, choosing a long contract with a given cadet sometimes and a short contract at other times when both are available is inconsistent with profit-maximization.

As I show in Appendix C, Proposition 1 follows immediately from the utility representation of Proposition 2 due to general results on quasi-linear utility functions induced by assignment valuations (Shapley 1962, Hatfield and Milgrom 2005).

B. *Isomorphism to a Kelso-Crawford Economy*

This section shows that, under mild conditions, the cadet-branch economy with substitutable BfYC priorities is *isomorphic* to a Kelso-Crawford economy.[26] More precisely, I take the salary corresponding to a contract (i, b, t) to be $g(t)$.

Proposition 2 shows that the branches' choice functions can be represented by quasi-linear utility functions. The cadets' preferences can be represented by utility

[24] Intuitively, short contracts correspond to high salaries because short contracts entail less service received by the military without change to the cost of educating a cadet.

[25] It is possible to choose such assignment values because the set of salaries—i.e., $\{g(\mathcal{T}_1), \ldots, g(\mathcal{T}_K)\}$—is finite. See Appendix C for the details.

[26] I formalize the notion of *isomorphism* in Appendix C.

functions. In order to ensure that cadets prefer high salaries, I need to assume that all cadets prefer short contracts.[27]

DEFINITION 2: *The preference of a cadet i is salary-monotonic if*

$$\left(i,b,t'\right) \notin C^i\left(\left\{\left(i,b,t\right),\left(i,b,t'\right)\right\}\right)$$

whenever $t < t'$ *and* $\left(i,b,t\right),\left(i,b,t'\right) \in X$.

In practice, cadets' preferences are likely to be salary-monotonic because cadets can choose to remain in the military after the expiry of their initial contracts. Under the assumption that cadets' preferences are salary-monotonic, their preferences can be represented by utility functions that are strictly increasing in salaries. It follows that the cadet-branch economy is isomorphic to a Kelso-Crawford economy.

THEOREM 3 (Informal statement): *If all cadets have salary-monotonic preferences, then*

(*i*) *The cadet-branch economy with substitutable BfYC choice functions is isomorphic to a Kelso-Crawford economy.*

(*ii*) *The Kelso-Crawford economy can be chosen so that so that* $g\left(t\right)$ *is the salary corresponding to contract* $\left(i,b,t\right)$ *for all* $\left(i,b,t\right) \in X$.

(*iii*) *The cadet-proposing deferred acceptance algorithm corresponds to the descending salary adjustment process under any isomorphism.*

See Appendix C for a formal statement and proof of Theorem 3.

In light of Theorems 2 and 3, the main mechanisms proposed by Sönmez (2013) and Sönmez and Switzer (2013) are descending salary adjustment processes in Kelso-Crawford economies. Therefore, cadet-branch matching does not require even the full generality of many-to-one matching with contracts and substitutable choice functions—only the Kelso-Crawford(1982) theory of many-to-one matching with salaries is needed to match cadets to branches.

C. *Related Literature on Contracts and Salaries*

Echenique (2012) showed that if all branches' choice functions are substitutatable, then a matching market with contracts can be embedded into a *matching market with salaries*. The Echenique (2012) embedding does not guarantee that branches' utility functions are quasi-linear. Moreover, the firms' utility functions cannot be quasi-linear in general. Indeed, Theorem 7 in Hatfield and Milgrom (2005) shows that the law of aggregate demand follows from substitutability and

[27] When cadets' preferences are salary-monotonic and the branches' priority profile is C_{sBfYC}, every contract is *Pareto optimal* in the sense of Roth (1984a) and the *generalized salary condition* of Roth (1985) is satisfied.

quasi-linearity. As the Echenique (2012) embedding preserves the law of aggregate demand (see Section IIE in Echenique 2012), the law of aggregate demand is necessary (but not in general sufficient) for the existence of a quasi-linear utility representation in matching with salaries. Schlegel (2015) showed that a matching market where branches' choice functions are unilaterally substitutable can be embedded (in a weaker sense than Echenique 2012) into a (potentially non-quasi-linear) matching market with salaries in which firms may be indifferent to paying a worker more.[28] This result applies in particular to the Sönmez (2013) cadet-branch market.

Proposition 2 and Theorem 3 offer salaries a more realistic interpretation than Echenique (2012) and Schlegel (2015) because the branches' utility functions are taken to be quasi-linear. Thus, the substitutable BfYC choice functions are the choices of profit-maximizing firms. Moreover, quasi-linearity is crucial to the proof that the substitutable BfYC choice functions satisfy the law of aggregate demand (Proposition 1). The law of aggregate demand is in turn critical to prove that the deferred acceptance mechanism is group strategy-proof (Hatfield and Kojima 2009, Hatfield and Kominers 2012) and to give a conceptual proof that the BfYC choice functions are unilaterally substitutable using Theorem 4. Thus, the interpretation of contract lengths as salaries offered by Theorem 3 is both conceptually appealing and technically useful.

VI. DA-Equivalence and Weakened Substitutability Conditions

This section studies the general implications of DA-equivalence to a substitutable choice function. Section VIA provides a conceptual explanation for the unilateral substitutability of the BfYC choice functions. Section VIB formalizes what it means for a choice function to be effectively substitutable from the perspective of deferred acceptance (DA-substitutable) and discusses the relationship between DA-substitutability and strategy-proofness results in the literature.

A. *DA-Equivalence and Unilateral Substitutability*

This section provides a conceptual proof of Sönmez's (2013) result that the BfYC choice functions are unilaterally substitutable and satisfy the law of aggregate demand (Lemmata 1 and 2 in Sönmez 2013). The proof relies on the following theorem, which asserts that feasibility and DA-equivalence to the choice function of a profit-maximizing firm together imply unilateral substitutability and the law of aggregate demand.

THEOREM 4: *Let $b \in B$ and let $\hat{C}^b$ be a choice function that is DA-equivalent to C^b. If C^b is feasible and satisfies the irrelevance of rejected contracts condition and $\hat{C}^b$ is feasible, substitutable, and satisfies the law of aggregate demand, then C^b is*

[28] That is, Schlegel (2015) did not require firms' utility functions to be strictly decreasing in salaries. This relaxed interpretation of salaries allows Schlegel (2015) to embed unilaterally substitutable priorities.

(*i*) *unilaterally substitutable and*

(*ii*) *satisfies the law of aggregate demand.*

In Section 1 of the online Appendix, I present examples to show that the hypotheses that $\hat{C}^b$ be feasible and satisfy the law of aggregate demand are necessary to both conclusions of Theorem 4. Intuitively, feasibility and the law of aggregate demand interact in Theorem 4 to constrain the number of different cadets that are chosen under $\hat{C}^b$, and hence under C^b as well.

Theorem 4 gives a conceptual explanation of why the BfYC choice functions are unilaterally substitutable: they induce the same deferred acceptance mechanisms as the substitutable BfYC choice functions (Theorem 2), which are the choice functions of profit-maximizing firms (Proposition 1). This unilateral substitutability condition is critical to Sönmez's (2013) and Sönmez and Switzer's (2013) approach to deriving stability and strategy-proofness. Previously, only a technical justification of this crucial condition was known (Sönmez 2013, Sönmez and Switzer 2013). Thus, the approach of constructing the Sönmez and Sönmez-Switzer mechanisms in Kelso-Crawford economies also helps shed light on the substitutability conditions involved in the Sönmez and Sönmez-Switzer models.

COROLLARY 2 (Sönmez 2013, Sönmez and Switzer 2013): *In cadet-branch matching, the choice function C^b_{BfYC} is unilaterally substitutable for all $b \in B$.*

REMARK 1: *Example* 4 *in the online Appendix shows that unilateral substitutability and the law of aggregate demand do not together imply DA-equivalence to a feasible, substitutable choice function. Thus, the existence of feasible, substitutable priorities that are DA-equivalent to the BfYC choice functions relies on additional structure present in cadet-branch matching beyond unilateral substitutability.*

B. *Substitutability from the Perspective of Deferred Acceptance*

As the example of cadet-branch matching shows, choice functions that exhibit complementarities may still be DA-equivalent to substitutable choice functions. I call such choice functions *DA-substitutable*.

DEFINITION 3: *A choice function C^b is DA-substitutable if there exists a choice function $\hat{C}^b$ that is DA-equivalent to C^b and substitutable.*

Substitutability alone is not sufficient to guarantee that the deferred acceptance mechanism is strategy-proof (Hatfield and Milgrom 2005)—the law of aggregate demand also plays a key role in deriving strategy-proofness (Hatfield and Milgrom 2005; Hatfield, Kominers, and Westkamp 2019). Theorem 4 also illustrates an important interaction between DA-equivalence and the law of aggregate demand. This motivates the consideration of *DA-strategy-proof* choice functions, which are defined to be the choice functions that are DA-equivalent to choice functions that are substitutable and satisfies the law of aggregate demand.

DEFINITION 4: *A choice function C^b is DA-strategy-proof if there exists a choice function $\hat{C}^b$ that is DA-equivalent to C^b, substitutable, and satisfies the law of aggregate demand.*

Recall that substitutable choice functions that satisfy the law of aggregate demand induce group strategy-proof deferred acceptance mechanisms (Hatfield and Kominers 2012). As DA-strategy-proof choice functions induce the same deferred acceptance mechanisms as certain substitutable choice functions that satisfy the law of aggregate demand, DA-strategy-proof choice functions induce group strategy-proof deferred acceptance mechanisms as well.

THEOREM 5: *If C^b is DA-strategy-proof for all $b \in B$, then $\mathcal{DA}_C$ is group strategy-proof.*[29]

PROOF:

For each b, let $\hat{C}^b$ be a choice function that is DA-equivalent to C^b, substitutable, and satisfies the law of aggregate demand. Theorem 1 implies that $\mathcal{DA}_C = \mathcal{DA}_{\hat{C}}$, while Theorem 10 in Hatfield and Kominers (2012) guarantees that $\mathcal{DA}_{\hat{C}}$ is group strategy-proof. ■

As was shown in Section IV, the branches' choice functions in cadet-branch matching satisfy a stronger condition than DA-strategy-proofness: the BfYC choice functions are DA-equivalent to *feasible*, substitutable choice functions that satisfy the law of aggregate demand (see Theorem 2 and Proposition 1). Feasibility plays a crucial role in deriving unilateral substitutability—as was seen in Section VIA—and is necessary to embed a matching market into a Kelso-Crawford economy—as will be done in Section V. Furthermore, the strategy-proofness results in cadet-branch matching rely only on strategy-proofness results in many-to-one matching with feasible, substitutable choice functions (Hatfield and Kojima 2009), while the proof of Theorem 5 uses results from many-to-many matching with contracts (Hatfield and Kominers 2012).

Unlike other weakened substitutability conditions in the literature (see, e.g., Hatfield and Kojima 2010; Hatfield and Kominers 2016; and Hatfield, Kominers, and Westkamp 2019), DA-substitutability and DA-strategy-proofness have interpretations in terms of being effectively substitutable from the perspective of deferred acceptance. As Theorem 4 shows, ideas similar to DA-substitutability also help provide intuition for unilateral substitutability.

DA-substitutability and DA-strategy-proofness relate to some of the weakened substitutability conditions in the matching literature. Theorem 4 shows that a strengthening of DA-strategy-proofness implies unilateral substitutability. However, Section 2 in the online Appendix explains that unilateral substitutability

[29] DA-strategy-proofness is not in general sufficient to ensure that $\mathcal{DA}_C$ is stable even if all branches' choice functions satisfy the irrelevance of rejected contracts condition, as Example 2 in the online Appendix shows. However, observable substitutability (in the sense of Hatfield, Kominers, and Westkamp 2019) and the irrelevance of rejected contracts condition together imply that the deferred acceptance mechanism is stable.

and *substitutable completability* (in the sense of Hatfield and Kominers 2015) imply DA-substitutability, but not vice versa. Similarly, the existence of a substitutable completion that satisfies the law of aggregate demand implies DA-strategy-proofness, but not vice versa.

VII. Extension to Slot-Specific Priorities

The DA-equivalence of Theorem 2 extends to the setting of *slot-specific priorities* (Kominers and Sönmez 2016), a class of choice functions that generalizes the branches' choice functions in cadet-branch matching. The choice functions associated to slot-specific priorities are defined by iterative processes that generalize the definitions of C^b_{BfYC} and C^b_{sBfYC}. For these choice functions, changing the relative priorities of contracts with individual cadets at each *slot* (sub-step) yields a DA-equivalent choice function. However, unlike in the case of Theorem 2, the modified choice function may be neither feasible nor substitutable. See Proposition B.1 in Appendix B for the details.[30,31]

VIII. Conclusion

Because military positions are jobs, it is natural to regard the cadet-branch matching market as a job market. This approach requires changes to the branches' choice functions, but Theorem 2 shows that the proposed modification does not affect the deferred acceptance mechanisms. Theorem 3 shows that the proposed matching mechanisms are simpler from the job-market viewpoint and clarifies the role of contracts as salaries. Along the way, Theorem 4 shows that the BfYC choice functions are unilaterally substitutable precisely because the substitutable BfYC choice functions are substitutable and consistent with profit-maximization.

Substitutability is crucial to matching with contracts (Hatfield and Kojima 2008; Hatfield and Kominers 2012, 2017; Hatfield et al. 2013; and Schlegel 2019). This paper shows that even choice functions that exhibit complementarities might be effectively substitutable from the perspective of matching mechanisms (DA-substitutable). Thus, reformulating a matching problem by modifying the priorities can simplify the analysis and clarify the roles of contracts, priority structures, and substitutability conditions.

Appendix A: Proofs of General Results on DA-Equivalence

A. *Proof of Theorem 1*

Formal Definition of the Deferred Acceptance Algorithm.—I use a simultaneous-proposal cadet-proposing deferred acceptance algorithm, following Gale and

[30] I use Proposition B.1 in Appendix B to prove Theorem 2 formally (see Appendix C).

[31] The same intuition extends to choice functions described by *capacity transfers* (Aygün and Turhan 2019), a class of priorities that generalizes distributional priorities (Kamada and Kojima 2015) and slot-specific priorities (Kominers and Sönmez 2016).

Shapley (1962). Initialize the set of "held" contracts to $G^{(0)} = \varnothing$ and the set of "unproposed" contracts to $J^{(0)} = X$. For $t = 1, 2, \ldots$, run the following iterative step.

Step t: Define the set of participating cadets to be

$$W^{(t)} = \left\{ i \in I \middle| J_i^{(t-1)} \text{ contains a contract that is acceptable to } d \right\} \backslash \iota\left(G^{(t-1)}\right).$$

If $W^{(t)} = \varnothing$, then terminate the process, set $T = t - 1$, and return $G^{(t-1)}$.

Otherwise, for all $d \in W^{(t)}$, let $x_i^{(t)}$ be the contract in $J_i^{(t-1)}$ that is most preferred by i. Each cadet $d \in W^{(t)}$ proposes $x_i^{(t)}$, so that the set of proposed contracts is

$$P^{(t)} = \left\{ x_i^{(t)} | d \in W^{(t)} \right\}.$$

The new set of held contracts is

$$G^{(t)} = \bigcup_{b \in B} C^b\left(G^{(t-1)} \cup P^{(t)}\right),$$

and the new set of unproposed contracts is

$$J^{(t)} = J^{(t-1)} \backslash P^{(t)}.$$

It is straightforward to the verify that the algorithm terminates in $T \leq |X|$ steps, because at least one contract is proposed at each step and no contract is proposed more than once.

Preliminaries.—The following claim formalizes the intuition that the set of available contracts is feasible at every step of deferred acceptance.

CLAIM A.1: *For all $1 \leq t \leq T$, the sets $G^{(t)}$ and $G^{(t-1)} \cup P^{(t)}$ are feasible. Here, we set $G^{(-1)} = P^{(0)} = \varnothing$.*

PROOF:

We prove the claim by induction on t. The base case of $t = 0$ is obvious.

Assume that the claim is true for $t = k$. If $T = k$, then there is nothing left to prove. Therefore, we can assume that $T > k$. By construction, the set $P^{(k+1)}$ is feasible and $P_i^{(k+1)} = \varnothing$ for all $i \in \iota\left(G^{(k)}\right)$. It therefore follows from the inductive hypothesis that $G^{(k)} \cup P^{(k+1)}$ is feasible. Because

$$G^{(k+1)} = \bigcup_{b \in B} C^b\left(G^{(k)} \cup P^{(k+1)}\right) \subseteq G^{(k)} \cup P^{(k+1)},$$

the set $G^{(k+1)}$ is also feasible, completing the proof of the inductive step. ∎

The following claim summarizes the key inductive argument in the proof of the "only if" direction of Theorem 1.

CLAIM A.2: *Let $\hat{C}$ be a branch priority profile that is DA-equivalent to C. Denote the analogues of the sets $G^{(t)}, J^{(t)}, W^{(t)}$, and $P^{(t)}$, and the integer T in the deferred acceptance algorithm with branch priority profile $\hat{C}$ by $\hat{G}^{(t)}, \hat{J}^{(t)}, \hat{W}^{(t)}, \hat{P}^{(t)}$, and $\hat{T}$, respectively. For all $0 \leq t \leq T$, we have $t \leq \hat{T}$ and $\left(\hat{G}^{(t)}, \hat{J}^{(t)}\right) = \left(G^{(t)}, J^{(t)}\right)$. Here, we set $W^{(0)} = \hat{W}^{(0)} = P^{(0)} = \hat{P}^{(0)} = \varnothing$.*

PROOF:

We proceed by induction on t. The base case of $t = 0$ is obvious.

Assume that the claim is true for $t = k$. If $T = k$, then there is nothing left to prove. Therefore, we can assume that $T > k$. The inductive hypothesis ensures that $\left(\hat{G}^{(k)}, \hat{J}^{(k)}\right) = \left(G^{(k)}, J^{(k)}\right)$. The formula for $W^{(k+1)}$ implies that $W^{(k+1)} = \hat{W}^{(k+1)}$. Because $T \geq k+1$, we have $W^{(k+1)} \neq \varnothing$ and therefore $\hat{W}^{(k+1)} \neq \varnothing$. As a result, the definition of $\hat{T}$ ensures that $\hat{T} \geq k+1$.

Because $W^{(k+1)} = \hat{W}^{(k+1)}$ and $J^{(k)} = \hat{J}^{(k)}$, we have that $P^{(k+1)} = \hat{P}^{(k+1)}$. It follows that

$$\hat{J}^{(k+1)} = \hat{J}^{(k)} \backslash \hat{P}^{(k+1)} = J^{(k)} \backslash P^{(k+1)} = J^{(k+1)}.$$

Because $\hat{G}^{(k)} = G^{(k)}$ (by the inductive hypothesis), we have that $\hat{G}^{(k)} \cup \hat{P}^{(k+1)} = G^{(k)} \cup P^{(k+1)}$. Claim A.1 guarantees that $G^{(k)} \cup P^{(k+1)}$ is feasible. Because $\hat{C}$ is DA-equivalent to C, it follows that

$$\begin{aligned} G^{(k+1)} &= \bigcup_{b \in B} \hat{C}^b\left(\hat{G}^{(k)} \cup \hat{P}^{(k+1)}\right) = \bigcup_{b \in B} \hat{C}^b\left(G^{(k)} \cup P^{(k+1)}\right) \\ &= \bigcup_{b \in B} C^b\left(G^{(k)} \cup P^{(k+1)}\right) = G^{(k+1)}, \end{aligned}$$

completing the proof of the inductive step. ∎

Completion of the Proof of Theorem 1.—We first prove the "only if" direction. Assume that $\hat{C}$ is DA-equivalent to C, fix a preference profile for cadets, and work in the notation of Claim A.2. Claim A.2 guarantees that $T \leq \hat{T}$ and, by symmetry, that $\hat{T} \leq T$. Thus, we have that $T = \hat{T}$. Claim A.2 also guarantees that

$$\hat{G}^{(\hat{T})} = \hat{G}^{(T)} = G^{(T)},$$

and therefore that the deferred acceptance algorithm with respect to $\hat{C}$ returns the same allocation as the deferred acceptance with respect to C. It follows that $\mathcal{DA}_C = \mathcal{DA}_{\hat{C}}$.

It remains to prove the "if" direction. Suppose that $\hat{C}$ is not DA-equivalent to C. Then, there exists a branch b and a feasible set $Y \subseteq X_b$ such that $C^b(Y) \neq \hat{C}^b(Y)$. Consider the cadet preference profile defined by $i: Y_i \succeq \varnothing$. The deferred acceptance algorithm returns $\hat{C}^b(Y)$ when the branch priority profile is $\hat{C}$ and returns $C^b(Y)$ when the branch priority profile is C. As a result, we can conclude that $\mathcal{DA}_C \neq \mathcal{DA}_{\hat{C}}$.

The contrapositive of the previous paragraph proves the "if" direction of Theorem 1. ∎

B. *Proof of Theorem 4*

Let $Y_1 \subseteq Y_2 \subseteq X_b$ be sets of contracts. Let $W_j = C^b(Y_j)$ and let $I_j = \iota(W_j)$ for $j = 1,2$. Because C^b is feasible, the set W_j is feasible for $j = 1,2$. Thus, we have that $\hat{C}^b(W_j) = W_j$ for $j = 1,2$.

We claim that

$$\text{(A1)} \qquad \hat{C}^b\left(W_j \cup \{z\}\right) = W_j \quad \text{for all } z \in Y_j \backslash (Y_j)_{I_j} \text{ and } j = 1,2.$$

Note that $W_j \subseteq W_j \cup \{z\} \subseteq Y_j$ for all $z \in Y_j \backslash (Y_j)_{I_j}$. Because C^b satisfies the irrelevance of rejected contracts condition, it follows that $C^b\left(W_j \cup \{z\}\right) = W_j$. Because W_j is feasible and $\iota(z) \notin I_j$, the set $W_j \cup \{z\}$ is feasible. Hence, we have that $\hat{C}^b\left(W_j \cup \{z\}\right) = W_j$ since C^b and $\hat{C}^b$ are DA-equivalent.

Because $\hat{C}^b$ is substitutable, (A1) implies that $\hat{C}^b(Y_2)_{I \backslash I_j} = \varnothing$. Since $\hat{C}^b$ was assumed to be feasible, we have that $|\hat{C}^b(Y_j)| \leq |I_j| = |W_j|$, with equality if and only if $\iota\left(\hat{C}^b(Y_j)\right) = I_j$. As $\hat{C}^b(W_j) = W_j$ and $W_j \subseteq Y_j$, the law of aggregate demand for $\hat{C}^b$ yields that

$$|W_j| = \left|\hat{C}^b(W_j)\right| \leq |\hat{C}^b(Y_j)| \leq |W_j|,$$

so the two inequalities must be equalities. Thus, we have that $\iota\left(\hat{C}^b(Y_2)\right) = I_2$ and thus $|\hat{C}^b(Y_2)| = |W_2|$.

Proof of Theorem 4(i).—Specialize to the case of $Y_2 = Y_1 \cup \{y\}$ and let $x \in Y_1$ satisfy $\left|(Y_1)_{\iota(x)}\right| = 1$. Suppose that $x \in W_2$. We divide into cases based on whether $\iota(x) = \iota(y)$ to prove that $x \in W_1$.

Case 1: $\iota(x) = \iota(y)$. Because C^b is feasible, we have $y \notin C^b(Y_2)$. By the irrelevance of rejected contracts, it follows that $x \in C^b(Y_2) = C^b(Y_1)$.

Case 2: $\iota(x) \neq \iota(y)$. In this case, note that $(Y_2)_{\iota(x)} = \{x\}$. Because $\iota\left(\hat{C}^b(Y_2)\right) = I_2$ and $x \in W_2$, it follows that $x \in \hat{C}^b(Y_2)$. As $\hat{C}^b$ is substitutable, we have $x \in \hat{C}^b\left(W_1 \cup \{x\}\right)$. The contrapositive of (A1) applied to $j = 1$ and $z = x$ guarantees that $\iota(x) \in I_1$. Because $(Y_2)_{\iota(x)} = \{x\}$, it follows that $x \in W_1$.

The casework clearly exhausts all possibilities and thus proves that $x \in W_1$. Taking $Y_1 = Y \cup \{x\}$ with $\iota(x) \notin \iota(Y)$ arbitrary yields that C^b is unilaterally substitutable.

Proof of Theorem 4(ii).—As $\hat{C}^b(W_1) = W_1$ and $W_1 \subseteq Y_2$, the law of aggregate demand for $\hat{C}^b$ yields that

$$|W_1| = \left|\hat{C}^b(W_1)\right| \leq |\hat{C}^b(Y_2)| \leq |W_2|.$$

Since $Y_1 \subseteq Y_2 \subseteq X_b$ were arbitrary, C^b must satisfy the law of aggregate demand. ∎

Appendix B: DA-Equivalence and Slot-Specific Priorities

This section describes a sufficient condition for two *slot-specific priorities* (in the sense of Kominers and Sönmez 2016) to be DA-equivalent. I begin by giving necessary and sufficient conditions for two unit-demand choice functions to be DA-equivalent and then generalize to slot-specific priorities, which are obtained by combining unit-demand "slot" priorities. In Appendix C, I use the sufficient condition to derive Theorem 2.

A. *Case of Unit-Demand Choice Functions*

Two unit demand choice functions are DA-equivalent if they make the same comparisons between contracts with different cadets. Intuitively, this occurs if and only if the preferences differ only by permuting consecutive contracts with a single cadet, as swapping the order of acceptable contracts with different cadets alters a trade-off between contracts with different cadets.

I formalize this intuition in the following lemma. Let $C_{\succ_j^b}$ denote the unit-demand choice function associated to a total order $\succ_j^b$ on $X_b \cup \{\varnothing\}$.

LEMMA B.1: *Let $\succ_j^b$ and $\hat{\succ}_j^b$ be priority orders on $X_b \cup \{\varnothing\}$. The following are equivalent:*

(*i*) *The choice functions $C_{\succ_j^b}$ and $C_{\hat{\succ}_j^b}$ are DA-equivalent.*

(*ii*) *We have that*

$$x \succ_j^b \varnothing \Leftrightarrow x \mathrel{\hat{\succ}_j^b} \varnothing$$

for all $x \in X_b$ and that

$$x \succ_j^b x' \quad and \quad x' \mathrel{\hat{\succ}_j^b} x \Rightarrow \iota(x) = \iota(x')$$

for all $x, x' \in X_b$ such that $x' \succ_j^b \varnothing$.

PROOF:

First, assume that Condition (i) is satisfied—that is, suppose that $C_{\succ_j^b}$ is DA-equivalent to $C_{\hat{\succ}_j^b}$. For all $x \in X_b$, we have that

$$x \succ_j^b \varnothing \Leftrightarrow C_{\succ_j^b}(\{x\}) = \{x\} \Leftrightarrow C_{\hat{\succ}_j^b}(\{x\}) = \{x\} \Leftrightarrow x \mathrel{\hat{\succ}_j^b} \varnothing$$

because $\{x\}$ is feasible. For all $x,x' \in X_b$ with $\iota(x) \neq \iota(x')$ and $x' \succ_j^b \varnothing$, we have that

$$x \succ_j^b \varnothing \Leftrightarrow C_{\succ_j^b}(\{x,x'\}) = \{x\} \Leftrightarrow C_{\hat{\succ}_j^b}(\{x,x'\}) = \{x\} \Leftrightarrow x \mathrel{\hat{\succ}_j^b} x'$$

because $\{x,x'\}$ is feasible. Therefore, Condition (ii) is satisfied.

Next, suppose that Condition (ii) is satisfied. Let $Y \subseteq X$ be a feasible set of contracts. Define a set

$$W = \{x \in Y | x \succ_j^b \varnothing\} = \{x \in Y | x \mathrel{\hat{\succ}_j^b} \varnothing\}.$$

Note that W is feasible. Condition (ii) and the assumption that W is feasible ensure that the restriction of $\succ_j^b$ to W is the same as the restriction of $\hat{\succ}_j^b$ to W. Therefore, we have that

$$C_{\succ_j^b}(Y) = C_{\succ_j^b}(W) = C_{\hat{\succ}_j^b}(W) = C_{\hat{\succ}_j^b}(Y).$$

Because Y was an arbitrary feasible set of contracts, it follows that $C_{\succ_j^b}$ is DA-equivalent to $C_{\hat{\succ}_j^b}$, which is Condition (i). ∎

B. *Extension to Slot-Specific Priorities*

Kominers and Sönmez (2016) defined a special class of choice functions for branches called the *choice functions associated to slot-specific priorities*. Let b be a branch and let $\succ^b = (\succ_i^b)_{i \leq k}$ be a profile of k total orders on $X_b \cup \{\varnothing\}$. The *choice function associated to slot-specific priority with slot priorities* $\succ^b$ is the choice function $C_{\succ^b}$ defined as follows. Fix a set $Y \subseteq X_b$, and run the following procedure for $1 \leq t \leq k$ to compute $C_{\succ^b}(Y)$.

Step t: If no available contract in Y is preferred under $\succ_t^b$ to $\varnothing$, then proceed to the next step. Otherwise, accept the available contract $x \in Y$ that is most preferred under $\succ_t^b$ and remove from consideration all other contracts with $\iota(x)$.

I now prove a sufficient condition for the DA-equivalence of the choice functions associated to two sequences of slot priorities.

PROPOSITION B.1: *Let b be a branch and let $\succ^b$ and $\hat{\succ}^b$ be profiles of k total orders on $X_b \cup \{\varnothing\}$. If $C_{\succ_j^b}$ and $C_{\hat{\succ}_j^b}$ are DA-equivalent for all $1 \leq j \leq k$, then $C_{\succ^b}$ and $C_{\hat{\succ}^b}$ are DA-equivalent.*

Intuitively, the slot priorities $\succ^b$ and $\hat{\succ}^b$ only differ in their trade-offs between contracts with individual cadets. As a result, only comparisons between contracts with individual cadets differ at each step of the computation of the corresponding slot-specific priorities. Since only comparisons between contracts with individual cadets differ at each step of the computation, only trade-offs between contracts with individual cadets differ between $C_{\succ^b}$ and $C_{\hat{\succ}^b}$. Thus, the choice functions $C_{\succ^b}$ and $C_{\hat{\succ}^b}$ are DA-equivalent.

PROOF:

Let $Y \subseteq X$ be a feasible set of contracts. We prove by induction on t that the first t steps of the computation of $C_{\succ^b}(Y)$ and $C_{\hat{\succ}^b}(Y)$ agree. The base case of $t \leq 0$ is obvious. Assume that the first m steps of the computations of $C_{\succ^b}(Y)$ and $C_{\hat{\succ}^b}(Y)$ agree, with $m < k$. Let $A \subseteq Y$ be the set of contracts that are available at Step $m+1$ in the computation of $C_{\succ^b}(Y)$. The inductive hypothesis guarantees that A is also the set of contracts that are available at Step $m+1$ in the computation of $C_{\hat{\succ}^b}(Y)$. Because $\succ^b_{m+1}$ is DA-equivalent to $\hat{\succ}^b_{m+1}$, we have that

$$C_{\succ^b_{m+1}}(A) = C_{\hat{\succ}^b_{m+1}}(A),$$

Hence, the computations of $C_{\succ^b}(Y)$ and $C_{\hat{\succ}^b}(Y)$ agree at Step $m+1$ as well, completing the proof of the inductive step.

Taking $t = k$ yields that $C_{\succ^b}(Y) = C_{\hat{\succ}^b}(Y)$. Because Y was an arbitrary feasible set of contracts, it follows that $C_{\succ^b}$ is DA-equivalent to $C_{\hat{\succ}^b}$. ∎

Theorem 1 and Proposition B.1 show that permuting consecutive sequences of contracts with a single cadet in slot priorities does not affect the deferred acceptance mechanism.

Appendix C: Proofs of Results on Cadet-Branch Matching

A. *Proof of Theorem 2*

Notice that the choice functions C^b_{BfYC} are slot-specific with $q^1_b + q^2_b$ slots, where each sub-step corresponds to a slot (Sönmez and Switzer 2013, Kominers and Sönmez 2016). Similarly, the substitutable choice functions C^b_{sBfYC} are associated to slot-specific priorities with $q^1_b + q^2_b$ slots, where each sub-step corresponds to a slot.

Note that the slot priorities in the first steps of the processes defining C^b_{sBfYC} and C^b_{BfYC} differ only in the relative orders of contracts with a given cadet. It follows from Lemma B.1 and Proposition B.1 that C_{sBfYC} is DA-equivalent to C_{BfYC}. Theorem 1 implies that $\mathcal{DA}_{\text{BfYC}} = \mathcal{DA}_{\text{sBfYC}}$. ∎

B. *Proof of Proposition 1*

Fix a branch b. Theorem 1 in Shapley (1962) guarantees that u_b is a grossly substitutable valuation (see also Theorem 13 in Hatfield and Milgrom 2005).

Theorem 2 in Hatfield and Milgrom (2005) and Proposition 2 show that C^b_{sBfYC} is substitutable. Because u_b is quasi-linear and induces a substitutable choice function C_{sBfYC}, Theorem 7 in Hatfield and Milgrom (2005) guarantees that C^b_{sBfYC} satisfies the law of aggregate demand.

The second part of the proposition follows from the first part due to Theorem 1 in Hatfield and Kojima (2009), which asserts that the deferred acceptance mechanism is group strategy-proof if all branches' choice functions are feasible, substitutable, and satisfy the law of aggregate demand.[32] ∎

C. *Proof of Proposition 2*

Fix a branch b. To prove Proposition 2, we begin by defining a matrix α^b of assignment values. We then prove several properties of value-maximizing assignments. We use these properties to show that the BfYC choice is the only possible value-maximizing assignment, and it is straightforward to conclude the proof from this observation.

Definition of the Assignment Value Matrix.—In order to define the assignment value matrix, we need to define a "small" quantity δ and a "large" quantity Δ. Let $\delta \in \mathbb{R}^+$ be such that

$$\delta < \inf_{1 \le \ell < \ell' \le K} \frac{1}{\mathcal{T}_\ell} - \frac{1}{\mathcal{T}_{\ell'}},$$

and let $\Delta \in \mathbb{R}^+$ be such that

$$\Delta > \frac{1}{\mathcal{T}_1}.$$

Any difference in contract inverse-lengths dominates a value difference of δ, while a value difference of Δ dominates any difference in contract inverse-lengths.[33]

The assignment values are formally defined as follows. Each branch b has $q_b^1 + q_b^2$ slots. Let

$$c_1 \succ^b_{\text{OML}} c_2 \succ^b_{\text{OML}} \cdots \succ^b_{\text{OML}} dc_M$$

be the set of cadets that are acceptable to a branch b in order of merit. For $i \in I$ and $1 \le j \le q_b^1 + q_b^2$, define the value of i to b in slot j as

$$\alpha^b_{i,j} = \begin{cases} (M+2-k)\Delta & \text{if } i = c_k \text{ and } j \le q_b^1 \\ \Delta + \frac{\delta}{k} & \text{if } i = c_k \text{ and } q_b^1 < j \le q_b^1 + q_b^2. \\ 0 & \text{if } \varnothing \succ^b_{\text{OML}} d \end{cases}$$

[32] Aygün and Sönmez (2012, 2013) showed that the irrelevance of rejected condition is crucial to the stability and strategy-proofness of deferred acceptance. However, as Aygün and Sönmez (2012, 2013) showed, substitutability and the law of aggregate demand together imply the *strong axiom of revealed preferences*, which in turn implies the irrelevance of rejected contracts condition. The fact that the substitutable BfYC choice functions satisfy the strong axiom of revealed preferences can easily be seen directly from Proposition 2.

[33] In Section 1, I take $\delta = 1/100$ and $\Delta = 1$.

The first q_b^1 slots give strong priority to cadets that are high on the order of merit, while the next q_b^2 slots give a slight priority to cadets that are high on the order of merit. All slots strongly dis-prefer unacceptable cadets. The remainder of this section is devoted to proving that

$$\arg\max_{Z \subseteq Y} u_b(Z) = \left\{C^b(Y)\right\}$$

for all $Y \subseteq X_b$.

Basic Properties of Optimal Assignments.—Note that $\alpha_{i,j}^b \geq \alpha_{i,k}^b$ for all $1 \leq j \leq k \leq q_b^1 + q_b^2$ and all cadets i. Therefore, we can assume that only the first $\min\left\{|E|, q_b^1 + q_b^2\right\}$ slots are used in an optimal assignment of a set of cadets in E to slots—we have that

$$\gamma_b(E) = \max_{\left\{i_1, i_2, \ldots, i_{m(E)}\right\} \subseteq A} \sum_{j=1}^{m(E)} \alpha_{i_j,j}^b,$$

where $m(E) = \min\left\{|E|, q_b^1 + q_b^2\right\}$. Note that whenever $i \succ_{\mathrm{OML}}^b i'$, we have that $\alpha_{i,j}^b > \alpha_{i',j}^b$ for all $1 \leq j \leq q_b^1 + q_b^2$ and $\alpha_{i,j}^b + \alpha_{i',j'}^b \geq \alpha_{i,j}^b + \alpha_{i',i}^b$ for all $1 \leq j < j' \leq q_b^1 + q_b^2$. As a result, we have that

$$\gamma_b(E) = \sum_{j=1}^{m(E)} \alpha_{i_j,j}^b,$$

where

$$E = \left\{i_1 \succ_{\mathrm{OML}}^b i_2 \succ_{\mathrm{OML}}^b \cdots \succ_{\mathrm{OML}}^b i_{|E|}\right\}.$$

Let $I_b = \left\{i \in I \,|\, d \succ_{\mathrm{OML}}^b \varnothing\right\}$ and define $f: I_b \to \left\{1, \ldots, M\right\}$ by $f(d_k) = k$. The discussion of the previous paragraph and the explicit definition of the assignment values ensure that

$$\gamma_b(E) = \min\left\{m'(E), q_b^1 + q_b^2\right\}\Delta + \sum_{t=1}^{\min\left\{m'(E), q_b^1\right\}} \left(M + 1 - k_t(E)\right)\Delta + \sum_{t=q_b^1}^{\min\left\{m'(E), q_b^1 + q_b^2\right\}} \frac{\delta}{k_t(E)},$$

where $m'(E) = |E \cap I_b|$ and

$$f(E \cap I_b) = \left\{k_1(E) < \cdots < k_{m'(E)}(E)\right\}.$$

We use this formula for γ_b implicitly during the remainder of the proof of the proposition.

Proof That Any Maximizer of u_b Must Be the BfYC Choice Set.—Let $Y \subseteq X$ and let $x_t \in Y \cup \{\varnothing\}$ be selected in the tth sub-step of the process defining C^b_{sBfYC}. Suppose that $A \subseteq Y$ and that $A \neq C^b_{\text{sBfYC}}(Y)$. We claim that there exists $A' \subseteq Y$ such that $u_b(A') > u_b(A)$. First, we show that we can make three simplifying assumptions.

(A) $A \subseteq Y_b$. Indeed, note that $u_b(A) \leq u_b(A \cap Y_b)$ with equality if and only if $A \subseteq Y_b$.

(B) A is feasible. Let $A' \subseteq A$ be such that $\iota(A'_b) = \iota(A_b)$ and A' is feasible. Then, $\gamma_b(\iota(A'_b)) = \gamma_b(\iota(A_b))$, so $u_b(A') \geq u_b(A)$ with equality if and only if $A = A'$.

(C) There do not exist $(i, b, \mathcal{T}_\ell) \in A$ and $(i, b, \mathcal{T}_{\ell'}) \in Y \backslash A$ with $\ell < \ell'$. If such i, b, ℓ, ℓ' exist, let $A' = A \cup \{(i, b, \mathcal{T}_{\ell'})\} \backslash \{(i, b, \mathcal{T}_\ell)\}$. Then, we have that $\gamma_b(\iota(A'_b)) = \gamma_b(\iota(A_b))$ and hence that $u_b(A') > u_b(A)$.

We can therefore assume that Conditions (A), (B), and (C) are all satisfied. To prove the claim in general, we divide into cases based on the first place in the process defining C_{sBfYC} at which A differs from $C^b_{\text{sBfYC}}(Y)$.

Case 1: There exists $1 \leq t \leq q^1_b$ such that $x_t \notin A \cup \{\varnothing\}$. Suppose that $x_t \in A \cup \{\varnothing\}$ for all $t \leq T$ and $x_{T+1} \notin A \cup \{\varnothing\}$. The definition of C^b_{sBfYC} ensures that $\iota(x_t) = k_t(\iota(Y_b))$ and that x_t is the longest contract with $k_t(\iota(Y_b))$ in Y for all $1 \leq t \leq T+1$. Condition (C) ensures that no contract with $k_{T+1}(\iota(Y_b))$ is in A.

Let $A' = A \cup \{x_{T+1}\}$. We claim that $u_b(A') > u_b(A)$. If $|A| = T$, then clearly we have that $\gamma_b(\iota(A')) \geq 2\Delta + u_b(A)$ and hence that $u_b(A') > u_b(A) + \Delta > u_b(A)$. Therefore, we can assume that $|A| > T$.

The definition of x_{T+1} ensures that $k_{T+1}(\iota(Y_b)) \succ^b_{\text{OML}} k_t(\iota(A))$ for all $t > T$. The definition of T ensures that $k_t(\iota(A)) \succ^b_{\text{OML}} k_{T+1}(\iota(Y_b))$ for all $t \leq T$. It follows that $k_{T+1}(\iota(A')) \succ^b_{\text{OML}} k_{T+1}(\iota(A))$ and $k_t(\iota(A')) \succ^b_{\text{OML}} k_t(\iota(A))$ for all $t \leq |\iota(A)|$. Because $T < q^1_b$ and due to Condition (A), it follows that $\gamma_b(\iota(A')) \geq \gamma_b(\iota(A)) + \Delta$. Since $\Delta > 1/\mathcal{T}_1$, it follows that $u_b(\iota(A')) > u_b(\iota(A))$, as desired.

Case 2: $x_t \in A \cup \{\varnothing\}$ for all $1 \leq t \leq q^1_b$ and there exists $q^1_b < t \leq q^1_b + q^2_b$ such that $x_t \notin A \cup \{\varnothing\}$. Suppose that $x_t \in A \cup \{\varnothing\}$ for all $t \leq T + q^1_b$ and $x_{T+q^1_b+1} \notin A \cup \{\varnothing\}$. The definition of C^b_{sBfYC} ensures that $x_{T+q^1_b+1}$ is the longest contract with $\iota(x_{T+q^1_b+1})$ in Y for all $1 \leq t \leq T+1$. Condition (C) ensures that no contract with $\iota(x_{T+q^1_b+1})$ is in A.

We now divide into cases based on the size of A to construct A'.

Subcase 2.1: $|A| \leq T + q^1_b$. Let $A' = A \cup \{x_{T+q^1_b+1}\}$. It is straightforward to verify that $\gamma_b(\iota(A')) > \Delta + u_b(A)$ and hence that $u_b(A') > u_b(A)$.

Subcase 2.2: $|A| > T + q_b^1$. Let

$$B = A \backslash \left\{ x_t \,|\, 1 \leq t \leq T + q_b^1 \right\}.$$

Because $|A| > T + q_b^1$, the set B is nonempty. Let $x' \in B$ be an arbitrary contract, and let $A' = A \cup \left\{ x_{T+q_b^1+1} \right\} \backslash \left\{ x' \right\}$.

By assumption, we have that

$$\left\{ x_t \,|\, 1 \leq t \leq T + q_b^1 \right\} \subseteq A.$$

As a result, we have that $k_t\big(\iota(A)\big) \neq \iota\big(x_{T+q_b^1+1}\big)$ for all $t \leq q_b^1$. It follows that $\gamma_b(A') \geq \gamma_b(A) - \delta$. If $x_{T+q_b^1+1}$ is longer than x', then we have that

$$\sum_{(i,b,t) \in A} \frac{1}{t} > \delta + \sum_{(i,b,t) \in A'} \frac{1}{t}.$$

It follows that

$$u_b(A') - u_b(A) > \gamma_b(A') - \gamma_b(A) + \delta > 0.$$

Therefore, we can assume that x' is at least as long as $x_{T+q_b^1+1}$. Conditions (A) and (B) and the definition of C^b_{sBfYC} imply that then $x_{T+q_b^1+1}$ and x' have the same length and that $x_{T+q_b^1+1} \succ^b_{\mathrm{OML}} x'$. It follows that $\gamma_b(A') \geq \gamma_b(A) + \delta$ and that

$$\sum_{(i,b,t) \in A} \frac{1}{t} = \sum_{(i,b,t) \in B} \frac{1}{t}.$$

Therefore, we have that

$$u_b(A') \geq u_b(A) + \delta > u_b(A),$$

as desired.

In either sub-case, we have constructed a set $A' \subseteq Y$ such that $u_b(A') > u_b(A)$. The sub-cases clearly exhaust all possibilities in the case under consideration.

Case 3: $C^b_{\mathrm{sBfYC}}(Y) \subsetneq A$. The definition of C_{sBfYC} guarantees that

$$\left| C^b_{\mathrm{sBfYC}}(Y) \right| = \min\left\{ |\iota(Y_b)|, q_b^1 + q_b^2 \right\}.$$

Conditions (A) and (B) imply that $|A| > q_b^1 + q_b^2$. Let $E = \left\{ k_1(A), \ldots, k_{q_b^1+q_b^2}(A) \right\}$. There exists a unique set of contracts $A' \subset A$ with $\iota(A') = E$. We have that $\gamma\big(\iota(A_b')\big) = \gamma\big(\iota(A_b)\big)$ and hence that $u_b(A') > u_b(A)$.

Because

$$C^b_{\mathrm{sBfYC}} = \left\{ x_t | 1 \leq t \leq q_b^1 + q_b^2 \right\} \backslash \left\{ \varnothing \right\}.$$

Cases 1 and 2 imply the claim if $C^b_{\text{sBfYC}}(Y) \not\subseteq A$. Case 3 implies the claim if $C^b_{\text{sBfYC}}(Y) \subsetneq A$. These cases exhaust all possibilities because $A \neq C^b_{\text{sBfYC}}(Y)$ by assumption.

Completion of the Proof.—We have proven that if

$$A \in \arg\max_{Z \subseteq Y} u_b(Z),$$

then $A = C^b_{\text{sBfYC}}(Y)$. Because $\arg\max_{Z\subseteq Y} u_b(Z)$ is nonempty, Proposition 2 follows. ∎

D. *Formal Statement and Proof of Theorem* 3

In order to state Theorem 3 formally, I need to define what it means for a matching market to be *isomorphic* to a Kelso-Crawford (1982) economy.

DEFINITION C.1: *A Kelso-Crawford economy* $(\mathcal{S}, u)$ *consists of*

- *a finite set of salaries* $\mathcal{S} \subseteq \mathbb{R}^+$ *with maximum* s_∞;
- *for each cadet* $i \in I$, *a utility function* $u_i : (H \times \mathcal{S}) \cup \{\varnothing\} \to \mathbb{R}$ *that is injective and increasing in salary*;
- *for each branch* $b \in B$, *a valuation function* $\gamma_b : \mathcal{P}(X) \to \mathbb{R}$, *which defines a quasi-linear utility function* $u_b : \mathcal{P}(I) \times \mathcal{S}^I \to \mathbb{R}$ *given by*

$$u_b(E, \mathbf{s}) = \gamma_b(E) - \sum_{d \in E} s_i;$$

such that the following conditions are satisfied:

- *for all branches* b, *the demand function* $D^b : \mathcal{S}^I \to \mathcal{P}(I)$ *defined by*

$$D^b(\mathbf{s}) = \arg\max_{E \subseteq D} u_b(E, \mathbf{s})$$

  is single-valued and grossly substitutable (*in the sense of Kelso and Crawford* 1982)—*if* $\mathbf{s} \leq \mathbf{s}'$ *and* $s_i = s_i'$, *then*

$$i \in D^b(\mathbf{s}) \Rightarrow i \in D^b(\mathbf{s}');$$

- *for all cadets* d, *branches* b, *and salary vectors* $\mathbf{s} \in \mathcal{S}^I$ *with* $s_i = s_\infty$, *we have that* $d \notin D^b(\mathbf{s})$.

Thus, a Kelso-Crawford economy is a discrete-salary market in the sense of Kelso and Crawford (1982) where non-integral salaries are allowed. Unlike Echenique (2012), I require cadets' utility functions to be strictly increasing in salary and branches' utility functions to be quasi-linear in salary.[34] These two additional requirements were assumed by Kelso and Crawford (1982) and offer a more realistic interpretation of salaries, as discussed in detail in Section VC.

The following definition of an *isomorphism* refines the definition of an *embedding* of a matching market with contracts into a matching market with salaries (Echenique 2012).

DEFINITION C.2: *An isomorphism of a matching market $\left(X, \hat{C}, \succ\right)$ with a Kelso-Crawford economy $\left(\mathcal{S}, u\right)$ is a function*

$$\varsigma : X \rightarrow \mathcal{S} \backslash \left\{ s_\infty \right\}$$

such that

- *the induced function $\left(\iota, \beta, \varsigma\right) : X \rightarrow I \times B \times \left(\mathcal{S} \backslash \left\{ s_\infty \right\}\right)$, defined by*

$$x \mapsto \left(\iota(x), \beta(x), \varsigma(x)\right),$$

  is bijective;
- *for all cadets $i \in I$ and all sets of contracts $Y \subseteq X_i$, we have that*

$$C^i(Y) = \underset{w \in Y' \cup \{\varnothing\}}{\arg\max}\, u_i(x),$$

  where Y' is the set of branch-salary pairs defined as

$$Y' = \left\{ \left(\beta(x), \varsigma(x)\right) \mid x \in Y \right\};$$

- *for all branches $b \in B$ and all sets of contracts $Y \subseteq X_b$, we have that*

$$C^b(Y) = \left\{ \left(d, s_i\right) \mid i \in D^b(\mathbf{s}) \right\},$$

  where $\mathbf{s}$ is the salary vector defined component-wise by

$$s_i = \min\left\{ s_\infty \right\} \cup \varsigma\left(Y_i \cap Y_b\right)$$

  for all $i \in I$.

We call $\varsigma(x)$ the salary corresponding to contract x.

An isomorphism exhibits a matching market as effectively identical to a Kelso-Crawford economy. More precisely, an isomorphism between a matching market and a Kelso-Crawford economy assigns salaries to contracts such that the agents' choice functions in the matching market maximize utility in the Kelso-Crawford economy. Moreover, every possible combination of a cadet, a branch, and a wage in

[34] Like Echenique (2012), Kominers (2012) and Schlegel (2015) did not require utility to be monotone or quasi-linear in salaries.

the Kelso-Crawford economy is required to be associated to a unique contract in the matching market. The precise statement of Theorem 3 builds on this formalism.

THEOREM 3 (Formal Statement): *Let* $g : \mathbb{R}^+ \to \mathbb{R}^+$ *be a strictly decreasing function. If all cadets have salary-monotonic preferences, then there exist a Kelso-Crawford economy* $(\mathcal{S}, u)$ *and an isomorphism* ς *of* $(X, C_{\text{sBfYC}}, \succ)$ *with* $(\mathcal{S}, u)$ *such that* $\varsigma(i,b,t) = g(t)$ *for all* $(i,b,t) \in X$. *The cadet-proposing deferred acceptance algorithm corresponds to the descending salary adjustment process under any such isomorphism.*

PROOF:

For all $b \in B$, let γ_b be the assignment valuation defined an assignment value matrix α^b satisfying the conditions of Proposition 2. Let

$$s_\infty > \sup_{b \in B} \sup_{I' \subseteq I} \gamma_b(I')$$

and let

$$\mathcal{S} = \left\{ \frac{1}{\mathcal{T}_\ell} \middle| 1 \leq \ell \leq \mathcal{K} \right\} \cup \{s_\infty\}.$$

The definition of $\mathcal{S}$ does not depend on b due to the second and third hypotheses of the proposition. Define $\varsigma : X \to \mathcal{S} \setminus \{s_\infty\}$ by $\varsigma(i,b,t) = 1/t$. Fix a cadet i. Because i has a salary-monotonic preference, there exists a utility function $u_i : (H \times \mathcal{S}) \cup \{\varnothing\} \to \mathbb{R}$ such that

- u_i is injective;
- $u_i(h, \varsigma(t)) < u_i(\varnothing)$ if $\varnothing \succ_i (i,b,t)$;
- $u_i((i,b,\mathcal{T}_\ell)) < u_i((i,b,\mathcal{T}_{\ell'}))$ for all cadets i and $1 \leq \ell < \ell' \leq \mathcal{K}$;
- for all $Y \subseteq X_i$, we have that

$$C^i(Y) = \operatorname*{arg\,max}_{w \in Y' \cup \{\varnothing\}} u_i(w),$$

where

$$Y' = \{(\beta(x), \varsigma(x)) \,|\, x \in Y\}.$$

The choice of s_∞ guarantees that for all cadets i, branches b, and salary vectors $\mathbf{s} \in \mathcal{S}^I$ with $s_i = s_\infty$, we have $i \notin D^b(\mathbf{s})$. Since γ_b is an assignment valuation, Theorem 1 in Shapley (1962) guarantees that the branches' demand functions D^b are grossly substitutable. Thus, $(\mathcal{S}, u)$ is a Kelso-Crawford economy.

The second and third hypotheses of the theorem ensure that the induced function from X to $I \times B \times (\mathcal{S} \setminus \{s_\infty\})$ defined by $(i,b) \mapsto (i,b,\varsigma(t))$ is bijective. The definition of u_i ensures the compatibility between C^i and u_i required by Definition C.2.

The definition of the valuations $(\gamma_b)_{b \in B}$ guarantees that for all branches $b \in B$ and all sets of contracts $Y \subseteq X_b$, we have that

$$C^b(Y) = \{(d, s_i) \mid i \in D^b(\mathbf{s})\},$$

where $\mathbf{s} \in \mathcal{S}^I$ is defined component-wise by

$$s_i = \max_{s \in \varsigma(Y_i \cap Y_b) \cup \{s_\infty\}} s$$

for all $i \in I$. Therefore, the function ς defines an isomorphism from $(X, C_{\text{sBfYC}}, \succ)$ to $(\mathcal{S}, u)$.

The last assertion of the theorem is clear, because both the cadet-proposing deferred acceptance algorithm and the descending salary adjustment process produce the cadet-optimal stable allocation by Theorem 4 in Hatfield and Milgrom (2005)(see also Section IID in Echenique 2012). ∎

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