Quantifying the Inefficiency of Multi-unit Auctions for Normal Goods

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Abstract

We study multi-unit auctions for homogenous goods in a private value setting where bidders have non-quasilinear preferences. Several recent impossibility results study this setting and find there is no mechanism that retains the Vickrey auction's desired incentive and efficiency properties without quasilinearity. While a fully efficient mechanism is impossible, we show that any undominated outcome of the Vickrey auction has a negligible inefficiency when bidder wealth effects are sufficiently small. In order to show this, we first place bounds on undominated bid behavior in the Vickrey auction when bidders have non-quasilinear preferences. We use (Marshallian) deadweight loss as our inefficiency metric, and we derive a tight upper bound on the inefficiency associated with the Vickrey auction in terms of the degree of bidder wealth effects. As wealth effects diminish, the bound continuously approaches zero. Other common multi-unit auction formats do not have this property, and their worst-case inefficiencies are higher than that of the Vickrey auction.

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1 Introduction

1.1 Motivation

The Vickrey Clarke Groves (henceforth VCG) mechanism is cited as an achievement of mechanism design because it implements an efficient resource allocation and also gives agents a dominant strategy to truthfully reveal their private information. However, this positive result crucially depends on the assumption that agents have quasilinear utility. Without the quasilinearity restriction on preferences, impossibility results abound. Perhaps the most notable is the Gibbard and Satterthwaite theorem, which gives conditions under which it is impossible to implement any non-trivial social choice function in dominant strategies when preferences are unrestricted.

More recent work has presented settings where Gibbard and Satterthwaite's impossibility theorem extends to relatively coarser payoff type spaces.[1] In particular, recent papers in economics and computer science have studied multi-unit auctions when bidders have non-quasilinear preferences. The emergence of this theoretical literature stems from frictions that commonly shape incentives in important real-world auction design problems – budgets, financial constraints, and bidder risk preferences are salient features of a bidders' decision problems in prominent multi-unit auctions for high value goods.[2] Many papers have shown that these violations of quasilinearity inhibit efficient implementation. For example, Dobzinski et al. (2012) show that there is no multi-unit auction that is Pareto efficient when bidders have private budgets and multi-units demands.[3] Consequently, the VCG mechanism no longer satisfies the properties that underpin its motivation – Pareto efficiency and dominant strategy implementability – in multi-unit auction settings when we relax the assumption that bidders have quasilinear preferences.

The main contribution of this paper is to quantify the inefficiency of the Vickrey auction, as well as other multi-unit auctions, when bidders have non-quasilinear preferences. In other words, we ask by how much does the Vickrey auction fall short of its efficiency objective when bidders can have non-quasilinear preferences? We show the magnitude of inefficiency associated with using the Vicrkey auction is scaled by the strength of bidders' wealth effects. Thus, while there is no auction that satisfies Vickrey's desired properties, if bidder wealth effects are sufficiently small, the inefficiency of any undominated Vickrey auction outcomes is

[1] Note that implementation problems become harder to solve, and more likely to yield impossibility results, when the agents' type spaces become finer.

[2] For example, Che and Gale (1998) and Dobzinski et al. (2012) note that budget constraints and imperfect credit markets are important considerations for bidders in many important real-world auction settings. Including these features violates the standard quasilinearity restriction.

[3] Kazumura and Serizawa (2016) show a similar result in a non-quasilinear setting where at least one bidder has multi-unit demand. Baisa (2020) shows these impossibility result extend to settings where bidders have non-quasilinear that do not satisfy single-crossing. See the related literature section for more detail.

negligible. We consider an auction where bidders with non-quasilinear preferences compete to win units of an indivisible homogeneous good.

We measure the inefficiency associated with an auction's outcome as its (Marshallian) deadweight loss. Essentially, our inefficiency metric is the consumer and producer surplus associated with a hypothetical perfectly competitive (and hence, efficient) resale market. This method of quantifying the inefficiency of an auction is related to notions of price of anarchy and liquid welfare maximization used by computer scientists studying related problems. However, neither price of anarchy nor liquid welfare are well-defined concepts in our settings that nests quasilinearity and hard budgets as special cases. Furthermore, our worst-case bounds are different from work on the price of anarchy because we do not assume bidders best respond according to a standard game theoretic equilibrium concept. Instead, we make a less restrictive assumption that bidders place undominated bids. Thus, our welfare bounds are robust to assumptions on what is common knowledge. We calculate the 'worst-case' inefficiency as the maximal deadweight loss associated with an undominated auction outcome.

We first illustrate the relationship between the degree of bidder wealth effects and undominated bid behavior in the Vickrey auction. While the Vickrey auction does not have a dominant strategy equilibrium without quasilinearity, many bidding strategies are weakly dominated and the set of undominated bidding strategies can be relatively small. When a bidder has non-quasilinear preferences, the amount she is willing to pay for her n^{th} unit depends on the amount she paid in order to win her first $n-1$ units. We get tighter bounds on range of a bidder's undominated bid curves in the Vickrey auction when the degree of wealth effects – i.e. the degree that an bidder's demand for additional units varies with her wealth – is small. This is because when wealth effects are small, the amount a bidder wants to pay for her later units is relatively invariant to the amount she paid to win the earlier units. In the limiting quasilinear case, where bidders' demands are invariant to their initial wealth, truthful bidding is the only undominated strategy.

Our main result uses these bounds on undominated bid behavior to quantify the inefficiency of the Vickrey auction. The upper bound is a function of the strength of bidder wealth effects. When all bidders have preferences where wealth effects are relatively small – like they do in the quasilinear benchmark – then the maximal amount of inefficiency possible is relatively small. If wealth effects are more salient, as they could be in the hard budget case of Dobzinski et al. (2012), for example, the worst case inefficiency is greater because the range of undominated bids is greater. We show our upper bound on inefficiency is tight by presenting an example of bidders with preferences and corresponding undominated bids that create the maximal possibly deadweight loss.

We then compare the maximal inefficiency of the Vickrey auction to the Uniform-price and Discriminatory auctions. We show that the Vickrey auction's inefficiency upper bound is below that of the other two auctions. In both the Uniform-price and Discriminatory auctions, the range of weakly undominated bid curves and the worst-case inefficiency do not scale with the degree of wealth effects.

Finally, we extend the measure of inefficiency to cases where there is additional information about bidder preferences or bids. First, we calculate the maximum inefficiency in the Vickrey auction when bidders are known to have soft budget constraint preferences and face a bounded interest rate. Second, we calculate the maximum inefficiency in the Vickrey auction when bid curves are known and bidder wealth effects are bounded. In both cases, the maximum inefficiency is less than the unconstrained maximum.

1.2 Related Literature

When agents have private values and quasilinear preferences, the VCG mechanism uniquely implements an efficient outcome in dominant strategies (Holmström, 1979). Bikhchandani et al. (2006) shows that we can implement any social choice function that satisfies weak monotonicity in quasilinear private value settings. However, the Gibbard-Satterthwaite impossibility theorem (Gibbard, 1973; Satterthwaite, 1975) shows that these positive implementation results do not extend to private value settings without quasilinearity.

A recent literature has focused the scope of dominant strategy implementation in multi-unit auctions where bidders have non-quasilinear preferences. These are settings where the existence of monetary compensation restricts the set of feasible preferences, unlike in the domain consider by the Gibbard-Satterthwaite impossibility theorem; and preferences are non-quasilinear, unlike in Bikhchandani et al. (2006). The "hard-budget" case is a well-studied special case where bidder preferences are described by a valuation and a budget. Borgs et al. (2005) shows that it is impossible to implement a non-trivial social choice function in this case. Dobzinski et al. (2012) show that efficient implementation requires that budgets be commonly known. Further impossibility results by Lavi and May (2012) and Goel et al. (2015) extend the impossibility of efficient implementation to cases where bidders have public budgets and marginal values for additional units that are not constant.

There are papers in the economics literature that establish similar results, but do not restrict to the hard budget case. For example, Morimoto and Serizawa (2015) show that the minimum price Walrasian rule is the unique mechanism that retains the desired properties of VCG when bidders have unit demands. Kazumura and Serizawa (2016) show that this positive implementation result fails when we allow bidders to have multi-unit demands. Baisa (2020) shows that the impossibility result extends to any multi-unit setting where bidder preferences do not satisfy single-crossing. Ma et al. (2018) and Kazumura et al. (2020) give further necessary and sufficient conditions on dominant strategy implementation of any social choice function in non-quasilinear settings.

A consistent theme emerges from the aforementioned research on auctions when bidders have non-quasilinear preferences: there is no mechanism that retains VCG's desired incentive and efficiency properties when bidders have multi-unit demands and bidder private information is sufficiently rich. Given these impossibility results, a subsequent literature has sought to understand what mechanisms are second-best, and what is the welfare loss associated with using a second-best mechanism. As Dobzinski and Paes Leme (2014) write "We start from the observation that a Pareto efficient solution is a binary notion: an allocation is either Pareto efficient or not, and there is no sense of one allocation being "more Pareto efficient" than the other. This is in contrast with efficiency in quasilinear environments where the traditional welfare objective induces a total order on the the allocations." Dobzinski and Paes Leme then use this observation to motivate the use of an efficiency metric they call liquid welfare for the case where bidders have hard budgets. Their metric of liquid welfare in their paper is related to inefficiency metric developed in this paper. They construct a dominant strategy implementable mechanism that generates a constant fraction of liquid welfare. Relatedly, Syrgkanis and Tardos (2013), Azar et al. (2017), and Caragiannis and Voudouris (2016) give bounds on the price of anarchy when budget constrained bidders place bids in many auctions simultaneously.

The analysis in this paper differs from papers on the price of anarchy, because we do not assume that bidders coordinate on placing equilibrium bids. Instead, we measure welfare by characterizing the outcome that is maximally inefficient when bidders place undominated strategies. Undominated implementation has been studied in a variety of mechanism design settings. Börgers (1991) shows that there cases where there exist non-dictatorial decision rules that are Pareto efficient when agents take undominated actions. Börgers and Smith (2012) suggest using elimination of weakly dominated strategies as a means of ranking mechanisms and show cases where undominated mechanisms can implement outcomes that can not be implemented through dominant strategy implementation. Yamashita (2015) similarly studies worst-case scenarios, assuming only that agents take undominated actions. He studies undominated implementation in a setting where agents have quasilinear utility and a single-dimensional type. Like this paper, Chiesa et al. (2015) studies undominated bid behavior in the Vickrey auction for indivisible homogeneous units. Also, their main result is an asymptotic efficiency result that is similar in spirit to our main result. However, their setting is quite different because they assume bidders have quasilinear preferences, but do not know their actual value. Instead, they model players as having Knightian uncertainty over their true value. Baisa (2017) considers undominated bid behavior when he studies a mechanism used to sell a single indivisible good in a multi-unit auction. Finally, Babaioff et al. (2006) consider undominated implementation in a combinatorial auction setting where bidders have quasilinear preferences.

2 Model

2.1 Bidder Preferences

Consider a seller with $M \geq 2$ indivisible homogeneous units and bidders $1, \ldots, N$. Bidder i's preferences are described by utility function u_i where

$$u_i : \{0, 1, \ldots, M\} \times \mathbb{R} \to \mathbb{R}.$$

Bidder i gets utility $u_i(q, t)$ when she owns q objects and has paid t. Without loss of generality, we assume that $u_i(0, 0) = 0 \ \forall i \in \{1, \ldots, N\}$. We assume that $u_i(q, \cdot)$ is strictly decreasing and continuous for all $q \in \{0, 1, \ldots N\}$. Also, we assume free disposal, and hence $u_i(\tilde{q}, t) \geq u_i(q, t) \ \forall i \in \{1, \ldots, N\}, t \in \mathbb{R}$, and $\tilde{q}, q \in \{0, 1, \ldots, M\}$ such that $\tilde{q} > q$.

In addition, we assume that bidders have bounded demands. That is, $\exists \overline{p} > 0$ such that $u_i(q, t) > u_i(q + 1, t + \overline{p})$ for any $i \in \{1, \ldots, N\}, t \in \mathbb{R}$, and $q \in \{0, 1, \ldots, M - 1\}$. It is without loss of generality to assume that $\overline{p} = 1$.

We make only two additional assumptions on bidder preferences. First, we assume weakly declining demand for additional units. That is, if a bidder is unwilling to pay p for her q^{th} unit, then she is unwilling to pay p for her $(q + 1)^{th}$ unit. The assumption ensures that bidders have (weakly) downward sloping (inverse) demand curves and generalizes the weakly declining marginal values assumption imposed in the benchmark quasilinear setting.

A.1. *(Weakly declining demand).* For any $t \in \mathbb{R}$, $p \in \mathbb{R}_+$, and $q \in \{1, \ldots, M - 1\}$, if

$$u_i(q - 1, t) \geq u_i(q, t + p),$$

then

$$u_i(q, t) \geq u_i(q + 1, t + p).$$

Second, we assume that bidders have weakly positive wealth effects. This means a bidder's demand does not decrease as her wealth increases. In words, suppose that bidder i chooses between two bundles. The first bundle has q units and costs p total, and the second bundle has q' unit and costs p' total, where $q > q'$. If bidder i weakly prefers the bundle with more units, then weakly positive wealth effects states that she would also weakly prefer the bundle with more units if her initial wealth increases. This is a multi-unit generalization of the definition of an indivisible, normal good in Cook and Graham (1977).

A.2. *(Weakly positive wealth effects).* Suppose $q > q'$ where $q, q' \in \{0, 1, \ldots, M\}$. Bidder i has weakly positive wealth effects:

$$u_i(q, t+p) \geq u_i(q', t+p') \Longrightarrow u_i(q, t'+p) \geq u_i(q', t'+p') \; \forall t > t'.$$

Let $\mathcal{U}$ denote the set of all utility functions u_i that satisfy A.1 and A.2. We call $\mathcal{U}$ the set of all positive wealth effects preferences.

We let d_i^1 be the amount that bidder i is willing to pay for her first unit of the good. Thus, d_i^1 implicitly solves

$$0 = u_i(1, d_i^1).$$

We similarly define $d_i^q(t)$ where $d_i^q : \mathbb{R} \to \mathbb{R}_+$ as bidder i's willingness to pay for her q^{th} unit, conditional on winning her first $q-1$ units for a cost of $t \in \mathbb{R}$. More precisely, $d_i^q(t)$ is implicitly defined as solving

$$u_i(q-1, t) = u_i(q, t + d_i^q(t)),$$

for all $q \in \{2, \ldots, M\}$ and $t \in \mathbb{R}$. We analogously define $s_i^q(t)$ as bidder i's willingness to sell her q^{th} unit conditional on having paid t in total. Thus, a bidder's willingness to sell her q^{th} unit $s_i^q(t)$ is implicitly defined as solving

$$u_i(q, t) = u_i(q-1, t - s_i^q(t)),$$

for all $q \in \{1, \ldots, M\}$ and $t \in \mathbb{R}$. Note that by construction,

$$s_i^q(t) = d_i^q(t - s_i^q(t)) \; \forall q \in \{1, \ldots, M\}, t \in \mathbb{R}. \tag{1}$$

Note also that our bounded demand assumption implies that $d_i^q(t), s_i^q(t) \leq 1, \forall i \in \{1, \ldots, N\}, t \in \mathbb{R}$. In addition, A.1 and A.2 imply:

1. $1 \geq d_i^q(t) \geq d_i^{q+1}(t)$ and $s_i^q(t) \geq s_i^{q+1}(t)$ for all $q \in \{1, \ldots, M-1\}$ and $t \in \mathbb{R}$.
2. $d_i^q(t)$ and $s_i^q(t)$ are continuous and weakly decreasing in t for all $t \in \mathbb{R}, q \in \{1, \ldots, M\}$.
3. $d_i^q(t') - d_i^q(t) \leq t - t' \forall t > t' \in \mathbb{R}$

The first point is implied by bounded and declining demand. The second point is implied by non-negative wealth effects. The final point follows because

$$\begin{aligned} & t > t' \implies u_i(q, t + d_i^q(t)) = u_i(q-1, t) \le u_i(q-1, t') = u_i(q, t' + d_i^q(t')) \\ \implies & t + d_i^q(t) \ge t' + d_i^q(t'). \end{aligned}$$

Intuitively, this says that if your initial wealth increases by $t - t'$ dollars, then your willingness to pay for your q^{th} unit cannot increase by more than the amount your wealth increased by. Thus, this is an upper bound on the degree of wealth effects; i.e. the rate at which a bidder's willingness to pay for additional units can vary with respect to her wealth.

We typically study a subset of the set of all positive wealth effects preferences $\mathcal{U}$. In particular, we focus on a subset of $\mathcal{U}$ that we call $\mathcal{U}(k)$, where $k \in [0, 1]$ is an upper bound on the degree of bidder wealth effects. The parameter k represents the maximal amount a bidder's demand for a marginal unit can increase when her wealth increases by a dollar. Intuitively, we can think of this parameter similarly to the marginal propensity to consume, but for an indivisible unit setting.

Definition 2.1. *(Wealth effects at most k).* The set $\mathcal{U}(k)$, where $k \in [0, 1]$, is the set of all $u_i \in \mathcal{U}$ such that

$$d_i^q(t') - d_i^q(t) \le k(t - t') \; \forall q \in \{1, \ldots, m\}, t > t'.$$

If u_i has wealth effects at most k, then k is the highest possible rate that bidder i's willingness to pay for a unit varies with wealth. By definition,

$$\mathcal{U}(k) \subset \mathcal{U}(k') \iff k < k'.$$

Note that the final point in the itemized list above shows us that $\mathcal{U}(1) = \mathcal{U}$. Note also that when k is relatively close to zero, then bidders have relatively inelastic Engel curves – their demand for additional units does not vary much with changes in the bidder's wealth – and when k is higher, then it is possible that bidders have relatively more elastic Engel curves. The extreme case where we assume $k = 0$, includes all quasilinear preferences, as well as the parallel domain studied by Ma et al. (2018). In this case, there are no wealth effects.

Example 2.1. *(Quasilinear Preferences).* Bidder i has quasilinear preferences where $u_i(q, t) = v_i(q) - t$ for some non-decreasing and weakly concave function $v_i : \{0, 1, \ldots, M\} \to \mathbb{R}_+$. Then $d_i^q(t) = s_i^q(t) = v_i(q) - v_i(q-1)$ for all $q \in \{1, \ldots, M\}$, $t \in \mathbb{R}$. This function is constant in t and thus $u_i \in \mathcal{U}(0)$.

Another instructive case is where bidders have budgets and can pay interest on bids in excess of their budget. Hafalir et al. (2012) call this the soft budget case.

Example 2.2. *(Soft Budget Constraint).* Bidder i has valuation function $v_i(q)$ where v_i is weakly increasing and weakly concave. Also, bidder i has budget $w_i \geq 0$ and pays interest rate $r \geq 0$ on the amount paid above budget,

$$u_i(q,t) = \begin{cases} v_i(q) - t & \text{if } t < w_i, \\ v_i(q) - t - r(t - w_i) & \text{if } t \geq w_i. \end{cases}$$

Then, $u_i \in \mathcal{U}(k) \iff k \geq \frac{r}{1+r}$. Note that when $r = 0$ this is the standard quasilinear setting. The limiting case where $r = \infty$ is the hard budget case that has been well studied by Dobzinski et al. (2012) and Pai and Vohra (2014), among many others.

2.2 Multi-unit auctions

We consider three well-studied auctions: the Vickrey auction, the Uniform-price auction, and the Discriminatory auction. We especially focus on the Vickrey auction. All three auctions have the same message space $\mathcal{B}$ where

$$\mathcal{B} := \{b \in \mathbb{R}^m_+ \mid \infty > b_1 \geq b_2 \geq \cdots \geq b_m \geq 0\}.$$

Each auction assigns units to the M highest bids (out of the total $M \times N$ submitted bids). For concreteness, we assume that ties are broken in favor of the lower numbered bidder, though this will not be essential to our analysis.

We follow Krishna (2009) to describe the payment rules in each auction. In the Vickrey auction, the price a bidder pays for objects is determined by other bidders' reported demands. Specifically, a bidder's payment is determined by the marginal price curve which is a residual demand curve formed by other bidders' reported demand curves.

Let c^{-i} be the vector of competing bids faced by bidder i, ordered from highest to lowest. Bidder i wins exactly $q \in \{0, 1, \ldots, M\}$ units when she submits q bids that rank in the top M. If bidder i wins q objects, she pays $\sum_{j=1}^{q} c^{-i}_{M+1-j}$. That is, bidder i faces an upward sloping marginal price curve because c^{-i}_{M+1-j} is increasing in j. The marginal price of acquiring the q^{th} unit is the $(M+1-q)^{th}$ highest bid made by i's competitors.

In the Uniform-price auction, bidders again submit an M dimensional (non-increasing) bid curve. All winning bidders pay the same price for all units won. The price is equal to the highest losing bid, which is the $M + 1^{st}$ highest bid submitted. In the Discriminatory auction, bidder i's payment is sum of all of her winning bids.

We let $y \in Y := \{y \in \{0, 1, \ldots, M\}^N | \sum_{i=1}^{N} y_i = M\}$ describe a feasible assignment of units. An auction outcome $(y, t_1, \ldots, t_N) \in Y \times \mathbb{R}^N$ describes both the feasible assignment of units and payments for all bidders. We use the notation $a(b, V) \in Y \times \mathbb{R}^N$ to describe the outcome of the Vickrey auction V when bids are $b \in \mathcal{B}$. We similarly use the notation $a(b, UP)$ and $a(b, D)$ to describe the outcomes of the Uniform-price and Discriminatory auctions, respectively.

We use the shorthand $V_i(b_i, b_{-i})$ to represent bidder i's utility when she participates in the Vickrey auction, has preferences $u_i \in \mathcal{U}$, bids $b_i \in \mathcal{B}$ and faces competing bids $b_{-i} \in \mathcal{B}^{N-1}$. Similarly, we let $UP_i(b_i, b_{-i})$ and $D_i(b_i, b_{-i})$ represent bidder i's payoffs in the Uniform-price and Discriminatory auctions, respectively.

We assume bidders place bids that are weakly undominated. We define weak dominance for the Vickrey auction below, and analogously extend the concept to study undominated bid behavior in the other two auctions as well.

Definition 2.2. *(Weakly Dominated Strategy).* Bidding $b^i \in \mathcal{B}$ is a weakly dominated strategy for bidder i if there exists a $\tilde{b}^i \in \mathcal{B}$ such that

$$V_i(\tilde{b}^i, b^{-i}) \geq V_i(b^i, b^{-i}) \ \forall b^{-i} \in \mathcal{B}^{N-1},$$

and

$$V_i(\tilde{b}^i, b^{-i}) > V_i(b^i, b^{-i}) \text{ for some } b^{-i} \in \mathcal{B}^{N-1}.$$

We let $\mathcal{B}^{UD}(u_i, V)$ be the set of all undominated bids for a bidder with preferences u_i in the Vickrey auction V. We use analogous notation for Uniform-price and Discriminatory auctions as well.

2.3 A Measure of Inefficiency

Recent papers show that there is no mechanism that is efficient and dominant strategy implementable when bidders have non-quasilinear preferences and a sufficiently rich type space.[4] For example, a corollary of Baisa (2020) is that is no such desirable mechanism when the type space is $\mathcal{U}(k)$, for any $k > 0$. However, as Dobzinski and Paes Leme (2014) note, Pareto efficiency is a binary concept. Thus, the aforementioned impossibility theorems do not inform us about the magnitude of the inefficiency associated with the commonly used and/or studied multi-unit auction formats. We develop bounds on the the magnitude of inefficiency, which we measure by calculating the maximal (Marshallian) deadweight loss associated with any undominated post auction outcome. Deadweight loss is defined as the hypothetical producer and consumer surplus gained by moving from the post auction outcome to the

[4]Dobzinski et al. (2012); Lavi and May (2012); Kazumura and Serizawa (2016) and Baisa (2020) all present impossibility results for efficient and dominant strategy implementable auctions in settings without quasilinearity.

outcome of a perfectly competitive resale market. Note that we are not assuming that bidders have access to a resale market.

We begin with a numerical example and then formally define the notion of deadweight loss in our setting.

Example 2.3. There are two bidders competing for three units. One bidder has quasilinear preferences and the other has soft budget constraint preferences with interest rate r and budget of one:

$$u_1(q,t) = q - t,$$
$$u_2(q,t) = q - t - r\max\{0, t-1\}.$$

Bidder 1 has no wealth effects and bidder 2 has wealth effects at most $k = r/(1+r)$, i.e. $u_2 \in \mathcal{U}(\frac{r}{1+r})$. Consider an outcome is such that bidder 1 wins zero units and pays nothing, and bidder 2 wins three units and pays three. In a hypothetical perfectly competitive resale market, bidder 2 would sell some of her units to bidder 1. Bidder 1, the buyer in this resale market, has demand equal to one for all three units because bidder 1 has quasilinear preferences. We use the notation RD to describe her resale demand curve:

$$RD(1) = RD(2) = RD(3) = 1.$$

Bidder 2's supply for the first unit in the resale market, which we will call $RS(1)$, equals her willingness to sell her third unit, which solves

$$u_2(3,3) = u_2(2, 3 - RS(1)) \Rightarrow RS(1) = 1/(1+r).$$

Similarly, her supply for the second unit $RS(2)$ and third unit $RS(3)$ solve

$$u_2(2, 3 - RS(2)) = u_2(1, 3 - 2RS(2)) \Rightarrow RS(2) = 1/(1+r),$$
$$u_2(1, 3 - 2RS(3)) = u_2(0, 3 - 3RS(3)) \Rightarrow RS(3) = 1.$$

In this case, the deadweight loss of the auction outcome is

$$DWL(a) = \sum_{j=1}^{3} \max\{RD(j) - RS(j), 0\} = 2\frac{r}{1+r} = 2k.$$

Formally, we define a bidder's (inverse, Marshallian) resale demand and supply curves and use these curves to calculate deadweight loss. The (inverse, Marshallian) resale demand curve states the highest per unit price a bidder where a bidder would demand exact $q \in$

$\{1, \ldots, M - y_i\}$ additional units relative to her post auction outcome (y_i, t_i),

$$RD_i(q, y_i, t_i) = \max\{p | u_i(q + y_i, t_i + qp) \geq u_i(\tilde{q} + y_i, t_i + \tilde{q}p), \forall \tilde{q} \in \{0, 1, \ldots, M - y_i\}\}.$$

We similarly define the (inverse, Marshallian) resale supply curve as the lowest per-unit price where a bidder would seek to sell exact q relative to her post auction outcome (y_i, t_i),

$$RS_i(q, y_i, t_i) = \min\{p | u_i(y_i - q, t_i - qp) \geq u_i(\tilde{q} + y_i, t_i - \tilde{q}p), \forall \tilde{q} \in \{0, 1, \ldots, y_i\}\}.$$

In introductory economics terms, deadweight loss is the region between the market demand and market supply curves in our hypothetical perfectly competitive resale market. Thus, if the post auction outcome is $a = (y, t) \in Y \times \mathbb{R}^N$, the deadweight loss is

$$\begin{aligned} DWL(a) = \max_{\{r_i\}} \sum_{i:r_i>0} \sum_{r=1}^{r_i} RD_i(r, a) - \sum_{i:r_i<0} \sum_{r=r_i}^{-1} RS_i(-r, a) \\ \text{s.t. } \sum_i r_i = 0 \text{ and } r_i \in \{-q_i, -q_i + 1, \ldots, m - q_i\} \forall i. \end{aligned}$$

The first term represents the market demand side of the resale market and the second term represents the market supply. Notice that $r_i > 0$ means bidder i buys units in the resale market, and conversely, $r_i < 0$ means bidder i sells units in the resale market.

We present efficiency bounds that are robust to both the precise realization of bidder preferences and actions. In particular, we find the highest deadweight loss associated with any undominated outcome of the given auction formats, as a function of the upper bound on bidder wealth effects k. For the case of the Vickrey auction, the upper bound is formally defined as

$$\overline{I}(k, V) = \sup_{\substack{u \in \mathcal{U}(k), \\ b \in \mathcal{B}^{UD}(u, V)}} DWL(a(b, V)).$$

We analogously define the inefficiency upper bound in the Uniform-price and Discriminatory auctions. Note that the well known desirable properties of the Vickrey auction in the quasilinear case shows us that $\overline{I}(0, V) = 0$. Or in other words, there is no deadweight loss in the undominated outcome of the Vickrey auction when we assume that bidders have no wealth effects. Conversely, the aforementioned impossibility results for settings without quasilinearity show us that $k > 0 \implies \overline{I}(k, V) > 0$. Also, observe that $\overline{I}(k, a) \leq M \forall k \in [0, 1]$ and for any auction a because of our bounded demands assumption. We confirm these results in our analysis.

3 Vickrey Auction

3.1 Characterizing Undominated Bid Behavior

In this section, we derive the inefficiency upper bound on the Vickrey auction $\overline{I}(k, V)$. We show that the upper bound converges to zero when the degree of wealth effects $k \in [0, 1]$ is sufficiently small. While no auction perfectly satisfies the desired VCG-properties, we show that any undominated outcome of the Vickrey auction has relatively little inefficiency, given that bidders have sufficiently small wealth effects.

Our first result, Theorem 1, gives upper and lower bounds on the set of undominated bids in the Vickrey auction for a bidder with preferences $u_i \in \mathcal{U}(k)$. The lower bound on undominated bids is the bidder's (inverse, Marshallian) demand curve. A bidder's (inverse) demand curve for her q^{th} unit is the highest price (per-unit) where a bidder where the bidder demands exactly $q \in \{1, \ldots, M\}$ units. Let $LB_i \in \mathcal{B}$ be bidder i's inverse demand curve, which is defined implicitly as

$$LB_i^q := d_i^q((q-1)LB_i^q) \forall q \in \{1, \ldots, M\}.$$

We show that upper bound on bidder i's bid for her q^{th}, which we call UB_i^q, is her willingness to pay for her q^{th} unit conditional on having paid nothing to win her first $q-1$ units,

$$UB_i^q := d_i^q(0) \forall q \in \{1, \ldots, M\}.$$

Note that by construction, $UB_1 = LB_1$ and $UB_q \geq LB_q \forall q \in \{1, \ldots, M\}$.

Theorem 3.1. In the Vickrey auction, the bid $b_i = (b_i^1, b_i^2, \ldots, b_i^M)$ is

1. undominated only if $b_i^1 = d_i^1$ and

$$b_i^q \in [LB_i^q, UB_i^q] \forall q \in \{2, \ldots, M\}.$$

2. (Baisa, 2016). weakly dominated by $\tilde{b_i}$ where $\tilde{b_i}^q = \max\{b_i^q, LB_i^q\} \forall q$ if b_i is such that $b_i^{q'} < LB_i^{q'}$ for some $q' \in \{1, \ldots, M\}$.

3. weakly dominated by $\tilde{b_i}$ where $\tilde{b_i}^q = \min\{b_i^q, UB_i^q\} \forall q$ if b_i is such that $b_i^{q'} > UB_i^{q'}$ for some $q' \in \{1, \ldots, M\}$.

Proof. The proof is shown in the appendix. □

The bounds on bid behavior described in Theorem 1 can be understood intuitively. Note that in the Vickrey auction, by construction, the marginal price of a unit weakly exceeds the inframarginal price. We study two extreme cases to understand why LB_i and UB_i bound bidder i's undominated bid behavior.

In the first case, suppose that bidder i pays nothing to win each of her first $q-1$ units, and we want to determine what is her best response bid for her q^{th} unit. Thus, we are considering a special case where bidder i faces at most $M-q$ competing bids that are not equal to zero, and hence she wins her first $q-1$ units for free. If bidder i paid nothing to win her first $q-1$ units, then her best response is to bid her willingness to pay for her q^{th} unit, conditional on having paid zero for the first $q-1$ units, which equals $d_i^q(0) = UB_i^q$.

However, in most cases, the gap between the marginal price bidder i pays to win her q^{th} unit and the inframarginal price she paid to win her first $q-1$ units is not so large. Furthermore, bidder i's willingness to pay for her q^{th} unit decreases when the inframarginal price she paid to win her first $q-1$ units increases, because she has positive wealth effects. Taking this logic to the other extreme, we see that bidder i wants to bid the lower bound on her undominated bids when she believes that the price she pays for her inframarginal units is as large as the price she pays for her marginal units. More precisely, consider the a second special case where all of bidder i's rivals bid the same price p^* for all units. Thus, bidder i faces perfectly elastic marginal price curve, and the price she pays to win her marginal q^{th} unit equals the price she paid for her first $q-1$ inframarginal units. In this case, bidder i best responds if she wins exactly the number of units she demands when the price is p^* per unit. Thus, she wants her bid for her q^{th} unit to equal the value of her inverse (Marshallian) demand LB_i^q, because she wants her q^{th} bid to be a winning bid if and only if she demands at least q units at price p^*.

Next, we relate the bounds on undominated bids in the Vickrey auction to the parameter describing the upper bound on the strength of bidder's wealth effects, $k \in [0,1]$.

Corollary 3.1. *(Bound on misreporting).* Suppose bidder i has wealth effects at most k (i.e. $u_i \in \mathcal{U}(k)$). Then,

$$UB_i^q - LB_i^q \leq k(q-1) \ \forall q \in \{1, 2, \ldots, M\}.$$

Proof. By construction of UB_i^q and LB_i^q, we have that

$$UB_i^q - LB_i^q = d_i^q(0) - d_i^q((q-1)LB_i^q) \leq k(q-1)LB_i^q \leq k(q-1),$$

where the penultimate inequality follows from the definition of wealth effects at most k, and the final inequality holds because bounded demands imply that $LB_i^q \leq 1$. □

There are two useful observations that follow from Corollary 3.1. First, note that the range of undominated bids is larger when wealth effects are larger. The intuition is that when wealth effects are larger, a bidder's willingness to pay for her q^{th} unit is more responsive to payment for the first $q-1$ units. Thus, the bidder best responds by placing a relatively high bid if she thinks her rivals placed relatively low bids, and hence, she paid little to win her first $q-1$ units. Similarly, the bidder best responds by placing a lower bid when she thinks her rivals placed relatively higher bids, and she must pay a lot to win her first $q-1$ units. In contrast, when wealth effects are near zero, such as when preferences are quasilinear, the range of undominated bids is small because the precise amount a bidder paid to win her first $q-1$ units has little impact on determining her demand for her q^{th} unit.

Second, we get tighter bounds on the range of undominated bids when looking at the bids for earlier units. This is because wealth effects compound. In the Vickrey auction, bidder i's willingness to pay for her second unit depends on the amount she pays to win her first unit. And similarly, her willingness to pay for her third unit depends on the amount she pays to win her first two units. The amount bidder i pays to win her first unit must be between her bid for her first unit and zero. The amount bidder i pays to win her first two units is between 0 and twice the value of her second bid. The latter is a larger range than the former. Thus, the range of possible values bidder i could have for her marginal willingness to pay for another unit will be greater for later units because there is a larger range of amounts she could pay for her first two units than there is for just her first unit.

Example 3.1. As in Example 2.3, there are two bidders competing for three units. One bidder has quasilinear preferences and the other has soft budget constraint preferences with interest rate r and budget of one:

$$u_1(q,t) = (1-\epsilon)q - t,$$
$$u_2(q,t) = q - t - r\max\{0, t-1\}.$$

Bidder 1's willingness to pay for an additional unit is always $d_1^q(t) = 1-\epsilon$ for all q and t because she has quasilinear preferences. Therefore, the upper and lower bounds on her bids are $1-\epsilon$:

$$LB_1^1 = LB_1^2 = LB_1^3 = 1-\epsilon,$$
$$UB_1^1 = UB_1^2 = UB_1^3 = 1-\epsilon.$$

Bidder 2's upper bound for q^{th} bid equals her willingness to pay for her q^{th} unit conditional on paying nothing for the first $q-1$ units that she won. Therefore, her upper bound solves the following equations:

$$\begin{aligned}
u_2(0,0) = u_2(1, UB_2^1) &\Rightarrow UB_2^1 = 1,\\
u_2(1,0) = u_2(2, UB_2^2) &\Rightarrow UB_2^2 = 1,\\
u_2(2,0) = u_2(3, UB_2^3) &\Rightarrow UB_2^3 = 1.
\end{aligned}$$

Bidder 2's lower bound equals her (inverse) Marshallian demand curve, which can be solved according to the following equations:

$$\begin{aligned}
&u_2(0,0) = u_2(1, LB_2^1) \Rightarrow LB_2^1 = 1,\\
&u_2(1, LB_2^2) = u_2(2, 2LB_2^2) \Rightarrow LB_2^2 = \frac{1+r}{1+2r},\\
&u_2(2, 2LB_2^3) = u_2(3, 3LB_2^2) \Rightarrow LB_2^3 = \frac{1}{1+r}.
\end{aligned}$$

Note that the difference between bidder 2's upper and lower bounds approaches zero as the interest rate r approaches zero. Bidder 1's only undominated bid is $b^1 = (1-\epsilon, 1-\epsilon, 1-\epsilon)$. Bidder 2 has positive wealth effects and has many undominated bids. Suppose bidder 2 bids her upper bound: $b^2 = (1,1,1)$. Then, if bidder 1 plays her dominant strategy, bidder 2 wins all three units and pays $3 - 3\epsilon$. As $\epsilon \to 0$, this allocation approaches allocation a from Example 2.3, and the resulting deadweight loss is approximately $2r/(1+r)$.

3.2 Inefficiency Bound

Our next step translates the bounds on undominated bids into bounds on the deadweight loss associated with undominated Vickrey auction outcomes. Our main theorem quantifies the worst case deadweight loss when bidders have wealth effects that are at most k (i.e. $u_i \in \mathcal{U}(k) \forall i \{1, \ldots, N\}$). We find a tight upper bound on the worst case inefficiency.

Theorem 3.2. *(Worst Case Inefficiency, Vickrey Auction).* Suppose there are M units and N bidders with preferences in $u \in \mathcal{U}(k)^N$. The maximal deadweight loss associated with any undominated outcome in the Vickrey auction is

$$\overline{I}(k, V) = \begin{cases} \sum_{j=1}^{M-1} \frac{(M-j)k}{1-jk} & \text{if } k < 1/M, \\ (M-1) & \text{otherwise.} \end{cases} \tag{2}$$

Also, there exist $(u_1, u_2, \ldots, u_N) \in \mathcal{U}(k)^N$ and undominated Vickrey auction bid curves $b^1, b^2, \ldots, b^N \in \beta$ such that the resulting deadweight loss associated with the post auction outcome exactly equals $\overline{I}(k, V)$.

Proof. The proof follows two steps. First, we fix an arbitrary N-touple of preferences and undominated bid curves and show that the value of the result post auction deadweight loss is at most $\overline{I}(k, V)$. This is shown in the appendix. In the second step, we consider a particular N-touple of preferences in $\mathcal{U}(k)^N$ and corresponding undominated bid curves. We verify that the resulting deadweight loss from the post auction outcome equals $\overline{I}(k, V)$ to demonstrate that our bound is tight.

To see this second step, consider the case where bidders have the following preferences:

$$u_1(q, t) = \begin{cases} \frac{1-(1-k)^q-kt}{k(1-k)^{q-1}} & \text{if } kM \leq 1 \\ \frac{1-(1-1/M)^q-t/M}{1/M(1-1/M)^{q-1}} & \text{otherwise.} \end{cases} \tag{3}$$

$$u_2(q, t) = (1-\epsilon)q - t, \tag{4}$$

where ϵ is arbitrarily close to zero and $k \in (0, 1)$. We may assume that all other bidders have negligible demands for units of the auctioned unit. For our proof, it is sufficient to show we can attain the maximal level of deadweight loss $\overline{I}(k, V)$ in a two bidder setting. In the appendix, we provide straightforward calculations proving that these preferences are in $\mathcal{U}(k)$.

Theorem 3.1 shows us that the upper bound on bidder 1's undominated bids equals 1 for all units. The undominated outcome that produces the most deadweight loss is when bidder 1 bids the upper bound on her undominated bids. Bidder 2's upper and lower bounds equal $1-\epsilon$ for all units, because she has quasilinear preferences. Thus, bidder 1 bids 1 for all M bids, and bidder 2 bids $1-\epsilon$ for all M bids. The auction outcome is that bidder 1 wins all M units and pays $M(1-\epsilon)$ in total. Bidder 2 wins nothing and pays nothing. In a hypothetical perfectly competitive resale market, bidder 2 would then have a demand of $1-\epsilon$ for each unit. Bidder 1 would sell $M-1$ of the units that she won in the Vickrey auction. Straightforward calculations show that bidder 1's inverse supply for her j^{th} unit approaches $\max\{0, (1-Mk)/(1-kj)\}$ as $\epsilon \to 0$. A final brute-force calculation shows that the consumer and producer surplus of this hypothetical resale market exactly equals right hand side of (2), which completes the proof. These calculations are in the appendix. □

To understand the proof intuitively, recall that we are looking for the undominated outcome of the Vickrey auction that generates the largest deadweight loss. We show that this worst-case outcome occurs when one bidder (without loss of generality, bidder 1) bids the upper bound on her undominated bids. In our example, bidder 1's upper bound is to bid 1 for each unit. Note that it is a best response for bidder 1 to bid her upper bound only when her rival bids zero on zero for all units other than her first unit (and hence, bidder 1 pays 0 to win each of her first $M-1$ units). However, in our example, bidder 2 has quasilinear preferences and her marginal value for each unit is arbitrarily close to the upper bound on bidder 1's bid curve. Therefore, in the undominated outcome, bidder 2 truthfully bids her marginal value, which is $1-\epsilon$, for each unit. Consequently, bidder 1 wins M units, but pays more than her willingness to pay for the last $M-1$ units that she wins, because her demand for later units is lower when she pays a relatively high price for early units, as she does in this particular outcome. Thus, bidder 1 is willing to sell her final $M-1$ units at price below bidder 2's willingness to pay for each unit. That gap between bidder 1's willingness to sell her final $M-1$ units and bidder 2's willingness to pay for those additional units is the missing surplus from the auction outcome that creates gainful trade in the resale market. The gains from trade in the resale market are larger when bidder 1 has stronger wealth effects, because in this case, bidder 1's relatively high auction payment for her earlier units reduces her willingness to sell her later units by a larger amount, and thus increases the surplus available in the hypothetical resale market.

Notice that when wealth effects are sufficiently strong, then bidder 1's willingness to sell her final $M-1$ units diminishes to an arbitrarily small amount. In this case, her resale supply curve is near zero for the first $M-1$ units that she sells. Consequently, the surplus gained in the hypothetical competitive resale market is approximately bidder 2's value for the $M-1$ units that she would buy, which is $(M-1)(1-\epsilon)$. Conversely, when wealth effects are sufficiently weak, we see that the maximal level of inefficiency becomes arbitrarily small.

4 Other Common Multi-unit Auctions

For comparison, we complete the same exercise for the Uniform-price and Discriminatory auctions. In both, we again find bounds on undominated bid behavior and use these bounds to find the maximal deadweight loss associated with any undominated auction outcome. Unsurprisingly, the bounds on bid behavior in these two auctions are less tight relative to the Vickrey auction. Thus, the inefficiency upper bound for both auctions strictly exceed the bound we found in the Vickrey auction $\overline{I}(k, V)$.

4.1 Uniform-price Auction

We first consider the Uniform-price auction. Recall, all winners pay a fixed price for each unit that they win, and this price equal to the lowest winning bid. Prior literature shows us that bidders do not overreport their demand curves, and instead often bid below their marginal value, in the uniform price auction. The intuition for why reporting a bid above a bidder's inverse demand curve is dominated is the same as the well-known intuition for why bidders underbid in equilibrium in the Uniform-price auction (see Ausubel et al. (2014), for example). If a bidder wins some units, then it is possible that her first losing bid is the $M+1^{st}$ highest bid submitted in total, and thus her bid sets the clearing price. In this case, the bidder benefits by underreporting her demand. However, a bidder can never benefit by submitting bids that are above her inverse demand curve. If submitting a bid above her if inverse demand changed the auction outcome (relative to the case where her bids equal her inverse demand curve), then she either increased the market clearing price and/or won more units than she desired at the market clearing price. However, note that a bidder always best responds by truthfully reporting her first bid, because bidders submit weakly decreasing bid schedules, and thus, it is impossible for a bidder's first bid to set the clearing price for any units that she wins.

In the space below, we cite a result fromBaisa (2016) that gives an upper bound on undominated bid behavior in the uniform-price auction, for our particular setting, where bidders have non-quasilinear preferences.

Proposition 4.1. (Baisa, 2016). In the Uniform-price auction, the bid $b_i = (b_i^1, b_i^2, \ldots, b_i^M)$ is

1. undominated only if $b_i^1 = d_i^1$ and

$$b_i^q \in [0, LB_i^q] \forall q \in \{2, \ldots, M\}.$$

2. weakly dominated by $\tilde{b_i}$ where $\tilde{b_i}^1 = LB_i^1$ and $\tilde{b_i}^q = \min\{b_i^q, LB_i^q\} \forall q > 1$ if b_i is such that $b_i^{q'} > LB_i^{q'}$ for some $q' \in \{1, \ldots, M\}$.

Unlike in the Vickrey auction, the bounds on undominated bid behavior do not depend on our upper bound on the strength of bidder wealth effects. Therefore, the worst case inefficiency is independent of the strength wealth effects.

Proposition 4.2. *(Worst Case Inefficiency, Uniform-price Auction).* Suppose there are M units and N bidders with preferences in $\mathcal{U}(k)$. The worst case inefficiency is

$$\overline{I}(k, UP) = (M - 1) \forall k \in [0, 1]$$

Furthermore, there exist $u_1, u_2, \ldots, u_N \in \mathcal{U}(k)$ and undominated Uniform auction bid curves $b^1, b^2, \ldots, b^N \in \beta$ such that the resulting resale market gains from trade are exactly $\overline{I}(k, UP)$.

Proof. The proof is provided in the appendix. □

The proof structure is similar to that of Theorem 3.2. First, we establish that $(M - 1)$ is an inefficiency upper bound. Then, example bid curves prove that this bound is tight. For this auction, an example where the worst case inefficiency is realized when there are two bidders, each with quasilinear preferences. One bidder has a marginal value of one for each unit and the other has marginal value equal to ϵ for each unit. The first bidder bid zero for every unit after her first unit, and the second bidder just outbids bidder 1 for each of these last $M-1$ units. Thus, the second, low demand bidder wins all but one of the units, and this yields a large surplus generating resale market. Moreover, since we present a quasilinear example in which we achieved the worst-case bound, we know the bound is invariant to any value of the the upper bound on wealth effects k.

4.2 Discriminatory Auction

Finally, we repeat the exercise for the Discriminatory auction, where payment from a winning bid is equal to the bid itself. Following a similar intuition as the Uniform-price auction, there is no overreporting in the Discriminatory auction.

The lower bound on undominated bid behavior is zero for all units, including the first. The intuition is the same intuition for why any bid between a zero and a bidder's willingness to pay is an undominated in a first price auction. We formalize this in the following proposition:

Proposition 4.3. In the Discriminatory auction, the bid $b_i = (b_i^1, b_i^2, \ldots, b_i^M)$ is

1. undominated only if
$$b_i^q \in [0, LB_i^q] \forall q \in \{1, \ldots, M\}.$$
2. weakly dominated by $\tilde{b_i}$ where $\tilde{b_i}^q = \min\{b_i^q, LB_i^q\} \forall q$ if b_i is such that $b_i^{q'} > LB_i^{q'}$ for some $q' \in \{1, \ldots, M\}$.

Proof. The first point follows from the second and the second point is shown in the appendix. □

These bounds on undominated bid behavior yield the following worst case inefficiency.

Proposition 4.4. *(Worst Case Inefficiency, Discrimanatory Auction).* Suppose there are M units and N bidders with preferences in $\mathcal{U}(k)$. The worst case inefficiency is

$$\overline{I}(k, D) = M.$$

Furthermore, there exist $u_1, u_2, \ldots, u_N \in \mathcal{U}(k)$ and undominated Discriminatory auction bid curves $b^1, b^2, \ldots, b^N \in \beta$ such that the resulting resale market gains from trade are exactly $\overline{I}(k, D)$.

Proof. The proof is provided in the appendix. □

An example where undominated bidding yields the worst case inefficiency outcome occurs can be seen with two bidders that have quasilinear preferences. The first bidder has a marginal value of one for all M units, and the other bidder has marginal value ϵ for all M

units. There is an undominated outcome where the first, high demand, bidder bids arbitrarily close to zero for each unit; while the other, low demand bidder, bids only slightly higher. In this case, the deadweight loss associated with the post auction outcome is arbitrarily close to M.

5 Extensions

Our main results calculate the worst case inefficiency given minimal information about preferences and strategies. However, we provide tighter bounds on the Vickrey auction's inefficiency if we have additional information on bidder preferences (beyond simply noting that bidders have preferences in $\mathcal{U}(k)$) or if we observe actual bid data. In this subsection, we present two cases where we can get a tighter bounds on the Vickrey auction's inefficiency. In the first case, we assume that bidders have quasilinear preferences, but also face a soft budget constraint. In the second case, we show how to calculate the worst case inefficiency after observing bids.

5.1 Soft Budget Constraint Preferences

Suppose that bidders have quasilinear preferences but face a soft budget constraint preferences as defined by Hafalir et al. (2012). That is, bidders are able to spend in excess of their budget, but they must pay interest rate r on all spending above their budget. Thus, their marginal utility of money is $1 + r$ whenever their spending exceeds their budget. Note that the degree of wealth effects is thus $\frac{r}{1+r}$. As the interest rate r approaches infinity, the bidders have hard budget constraints and the degree of wealth effects approaches one.

For simplicity, we restrict attention to a two bidder case. The following proposition shows us that our worst-case inefficiency of the Vickrey auction within the soft budget setting is below the maximal inefficiency described in Theorem 3.2.

Proposition 5.1. *(Worst Case Inefficiency, Vickrey Auction, Soft Budget Constraint Preferences).* Suppose there are M units and 2 bidders with soft budget constraint preferences facing interest rate at most r, meaning wealth effects are at most $k = r/(1 + r)$. The worst case inefficiency is

$$\overline{I}(k, V) \in [k(M - 1), \max\{k(M - 1), k(M - 2)(2 - k)\}]. \quad (5)$$

Proof. See appendix. □

With some case checking, it is straightforward to show that the upper limit of (5) is always smaller than (2)

The worst case inefficiency is achieved when two high valuation bidders compete. One of the bidders is not budget constrained and has constant valuation $1 - \epsilon$. The second bidder has a budget constraint starting at one. The first bidder bids her only dominant strategy, her valuation of $1 - \epsilon$. The second bidder falsely believes that she will not pay anything for early units, and thus bids one for all units, winning them all. The second bidder is willing to sell her units in the hypothetical resale market because she far exceeded her budget in the Vickrey auction and needs liquidity to pay down her interest.

The worst case inefficiency in (5) smoothly increases with the interest rate. If bidders do not face a positive interest rate, then the Vickrey auction efficiently allocates the units. In contrast, as the interest rate grows, the auction inefficiency approaches the maximal value of all but one unit $M - 1$.

5.2 Known Bid Curves

The other way in which there might be more available information is with respect to equilibrium strategies. For example, suppose that the realized equilibrium bid curves and allocation in the Vickrey auction are known. A natural question is to evaluate the success of the auction outcome from an efficiency standpoint. In general, it is not possible to evaluate the success of an allocation without knowledge about preferences. Therefore, one can calculate the worst case inefficiency given observed bid curves and knowledge that wealth effects are at most k.

For simplicity, consider a Vickrey auction with two bidders and two units. The following proposition relates observed bid curves (b_1^1, b_1^2) and (b_2^1, b_2^2) to the maximum possible size of a hypothetical resale market.

Proposition 5.2. *(Worst Case Inefficiency, Vickrey Auction, Two Bidders, Two Units, Known Bid Curves).* Suppose there are 2 units and 2 bidders with preferences in $\mathcal{U}(k)$. The equilibrium bid curve for bidder 1 is restricted to (b_1^1, b_1^2) and the equilibrium bid curve for bidder 2 is restricted to (b_2^1, b_2^2), where $1 = b_1^1 \geq b_2^1$. The the worst case inefficiency is

$$I^\star \leq \max \left\{ \frac{b_2^1 - b_1^2 - k b_2^2}{1 + k}, \frac{b_1^2 - b_2^2}{1 + k}, 0 \right\}. \tag{6}$$

Proof. See appendix. □

The three terms over which the maximum is taken correspond to, respectively, resale markets where bidder 1 sells one of her two goods goods, bidder 1 buys a second good, and no trade occurs. At most one unit is traded in the resale market because the Vickrey auction always allocates one unit efficiently.

6 Conclusion

In this paper, we consider a multi-unit auction where bidders may have non-quasilinear preferences. Many recent impossibility results show us that there is no mechanism that retains VCG's desired incentive and efficiency properties without quasilinearity. This paper shows us that in non-quasilinear settings, the degree to which the Vickrey auction fails to robustly meet its commonly cited objective depends on the degree of bidder wealth effects. When the degree of wealth effects are sufficiently small, the Vickrey auction is arbitrarily close to meeting the commonly cited efficiency and incentive properties, even though no mechanism satisfies those properties precisely. Other multi-unit auctions do not preform as well.

In order to make this conclusion, we develop a deadweight loss measure of the inefficiency and apply it to common multi-unit auctions. We derive a tight upper bound on this inefficiency in the Vickrey auction as a function of bidder wealth effects. The worst case inefficiency scales with the degree of bidder wealth effects. To explain this phenomenon, we show that the range of undominated bids in the Vickrey auction increases with the degree of bidder wealth effects. We extend the analysis to consider cases where bidders have soft budget constraint preferences or where bid curves are known.

References

Ausubel, L. M., P. Cramton, M. Pycia, M. Rostek, and M. Weretka (2014). Demand reduction and inefficiency in multi-unit auctions. *The Review of Economic Studies 81*(4), 1366–1400.

Azar, Y., M. Feldman, N. Gravin, and A. Roytman (2017). Liquid price of anarchy. In *International Symposium on Algorithmic Game Theory*, pp. 3–15. Springer.

Babaioff, M., R. Lavi, and E. Pavlov (2006). Single-value combinatorial auctions and implementation in undominated strategies. In *SODA*, Volume 6, pp. 1054–1063. Citeseer.

Baisa, B. (2016). Overbidding and inefficiencies in multi-unit vickrey auctions for normal goods. *Games and Economic Behavior 99*, 23–35.

Baisa, B. (2017). Auction design without quasilinear preferences. *Theoretical Economics 12*(1), 53–78.

Baisa, B. (2020). Efficient multiunit auctions for normal goods. *Theoretical Economics 15*(1), 361–413.

Bikhchandani, S., S. Chatterji, R. Lavi, A. Mu'alem, N. Nisan, and A. Sen (2006). Weak monotonicity characterizes deterministic dominant-strategy implementation. *Econometrica 74*(4), 1109–1132.

Börgers, T. (1991). Undominated strategies and coordination in normalform games. *Social Choice and Welfare 8*(1), 65–78.

Börgers, T. and D. Smith (2012). Robustly ranking mechanisms. *American Economic Review 102*(3), 325–29.

Borgs, C., J. Chayes, N. Immorlica, M. Mahdian, and A. Saberi (2005). Multi-unit auctions with budget-constrained bidders. In *Proceedings of the 6th ACM Conference on Electronic Commerce*, pp. 44–51.

Caragiannis, I. and A. A. Voudouris (2016). Welfare guarantees for proportional allocations. *Theory of Computing Systems 59*(4), 581–599.

Che, Y.-K. and I. Gale (1998). Standard auctions with financially constrained bidders. *The Review of Economic Studies 65*(1), 1–21.

Chiesa, A., S. Micali, and Z. A. Zhu (2015). Knightian analysis of the vickrey mechanism. *Econometrica 83*(5), 1727–1754.

Cook, P. J. and D. A. Graham (1977). The demand for insurance and protection: The case of irreplaceable commodities. In *Foundations of Insurance Economics*, pp. 206–219. Springer.

Dobzinski, S., R. Lavi, and N. Nisan (2012). Multi-unit auctions with budget limits. *Games and Economic Behavior 74*(2), 486–503.

Dobzinski, S. and R. Paes Leme (2014). Efficiency guarantees in auctions with budgets. In *International Colloquium on Automata, Languages, and Programming*, pp. 392–404. Springer.

Gibbard, A. (1973). Manipulation of voting schemes: a general result. *Econometrica: journal of the Econometric Society*, 587–601.

Goel, G., V. Mirrokni, and R. Paes Leme (2015). Polyhedral clinching auctions and the adwords polytope. *Journal of the ACM (JACM) 62*(3), 1–27.

Hafalir, I. E., R. Ravi, and A. Sayedi (2012). A near pareto optimal auction with budget constraints. *Games and Economic Behavior 74*(2), 699–708.

Holmström, B. (1979). Groves' scheme on restricted domains. *Econometrica: Journal of the Econometric Society*, 1137–1144.

Kazumura, T., D. Mishra, and S. Serizawa (2020). Mechanism design without quasilinearity. *Theoretical Economics 15*(2), 511–544.

Kazumura, T. and S. Serizawa (2016). Efficiency and strategy-proofness in object assignment problems with multi-demand preferences. *Social Choice and Welfare 47*(3), 633–663.

Krishna, V. (2009). *Auction theory*. Academic press.

Lavi, R. and M. May (2012). A note on the incompatibility of strategy-proofness and pareto-optimality in quasi-linear settings with public budgets. *Economics Letters 115*(1), 100–103.

Ma, H., R. Meir, and D. C. Parkes (2018). Social choice with non quasi-linear utilities. *arXiv preprint arXiv:1804.02268*.

Morimoto, S. and S. Serizawa (2015). Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price walrasian rule. *Theoretical Economics 10*(2), 445–487.

Pai, M. M. and R. Vohra (2014). Optimal auctions with financially constrained buyers. *Journal of Economic Theory 150*, 383–425.

Satterthwaite, M. A. (1975). Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of economic theory 10*(2), 187–217.

Syrgkanis, V. and E. Tardos (2013). Composable and efficient mechanisms. In *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*, pp. 211–220.

Yamashita, T. (2015). Implementation in weakly undominated strategies: Optimality of second-price auction and posted-price mechanism. *The Review of Economic Studies 82*(3), 1223–1246.

A Appendix: Proofs

A.1 Proof of Theorem 3.1

It suffices to show that, in the Vickrey auction, the bid $b_i = (b_i^1, b_i^2, \dots, b_i^M)$ is weakly dominated by $\tilde{b}_i$ where $\tilde{b}_i^q = \min\{b_i^q, UB_i^q\} \forall q$ if b_i is such that $b_i^{q'} > UB_i^{q'}$ for some $q' \in$

$\{1,\ldots,M\}$. First, we show $V_i(\tilde{b}^i, b^{-i}) \geq V_i(b^i, b^{-i})$ for all $b^{-i} \in \mathcal{B}^{N-1}$. Let x be the number of units bidder i wins when she bids b^i and $\tilde{x}$ be the number of units that bidder i wins when she bids $\tilde{b}^i$.

Case 1: Bidder i wins the same number of objects if when she bids b^i or $\tilde{b}^i$. Under each bid curve, the bidder pays the same amount. This is because the marginal price of an additional unit is based on other bidders' bids. Thus, her payoff is the same.

Case 2: Bidder i wins more objects by bidding $\tilde{b}^i$ than bidding b^i. Here we assume $\tilde{x} > x$. By the argument in Case 3 of Proposition 2 in Baisa (2016), $\tilde{b}^i_j > b^i_j \;\forall j \in \{x+1,\ldots,\tilde{x}\}$. Thus, by construction of $\tilde{b}^i$, $x = 0, \tilde{x} = 1$, and $\tilde{b}^i_1 = d^1_i$. Because $\tilde{b}^i$ wins a unit but b^i does not, it must be that $b^i_1 < c^{-i}_M < \tilde{b}^i_1 = d^1_i$ and $c^{-i}_{M-1} > \tilde{b}^i_2$. Bidder i pays c^{-i}_M when bidding $\tilde{b}^i$. Thus by the definition of d^1_i,

$$V_i(\tilde{b}^i, b^{-i}) = u_i(1, c^{-i}_M) \geq u_i(1, d^1_i) = u^i(0,0) = V_i(b^i, b^{-i}).$$

Case 3: Bidder i wins fewer objects by bidding $\tilde{b}^i$ than bidding b^i. Here we assume $x > \tilde{x}$. By the argument in Case 3 of Proposition 2 in Baisa (2016), $b^i_j > \tilde{b}^i_j \;\forall j \in \{\tilde{x}+1,\ldots,x\}$. By construction, this implies $\tilde{b}^i_j = UB^j_i$ for $j \in \{\tilde{x}+1,\ldots,x\}$. By the definition of UB_i and the fact that $d^q_i(\cdot)$ is decreasing,

$$\begin{aligned} V_i(\tilde{b}^i, b^{-i}) &= u_i\left(\tilde{x}, \sum_{j=1}^{\tilde{x}} c^{-i}_{M+1-j}\right) = u_i\left(\tilde{x}+1, \sum_{j=1}^{\tilde{x}} c^{-i}_{M+1-j} + d^{\tilde{x}+1}_i\left(\sum_{j=1}^{\tilde{x}} c^{-i}_{M+1-j}\right)\right) \\ &\geq u_i\left(\tilde{x}+1, \sum_{j=1}^{\tilde{x}} c^{-i}_{M+1-j} + d^{\tilde{x}+1}_i(0)\right) = u_i\left(\tilde{x}+1, \sum_{j=1}^{\tilde{x}} c^{-i}_{M+1-j} + \tilde{b}^i_{\tilde{x}+1}\right). \end{aligned}$$

Because bidding $\tilde{b}^i$ means that bidder i loses the $\tilde{x}+1$st unit, $c^{-i}_{M-\tilde{x}} \geq \tilde{b}^i_{\tilde{x}+1}$. Therefore,

$$u_i\left(\tilde{x}+1, \sum_{j=1}^{\tilde{x}} c^{-i}_{M+1-j} + \tilde{b}^i_{\tilde{x}+1}\right) \geq u_i\left(\tilde{x}+1, \sum_{j=1}^{\tilde{x}+1} c^{-i}_{M+1-j}\right).$$

Continuing this argument inductively, we have that

$$V_i(\tilde{b}^i, b^{-i}) \geq u_i\left(x, \sum_{j=1}^{x} c^{-i}_{M+1-j}\right) = V_i(b^i, b^{-i}).$$

Second, we show that there exists $b^{-i} \in \mathcal{B}^{N-1}$ such that $V_i(\tilde{b}^i, b^{-i}) > V_i(b^i, b^{-i})$. We consider two cases. First suppose that $b^i_1 < \tilde{b}^i_1$. Take $c^{-i}_j = \gamma \;\forall j \in \{1,\ldots,M\}$ where $\gamma \in (b^i_1, \tilde{b}^i_1)$. We know that $\tilde{b}^i_1 = d^1_i$. In addition, because bids are decreasing and $b^i_1 < \gamma$, strategy b^i wins zero units. Since $\tilde{b}^i_2 \leq b^i_2 \leq b^i_1 < \gamma$, bidding $\tilde{b}^i$ wins exactly one unit. Finally, because $\gamma < d_i^1$,

$$V_i(\tilde{b}^i, b^{-i}) = u_i(1, \gamma) > u_i(1, d_i^1) = u^i(0,0) = V_i(b^i, b^{-i}).$$

Now, instead suppose that $\tilde{b}_1^i = d_i^1 \leq b_1^i$. Then $\tilde{b}_x^i \leq b_x^i \ \forall x \in \{1, \dots, M\}$ by construction and there exists an x such that $\tilde{b}_x^i = UB_i^x < b_x^i$. Let x^* be the minimum such x. Let $c_j^{-i} = 0$ for $j \in \{1, \dots, x^* - 1\}$, let $c_{x^*}^{-i} = \gamma \in (\tilde{b}_{x^*}^i, b_{x^*}^i)$, and let $c_j^{-i} = \omega > b_{x^*}^i$ for $j > x^*$. Bidding $\tilde{b}^i$ means the bidder wins exactly $x^* - 1$ units and pays nothing and bidding b^i means the bidder wins exactly x^* units by construction, paying γ. Thus,

$$V_i(\tilde{b}^i, b^{-i}) = u_i(x^* - 1, 0) = u_i(x^*, UB_i^{x^*}) > u_i(x^*, \gamma) = V_i(b^i, b^{-i}).$$

Therefore, $\tilde{b}^i$ weakly dominates b^i. □

A.2 Proof of Theorem 3.2

For the main part of the proof, it suffices to show that for any N-touple of preferences $u \in \mathcal{U}(k)^N$ and weakly undominated bid curves $b \in \mathcal{B}^N$, the resulting deadweight loss $DWL(a(b, V))$ is at most the right hand side of (2),

$$\begin{cases} \sum_{j=1}^{M-1} \frac{(M-j)k}{1-jk} & \text{if } k < 1/M, \\ (M-1) & \text{otherwise.} \end{cases}$$

Fix admissible preferences $u \in \mathcal{U}(k)^N$ and undominated Vickrey auction bid curves $b \in \mathcal{B}^N$.

Step 1: Definitions and plan for proof. The deadweight loss $DWL(a(b, V))$ equals the maximum size of a hypothetical perfectly competitive resale market. Hereafter, we omit mention of 'hypothetical' and 'perfectly competitive' to facilitate exposition. In the resale market, let r_i denote the number of units that bidder i (hypothetically) buys or sells. Let $D \subseteq [N]$ denote the set of bidders with $r_i \geq 0$ and $S \subseteq [N]$ denote the set of bidders with $r_i < 0$ (note we include bidders who do not trade in the resale market with the demand side for convenience). We refer to bidders $i_D \in D$ as 'buyers' and bidders $i_S \in S$ as 'sellers' (though we emphasize these are categorizations in a *hypothetical* resale market). Suppose the bid curves b are such that b_D is the aggregate bid curve among all buyers $i_D \in D$ and b_S as the aggregate bid curve among all sellers $i_S \in S$. We similarly define LB_D and LB_S from the bidder lower bounds LB_i. Under allocation $a(b, V)$, suppose that the sellers win q_S units in the Vickrey auction and the buyers win the remaining $q_D = M - q_S$ units.

We will bound the size of the resale market unit by unit. That is, we imagine that buyers line up in decreasing order of (inverse) resale demand and sellers line up in increasing order of (inverse) resale supply. We bound the difference between the j^{th} largest resale demand

and the j^{th} smallest resale supply. We refer to this difference as 'the gains from trade of unit j.' More precisely, we define the market (inverse) resale demand curve $RD(j)$ as the horizontal sum of individual buyer (inverse) resale demand curves $RD_i(j)$ over buyers i in D (in an abuse of notation, we suppress that this is dependent on the allocation $a(b, V)$). We then calculate the deadweight loss as

$$\sum_{j=1}^{M} \max\{RD(j) - RS(j), 0\}.$$

This expression equals $DWL(a)$ because it represents the area between the demand and supply curves in the resale market.

The plan for proof is as follows. We begin by establishing two basic facts: the gains from trade of unit j is at most one, and the gains from trade of unit M is non-positive. These two facts imply it suffices to show that if $kM < 1$, then $RD(j) - RS(j) \leq \frac{(M-j)k}{1-jk}$ for $j \in \{1, 2, \ldots, M-1\}$. The later steps of the proof bound $RD(j) - RS(j)$ assuming $kM < 1$.

Step 2: The gains from trade of of unit j is at most one. Consider the gains from trade of unit j. The supply $RS(j)$ is at least zero because the units are goods. Furthermore, demand is at most one, including in the resale market. Therefore, the gains from trade are at most $1 - 0 = 1$.

Step 3: The gains from trade of unit M is non-positive. In other words, it will never be the case that the hypothetical resale market ends with the buyers buying all of the units from the sellers. The gains from trade of unit M can only be positive if the sellers win M units in the Vickrey auction and the buyers win zero units. Accordingly, suppose $q_S = M$ and $q_D = 0$.

First, consider the demand side. In the resale market, the demand for unit M $RD(M)$ is the Mth largest resale demand, which is at most the largest resale demand $RD(1)$. Because $q_D = 0$, the largest resale demand equals the buyer's demand for the first object in the Vickrey auction LB_D^1. For every bidder, the only weakly undominated bid for the first item is demand itself. Therefore, $RD(M) \leq LB_D^1 = b_D^1$.

Second, consider the supply side. The supply for unit M is the Mth smallest resale supply. By definition, there exists a seller $i_S \in S$ such that $RS(M) = RS_{i_S}(q_{i_S})$. Since the only undominated bid for the first unit is $LB_{i_S}^1$, the most that seller i_S pays in the Vickrey auction is $q_{i_S} LB_{i_S}^1$. Let $RS_{i_S}^{\star}(q_{i_S})$ denote seller i_S's resale supply for unit q_{i_S} under a counterfactual allocation where seller i_S pays $q_{i_S} LB_{i_S}^1$ in the Vickrey auction to win q_{i_S} units. Because units are normal, decreasing seller i_S's wealth can only decrease her supply curve, so $RS_{i_S}(q_{i_S}) \geq RS^\star_{i_S}(q_{i_S})$. $RS^\star_{i_S}(q_{i_S})$ solves the following equation:

$$u_{i_S}(0, q_{i_S} LB^1_{i_S} - q_{i_S} RS^\star_{i_S}(q_{i_S})) = u_{i_S}(1, q_{i_S} LB^1_{i_S} - (q_{i_S} - 1) RS^\star_{i_S}(q_{i_S})).$$

Setting $RS^\star_{i_S}(q_{i_S})$ equal to $LB^1_{i_S}$, this equation reduces to

$$u_{i_S}(0,0) = u_{i_S}(1, LB^1_{i_S}),$$

which is true by the definition of $LB^1_{i_S}$. Therefore, $RS^\star_{i_S}(q_{i_S} = LB^1_{i_S} = b^1_{i_S}$. Because the buyers did not win any units, $b^1_{i_S} \geq b^1_D$. Therefore, $RS(M) = RS_{i_S}(q_{i_S}) \geq RS^\star_{i_S}(q_{i_S}) = LB^1_{i_S} = b^1_{i_S} \geq b^1_D$. Combining the demand and supply side,

$$RD(M) - RS(M) \leq b^1_D - b^1_D = 0,$$

completing this step of the proof.

Step 4: Preliminary Bounds on the Gains from Trade of Unit j. Using Step 3, it suffices to calculate $RD(j) - RS(j)$ for each unit $j \in \{1, 2, \ldots, M-1\}$ because at most $M-1$ units have positive gains from trade. Using Step 2, $RD(j) - RS(j) \leq 1$ for all j so $DWL(a) \leq M - 1$. Note that $\frac{(M-j)k}{1-jk} < 1$ if and only if $kM < 1$. Therefore, it suffices to show that, whenever $kM < 1$, $RD(j) - RS(j) \leq \frac{(M-j)k}{1-jk}$. Assume $kM < 1$ and let $j \in \{1, 2, \ldots, M-1\}$ for the remainder of the proof.

We place a lower bound on the supply for unit j, $RS(j)$. By the definition of market supply as the horizontal sum of individual supply curves, there exists a seller $i_S \in S$ and unit $\ell \leq j$ such that

$$RS(j) = s^{q_{i_S} - \ell + 1}_{i_S}(t_{i_S} - (\ell - 1)RS(j)) = d^{q_{i_S} - \ell + 1}_{i_S}(t_{i_S} - \ell RS(j)), \tag{7}$$

by the characterization of the willingness to sell function. Because bidders have positive wealth effects at most k, we get that

$$\begin{aligned} d^{q_{i_S}-\ell+1}_{i_S}(t_{i_S} - \ell RS(j)) &\geq d^{q_{i_S}-\ell+1}_{i_S}(t_{i_S} - jRS(j)) \\ &= d^{q_{i_S}-\ell+1}_{i_S}(0) - (d^{q_{i_S}-\ell+1}_{i_S}(0) - d^{q_{i_S}-\ell+1}_{i_S}(t_{i_S} - jRS(j))) \\ &\geq d^{q_{i_S}-\ell+1}_{i_S}(0) - k(t_{i_S} - jRS(j)). \end{aligned} \tag{8}$$

Note that the first inequality holds because we assume bidders have weakly positive wealth effects and $\ell \leq j$. The final inequality holds from the implication of wealth effects being at most k. Combining (7) and (8),

$$RS(j) \geq d_{i_S}^{q_{i_S}-\ell+1}(0) - k(t_{i_S} - jRS(j))$$
$$\Rightarrow RS(j) \geq \frac{d_{i_S}^{q_{i_S}-\ell+1}(0) - kt_{i_S}}{1-kj},$$

using the fact that $kj < kM < 1$ by assumption. Next, by construction of the Vickrey auction, bidder i_S (who is a seller in the hypothetical post auction perfectly competitive resale market) paid at most her bid for last unit, $b_{i_S}^{q_{i_S}}$, on every unit she won in the auction, meaning $t_{i_S} \leq b_{i_S}^{q_{i_S}} q_{i_S}$, implying

$$RS(j) \geq \frac{d_{i_S}^{q_{i_S}-\ell+1}(0) - kb_{i_S}^{q_{i_S}} q_{i_S}}{1-kj} \geq \frac{b_{i_S}^{q_{i_S}} - kb_{i_S}^{q_{i_S}} q_{i_S}}{1-kj}. \tag{9}$$

The last inequality follows because, by definition, $d_{i_S}^{q_{i_S}-\ell+1}(0) = UB_{i_S}^{q_{i_S}-\ell+1}$, which is weakly larger than the bid for unit q_{i_S} by Theorem 3.1. Because $q_{i_S} \leq q_S$, we can rewrite (9) as:

$$RS(j) \geq \frac{b_{i_S}^{q_{i_S}}(1-kq_S)}{1-kj}. \tag{10}$$

Note that (10) is non-negative because $kq_S \leq kM < 1$.

Next, we incorporate information about the demand curve to bound the gains from trade. To relate the supply and demand curves, we replace $b_{i_S}^{q_{i_S}}$ with a model primitive. Bid $b_{i_S}^{q_{i_S}}$ is weakly larger than the last winning seller bid, $b_S^{q_S}$, which is in turn larger than the first losing buyer bid, $b_D^{q_D+1}$. Therefore,

$$b_{i_S}^{q_{i_S}} \geq b_S^{q_S} \geq b_D^{q_D+1} \geq LB_D^{q_D+1}, \tag{11}$$

using the fact that all bids are at least the lower bound. The following equation combines (10) with (11) to place a bound on the difference between demand and supply (this is the object of interest):

$$\begin{aligned}
RD(j) - RS(j) &\leq RD(j) - \frac{b_{i_S}^{q_{i_S}}(1-kq_S)}{1-kj} \\
&\leq LB_D^{q_D+1} - \frac{LB_D^{q_D+1}(1-kq_S)}{1-kj} + (RD(j) - LB_D^{q_D+1}) \\
&= \frac{LB_D^{q_D+1}(q_S - j)k}{1-kj} + (RD(j) - LB_D^{q_D+1}) \\
&\leq \frac{(q_S - j)k}{1-kj} + (RD(j) - LB_D^{q_D+1}),
\end{aligned} \tag{12}$$

where the last inequality follows from the fact that demand is bounded by one.

The next two steps place bounds on the second term of (12). This term is the difference between the resale demand curve and the lower bound curve in the Vickrey auction. Step 5 bounds this difference for an individual buyer and Step 6 extrapolates this bound to the market-level curve.

Step 5. Resale Demand $RD_{i_D}(j)$ Is Not Much Larger than $LB_{i_D}^{q_{i_D}+1}$. Consider a bidder who would be buyers in the hypothetical resale market, $i_D \in D$. We show that the buyers resale demand for her j unit is bounded and $RD_{i_D}(j) \leq (1 + kq_{i_D})LB_{i_D}^{q_{i_D}+1}$.

Applying the definitions of demand and resale demand,

$$\begin{aligned} RD_{i_D}(j) - LB_{i_D}^{q_{i_D}+j} &= d_{i_D}^{q_{i_D}+j}(t_{i_D} + (j-1)RD_{i_D}(j)) - d_{i_D}^{q_{i_D}+j}((q_{i_D} + j - 1)LB_{i_D}^{q_{i_D}+j}) \\ &\leq d_{i_D}^{q_{i_D}+j}((j-1)RD_{i_D}(j)) - d_{i_D}^{q_{i_D}+j}((q_{i_D} + j - 1)LB_{i_D}^{q_{i_D}+j}) \\ &\leq k((q_{i_D} + j - 1)LB_{i_D}^{q_{i_D}+j} - (j-1)RD_{i_D}(j)). \end{aligned} \tag{13}$$

The first inequality follows because the units are normal. The second inequality follows from the implications of wealth effects being at most k. Solving (13) for $RD_{i_D}(j)$ yields:

$$\begin{aligned} RD_{i_D}(j) &\leq \frac{1 + k(q_{i_D} + j - 1)}{1 + (j-1)k} LB_{i_D}^{q_{i_D}+j} \\ &\leq (1 + kq_{i_D})LB_{i_D}^{q_{i_D}+1}. \end{aligned}$$

The second inequality follows because both the fraction and $LB_{i_D}^{q_{i_D}+j}$ are decreasing in j.

Step 6. Market Resale Demand $RD_D(j)$ Is Not Much Larger than $LB_D^{q_D+1}$. In analogy to the bound for individual buyers, we show $RD(j) \leq (1 + kq_D)LB_D^{q_D+1}$. Using Step 5, for all buyers $i_D \in D$,

$$RD_{i_D}(j) \leq (1 + kq_D)LB_D^{q_D+1} \tag{14}$$

because $q_{i_D} \leq q_D$ and $LB_D^{q_D+1}$ is equal to the maximum of $LB_{i_D}^{q_{i_D}+1}$ across all buyers $i_D \in D$. Therefore, since individual demand for all units j is at most the right hand side of (14), market demand for all units must be at most the right hand side of (14).

Step 7: Final Bounds on the Gains from Trade of Unit j. Applying Step 6 to (12),

$$\begin{aligned} RD(j) - RS(j) &\leq \frac{(q_S - j)k}{1 - kj} + ((1 + kq_D)LB_D^{q_D+1} - LB_D^{q_D+1}) \\ &\leq \frac{(q_S - j)k}{1 - kj} + kq_D = \frac{(q_S - j)k}{1 - kj} + k(M - q_S), \end{aligned} \tag{15}$$

by the fact that demand is at most one and the definition of q_D. Because the right hand side of (15) is non-decreasing in q_S and $q_S \leq M$,

$$RD(j) - RS(j) \leq \frac{(M-j)k}{1-kj}.$$

Step 8: The bound is tight. To prove that the bound is tight, we have to find preferences $u_1, u_2, \ldots, u_N \in \mathcal{U}(k)$ and undominated bids $b \in \mathcal{B}$ such that $DWL(a(b,V))$ equals the right hand side of (2). Consider the following preferences:

$$u_1(q,t) = \begin{cases} \frac{1-(1-k)^q - kt}{k(1-k)^{q-1}} & \text{if } kM \leq 1 \\ \frac{1-(1-1/M)^q - t/M}{1/M(1-1/M)^{q-1}} & \text{otherwise.} \end{cases}$$
$$u_2(q,t) = (1-\epsilon)q - t$$
$$u_3(q,t) = u_4(q,t) = \cdots = u_N(q,t) = -t.$$

Without loss, we can suppose that $k \leq 1/M$ because the second case in $u_1(q,t)$ replaces k with $1/M$ and also the second case in $\overline{I}(k,V)$ is the limit of the first case as $k \to 1/M$.

To begin, we derive bidder 1's willingness to sell function $s_1^q(t)$ which solves:

$$\begin{aligned} & u_1(q,t) = u_1(q-1, t - s_1^q(t)) \\ \Rightarrow & \frac{1-(1-k)^q - kt}{k(1-k)^{q-1}} = \frac{1-(1-k)^{q-1} - k(t - s_1^q(t))}{k(1-k)^{q-2}} \\ \Rightarrow & 1-(1-k)^q - kt = 1-(1-k)^q - k(t - s_1^q(t)) - k + k^2(t - s_1^q(t)) \\ \Rightarrow & s_1^q(t) = \frac{1-kt}{1-k}. \end{aligned}$$

Suppose that bidder 1 bids her upper bound of one for all units and bidder 2 bids her demand of $1-\epsilon$ for all units. Bidder 1 wins all M units and pays $M(1-\epsilon)$. Consider a hypothetical resale market where bidder 2 can buy some units from bidder 1. Bidder 2 has quasilinear preferences and thus demand $RD_2(j)$ equal to $(1-\epsilon)$ for all units. Bidder 2's supply for the j^{th} unit that bidder 2 sells in the resale market $RS_1(j)$ solves:

$$\begin{aligned} RS_1(j) &= s_1^{M-j+1}(M(1-\epsilon) - (j-1)RS_1(j)) \\ \Rightarrow RS_1(j) &= \frac{1 - k(M(1-\epsilon) - (j-1)RS_1(j))}{1-k} \\ \Rightarrow RS_1(j) &= \frac{1-kM(1-\epsilon)}{1-jk}. \end{aligned}$$

The deadweight loss thus equals

$$\begin{aligned} DWL(a(b,V)) &= \sum_{j=1}^{M} \max\{0, RD_2(j) - RS_1(j)\} \\ &= \sum_{j=1}^{M} \max\{0, 1-\epsilon - \frac{1-kM(1-\epsilon)}{1-jk}\} \\ &\to \sum_{j=1}^{M-1} \frac{(M-j)k}{1-jk}, \end{aligned}$$

completing the proof. □

A.3 Proof that (3) and (4) are in $\mathcal{U}(k)$

Consider the following preferences:

$$u_1(q,t) = \begin{cases} \frac{1-(1-k)^q - kt}{k(1-k)^{q-1}} & \text{if } kM \leq 1 \\ \frac{1-(1-1/M)^q - t/M}{1/M(1-1/M)^{q-1}} & \text{otherwise.} \end{cases}$$

The second case of u_1 simply replaces k with $1/M$ when $k > 1/M$. Since $k < k' \Rightarrow \mathcal{U}(k) \subseteq \mathcal{U}(k')$, it suffices to assume that $kM \leq 1$ and consider only the first case.

To show that $u_1 \in \mathcal{U}(k)$, we conduct five checks:

1. $u_1(q, \cdot)$ is strictly decreasing and continuous.
2. If $\tilde{q} > q$, then $u_1(\tilde{q}, t) \geq u_1(q, t)$.
3. Demand is bounded by one.
4. Demand is weakly declining.
5. Wealth effects are weakly positive and at most k.

The first item follows immediately. To show the second item, we note that we can assume without loss that $t \leq M$. This is true because in the Vickrey auction and in a hypothetical perfectly competitive resale market, payment for any one of the M goods is bounded by one. When $t \leq M$,

$$\begin{aligned} &1-k \leq 1 \Rightarrow (1-k)(1-(1-k)^q - kt) \leq 1-(1-k)^{q+1} - kt \\ \Rightarrow &u_1(q,t) \leq u_1(q+1,t). \end{aligned}$$

Therefore, the second item holds.

To show the third item, we first calculate the willingness to pay function $d_1^q(t)$, which solves:

$$\begin{aligned}
&u(q, t + d_1^q(t)) = u_1(q-1, t) \\
\Rightarrow &\frac{1-(1-k)^q - k(t + d_1^q(t))}{k(1-k)^{q-1}} = \frac{1-(1-k)^{q-1} - kt}{k(1-k)^{q-2}} \\
\Rightarrow &1-(1-k)^q - k(t + d_1^q(t)) = (1-k)(1-(1-k)^{q-1} - kt) \\
\Rightarrow &d_1^q(t) = 1 - tk.
\end{aligned}$$

The (inverse) demand function LB_1^q solves:

$$\begin{aligned}
&LB_1^q = d_1^q((q-1)LB_1^q) \\
\Rightarrow &LB_1^q = \frac{1}{1+k}
\end{aligned}$$

Demand is at most one, so the third item holds. Similarly, demand is weakly declining in q, so the fourth item holds.

To prove the fifth item, it suffices to show that the slope of $d_1^q(t)$ is in $[-k, 0]$. This holds because the slope of $d_1^q(t)$ is $-k$. Therefore, $u_1(q,t) \in \mathcal{U}(k)$.

Finally, we move to the second utility function $u_2(q,t) = (1-\epsilon)q - t$. These are quasilinear preferences and thus an element of $\mathcal{U}(0) \subseteq \mathcal{U}(k)$.

A.4 Proof of Proposition 4.2

The proof proceeds in two steps. First, we show that for any N-touple of preferences $u \in \mathcal{U}(k)^N$ and undominated Uniform-price auction bid curves $(b_1, b_2, \ldots, b_N) \in \mathcal{B}^N$, and resulting allocation $a(b, UP)$, the resulting deadweight loss $DWL(a(b, UP))$ is at most the right hand side of (2). Second, we show that there exist preferences $u \in \mathcal{U}(k)^N$ and undominated bid curves such that the inefficiency equals the right hand side of (2).

Step 1. Consider the same basic definitions from the proof of Theorem 3.2. We begin by establishing two preliminary facts about the equilibrium. First, for all units $j \in \{1, 2, \ldots, M\}$, the resale gains from trade $RD(j) - RS(j)$ are at most one. This is true for the same reason as provided in the proof of Theorem 3.2. Second, the gains from trade of unit M, $RD(M) - RS(M)$, are not positive. This is true because the only undominated bid for the first unit is demand d_i^1 for all i. The same is true in the Vickrey auction, and this is the only property that we used to establish the same fact in the proof Theorem 3.2. Therefore, it extends to this case as well. Combining these two facts, we have that the resale gains from trade are at most $(M-1)$, completing the first step of the proof.

Step 2. To show that the deadweight loss is at least $M-1$, it suffices to find preferences

$u \in \mathcal{U}(k)^N$, undominated bids, the corresponding allocation $a(b, UP)$, and a resale market such that the gains from trade equal $M-1$. Consider the following preferences:

$$u_1(q,t) = q - t,$$
$$u_2(q,t) = u_3(q,t) = \cdots = u_N(q,t) = -t.$$

These preferences are quasilinear and thus in in $\mathcal{U}(k)$. Bidder 1's necessarily bids one for the first unit but may bid anything at most one for the subsequent units. All other bidders necessarily bid zero for all units. Suppose that Bidder 1 bids zero for all but the first unit and only wins one unit. Since bidders are quasilinear, it is clear that Bidder 1 has demand of B for the first $M-1$ units sold in the resale market and that the supply curve equals zero, meaning that resale gains from trade are equal to $M-1$. □

A.5 Proof of Proposition 4.3

First, I will show that $D_i(\tilde{b}_i, b_{-i}) \geq D_i(b_i, b_{-i})$ for all $b_{-i} \in \mathcal{B}^{N-1}$. Let $b_{-i} \in \mathcal{B}^{N-1}$. Let $j \in \{1, 2, \dots, M\}$ denote the largest index where $b_i^j > \tilde{b}_i^j$. Such a j exists because $\tilde{b}_i^{j'} \leq b_i^j$ for all $j' \in \{1, 2, \dots, M\}$ and, without loss, $b_i \neq \tilde{b}_i$. We will show that buyer i is weakly better off replacing b_i^j with $\tilde{b}_i^j$. Such a bid is still well-defined. Therefore, iterating this argument until $b_i = \tilde{b}_i$, we conclude that $D_i(\tilde{b}_i, b_{-i}) \geq D_i(b_i, b_{-i})$. Let $b_i^{(j)}$ denote the modified bid where b_i^j is replaced with $\tilde{b}_i^j$: $(b_i^1, b_i^2, \dots, b_i^{j-1}, \tilde{b}_i^j, b_i^{j+1}, \dots, b_i^m)$. Let q_i denote the number of units that i wins by bidding b_i and $q_i^{(j)}$ denote the number of units that i wins by bidder $b_i^{(j)}$. Note that $q_i \geq q_i^{(j)}$. There are three cases.

Case 1: $q_i < j$. If bidder i wins fewer than j units by bidding b_i, then by decreasing the j'th component of her bid, she will still win q_i units. She will also pay the same amount. Therefore, her payment remains the same and $D_i(\tilde{b}_i^{(j)}, b_{-i}) \geq D_i(b_i, b_{-i})$.

Case 2: $q_i \geq j$ *and* $q_i^{(j)} = q_i$. In this case, bidder i wins the same number of units but pays strictly less under $b_i^{(j)}$. Therefore, since utility is strictly decreasing in the payment conditional on a fixed number won, $D_i(\tilde{b}_i^{(j)}, b_{-i}) \geq D_i(b_i, b_{-i})$.

Case 3: $q_i \geq j$ *and* $q_i^{(j)} < q_i$. Since only one bid changed, it must be that $q_i^{(j)} = q_i - 1$. It suffices to show that bidder i's willingness to pay for her j'th unit, having paid the sum of her first $j-1$ bids for her first $j-1$ units, is at most $\tilde{b}_i^j = LB_i^j$. This will imply that i has a lower utility than she would have without buying the j^{th} unit for a price of b_i^j, which is greater than her willingness to pay. Indeed, such a claim holds:

$$d_i^j\left(\sum_{j'=1}^{j-1} b_i^{j'}\right) \leq d_i^j((j-1)b_i^j) \leq d_i^j((j-1)LB_i^j) = LB_i^j.$$

The first inequality follows from the fact that bid curves are non-increasing and the willingness to pay function decreases in its argument. The second inequality holds because $b_i^j > \tilde{b}_i^j = LB_i^j$. The final equality holds by the definition of LB_i^j. Therefore, in all three cases, $D_i(\tilde{b}_i^{(j)}, b_{-i}) \geq D_i(b_i, b_{-i})$.

The second step is to show that there exists $b_{-i} \in \mathcal{B}^{N-1}$ such that $D_i(\tilde{b}_i, b_{-i}) > D_i(b_i, b_{-i})$. Choose b_{-i} to equal zero such that bidder i wins all the units under both b_i and $\tilde{b}_i$. Then by bidding $\tilde{b}_i$, bidder i's payment strictly falls but she still wins all M units. Because utility is strictly decreasing in the amount paid, $D_i(\tilde{b}_i, b_{-i}) > D_i(b_i, b_{-i})$, completing the proof. □

A.6 Proof of Proposition 4.4

The proof proceeds in two steps. First, we show that for any N-touple of preferences $u \in \mathcal{U}(k)^N$ and undominated bids in $\mathcal{B}$, the resulting deadweight loss is at most M. This result follows immediately because there are M units for sale and demand is bounded by one. Second, we show there exist preferences $u \in \mathcal{U}(k)^N$ and undominated bids in $\mathcal{B}$ such that the resulting deadweight loss equals M. Consider the following preferences:

$$u_1(q,t) = q - t,$$
$$u_2(q,t) = u_3(q,t) = \cdots = u_N(q,t) = -t.$$

These preferences are quasilinear and thus in in $\mathcal{U}(k)$. Bidder 1 bids at most one for all units. All other bidders necessarily bid zero for all units. Suppose that Bidder 1 bids zero for all units and only wins no units. Since bidders are quasilinear, it is clear that Bidder 1 has demand of one for all M units sold in the resale market and that the supply curve equals zero, meaning that resale gains from trade are equal to M, completing the proof. □