Ordinal One-Switch Utility Functions

Ali E. Abbas

University of Southern California, Los Angeles, California 90089, aliabbas@usc.edu

David E. Bell

Harvard Business School, Boston, Massachusetts 02163, dbell@hbs.edu

Abstract

We study the problem of finding ordinal utility functions to rank the outcomes of a decision when preference independence between the attributes is not present. We propose that the next level of complexity is to assume that preferences over one attribute can switch at most once as another attribute varies from low to high. We refer to this property as ordinal one-switch independence. We present both necessary and sufficient conditions for this ordinal property to hold and provide families of functions that satisfy this new property.

1. Introduction

An ordinal utility function reflects a decision maker's rank order for the consequences of a decision, as described by a set of attributes, $X_1, ..., X_n$. For example, a function $u(x, y)$ over two attributes X and Y is an ordinal utility function when it returns a higher value for a more preferred prospect and returns equal values when two prospects are equally preferred, i.e.

$$u(x_1, y_1) > u(x_2, y_2) \Leftrightarrow (x_1, y_1) \succ (x_2, y_2) \text{ and } u(x_1, y_1) = u(x_2, y_2) \Leftrightarrow (x_1, y_1) \sim (x_2, y_2).$$

An ordinal utility function is sufficient to determine the best decision alternative if there is no uncertainty about the outcomes. The best decision alternative corresponds to the prospect with the highest ordinal utility. When uncertainty is present, a cardinal utility function is needed, and the best decision alternative is the one with the highest expected utility.

Identifying a suitable utility function to determine the best decision alternative (whether ordinal or cardinal) can be challenging. The standard approach is to decompose a multiattribute utility function into lower-order components by finding appropriate simplifications.

A wealth of literature has provided conditions for preferences over lotteries to characterize the functional form of the cardinal utility function. See for example Pfanzagl (1959), Bell (1988) and Abbas (2007) for conditions on univariate lotteries, and Farquhar (1975), Fishburn (1974 and

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1975), Keeney and Raiffa (1976), Bell (1979), Abbas (2009), and Abbas and Bell (2011 and 2012) for conditions on multivariate lotteries. Much less literature has provided ordinal conditions that characterize the utility function.

The best-known condition on ordinal preferences is based on the idea of preferential independence (Debreu 1960), which requires that ordinal preferences for consequences of a decision characterized by any subset of the attributes do not depend on the levels at which the remaining attributes are fixed. If three or more attributes satisfy this ordinal property, then the ordinal utility function must be a monotone transformation of an additive function of the attributes, i.e.

$$u(x_1,...,x_n) = g\left(\sum_{i=1}^{n} u_i(x_i) \right),$$

where $n \geq 3$, g is a monotonic function and u_i, $i = 1,...,n$ are arbitrary functions.

The condition of mutual preferential independence does not determine the ordinal functions u_i, $i = 1,...,n$, but it does decompose the functional form into univariate assessments over each of the individual attributes, which reduces the search space for the structure of the ordinal utility function significantly. Debreu's result only works if there are three or more attributes. For $n=2$ there are classic conditions which dictate the additive form, for example Karni and Safra (1998). One of the contributions of this paper is to derive the general form of preference independence for two attributes.

While ordinal preferences described by mutual preferential independence may suffice in a variety of problems, it is natural to consider how to proceed with the construction of the utility function if it does not hold.

Keeney and Raiffa (1976) introduced the notion of utility independence, where preferences for lotteries over a subset of the attributes do not depend on the levels of the remaining attributes. They derived a family of cardinal utility functions that satisfies this property. In Abbas and Bell (2011, 2012), we generalized the notion of utility independence for cardinal utility functions. The idea is to consider how many times preferences over pairs of gambles on one attribute can switch

as another attribute ranges from small to large. That is, for any two lotteries $\tilde{x}_1, \tilde{x}_2$ in X, how often can the difference in expected utilities of the lotteries, $u(\tilde{x}_1, y) - u(\tilde{x}_2, y)$, switch sign? Utility independence assumptions correspond to "no switch". In our work, we proposed a functional form that allows for preferences over lotteries to "switch at most once", and we also generalized the result to functional forms that allow for preferences to switch any number of times. But if pairwise preferences can switch multiple times as y varies one has to wonder whether the attributes have been chosen appropriately.

The purpose of this paper is to provide new conditions on ordinal preferences to help the analyst identify suitable ordinal utility functions that may be used when preferential independence conditions are not present. We start our discussion with the case of a single attribute, and discuss the implications of ordinal one-switch and zero-switch independence for wealth. Next, we consider the case of two attributes. We consider the functional form of an ordinal utility function where only one attribute is preferentially independent of the other. We then consider our new ordinal property. We provide families of functions that satisfy this ordinal one-switch property and also discuss conditions (Theorem 5.1) to identify when this property does not hold. We conclude with a new formulation (Theorem 7.3) that can be used to construct ordinal one-switch utility functions using two curves on the surface of the ordinal function, subject to reasonable conditions on its derivatives. Throughout the paper we will assume utility functions are continuous, and on occasion, differentiable.

2. Basic Notation and Definitions

To simplify the exposition, we start our analysis by considering the case of a one-attribute utility function, $u(x+w)$, where w is initial wealth and x is the increment to be considered. In our exposition, we first consider <u>lotteries</u> on X, which we write as $\tilde{x}$, denoting the expected utility of those lotteries as $u(\tilde{x} + w)$.

A utility function, $u(x+w)$ is <u>cardinal zero-switch</u> if for any two lotteries over X, say $\tilde{x}_1$ and $\tilde{x}_2$, the difference in expected utilities $D(w) = u(\tilde{x}_1 + w) - u(\tilde{x}_2 + w)$ does not change sign as w varies.

Throughout the paper, when we say that a function "does not change sign", we mean that it is either always positive, always negative, or always zero.

As has been shown by Pfanzagl (1959), the cardinal zero-switch condition is satisfied for all lotteries if and only if the utility function is a linear transform of either a linear or an exponential utility function, i.e. the utility function is one of the forms $u(x) = a + be^{-cx}$ or $u(x) = a + bx.$

Having discussed the condition of zero-switch utility functions over <u>lotteries</u>, we now define the corresponding property for ordinal utility functions over <u>known consequences</u>.

Definition 2.1 *A utility function, $u(x+w)$ is <u>ordinal zero switch</u> if for any two fixed values x_1, x_2 of attribute X the difference $\Delta(w) = u(x_1 + w) - u(x_2 + w)$ does not change sign as w varies.*

Theorem 2.1 *A utility function $u(x)$ is ordinal zero-switch if and only if it is strictly monotone.*

The proof is evident. As we have seen, the equivalent cardinal property for zero-switch utility functions requires the linear or exponential functions (both of which are monotone functions). The ordinal zero-switch property provides less specificity than the cardinal zero-switch property (requiring the function only to be monotone and not necessarily linear or exponential). This is a general theme that we shall observe throughout this paper: ordinal utility functions provide less specification than the corresponding properties over lotteries as they impose milder conditions. However, they are also easier to assert than preferences over lotteries.

What if ordinal preferences for increments of X <u>can</u> switch, but at most once? Once again let us first start with the cardinal case.

A utility function, $u(x+w)$ is a <u>cardinal one-switch</u> utility function over wealth if for any two lotteries, $\tilde{x}_1$ and $\tilde{x}_2$, over X the difference in expected utilities, $D(w) = u(\tilde{x}_1 + w) - u(\tilde{x}_2 + w)$, is either never zero, is zero for all *w,* or is zero for at most one *w.*

In this and later switching definitions, the one-switch condition will include the zero-switch condition as a special case, so that zero-switch utility functions will automatically qualify as one-switch functions.

Bell (1988) showed that the only utility functions that satisfy this cardinal one-switch condition are the four functions: $u(x) = ax^2 + bx + c$; $u(x) = (a+bx)e^{-cx} + d$; $u(x) = ax + be^{-cx} + d$; and $u(x) = ae^{bx} + ce^{dx} + f$.

We now turn to the ordinal case.

Definition 2.2 *A utility function u(x+w) is <u>ordinal one switch</u> if for any pair of X consequences either one is preferred to the other for all w, or they are indifferent for all w, or there exists a unique wealth level above which one consequence is preferred, below which the other is preferred.*

Evidently the ordinal one-switch condition implies that $\Delta(w) = u(x_1 + w) - u(x_2 + w)$ is either always zero, never zero, or zero for exactly one *w*. In particular it excludes cases where two values of *X* are indifferent on an interval but strict preference holds on another, for if $x_1 \sim x_2$ at any two wealth levels then our definition requires that $x_1 \sim x_2$ for all *w*.

Theorem 2.2 *A utility function $u(x)$ is ordinal one-switch if and only if it is unimodal (i.e. has at most one turning point, which may be infinite).*

Again the proof is evident. The ordinal zero-switch condition required strictly monotone functions, and the ordinal one-switch condition requires unimodal functions. Theorem 2.2 also illustrates the generality of the ordinal one-switch condition in comparison to its corresponding cardinal form. Note that the four cardinal one-switch utility families shown above are all unimodal functions.

3. Two Attributes: Ordinal Zero-Switch Independence

So far we have considered ordinal and cardinal switching properties for a single attribute. Specifying similar properties for more than one attribute enables a decomposition of multiattribute utility functions that simplifies their assessment. For the sake of completeness we review preference independence.

Definition 3.1: *Attribute X is <u>preferentially independent</u> of attribute Y if the preference ordering of any two levels of X is independent of the fixed level of Y, that is, the difference $\Delta(y) = u(x_1, y) - u(x_2, y)$ has constant sign as y varies.*

As we noted, Debreu (1960) discussed the implications of mutual preferential independence for three or more attributes, where every subset of the attributes is preferentially independent of its complement. Surprisingly, we have not found any explicit discussion of this ordinal property for two attributes. The following theorem characterizes this property.

Theorem 3.1 *X is preferentially independent of Y if and only if $u(x, y) = \phi\big(v_1(x), y\big)$ where ϕ is a strictly monotonic function of v_1 for all y.*

Note that the function $v_1(x)$ itself need not be monotone. For example $u(x, y) = e^y \sin(x)$ satisfies the condition of *X* being preferentially independent of *Y*.

The following equivalent definition of preferential independence, using the notion of zero-switch preferences, serves to underline our view of one-switch as a natural extension of preference independence.

Definition 3.2 *An attribute X is <u>ordinal zero-switch independent</u> of attribute Y, if for any two values of X, say x_1 and x_2, the difference $\Delta(y) = u(x_1, y) - u(x_2, y)$ does not change sign as y varies.*

Proposition 3.1 *X is zero-switch independent of Y if and only if it is preferentially independent of Y.*

Proposition 3.1 and Theorem 3.1 imply that *X* is zero-switch independent of *Y* if and only if $u(x, y) = \phi\big(v_1(x), y\big)$. The equivalent (stronger) condition for lotteries is the notion of utility independence (Keeney and Raiffa 1976) where preferences for lotteries over *X* do not change for any value of *Y* and so the difference $D(y) = u(\tilde{x}_1, y) - u(\tilde{x}_2, y)$ does not change sign with *y*.

Keeney and Raiffa (1976) show that this condition implies that $u(x, y) = g_0(y) + g_1(y)v(x)$,

where $g_1(y)$ does not change sign. It is clear that the functional form corresponding to utility independence is a special case of that corresponding to preferential independence, where the function ϕ is an affine function of $v(x)$, i.e. $f(v, y) = g_0(y) + g_1(y)v(x)$.

Once again, we observe the generality of the functional form corresponding to the ordinal property. Note for example, that the function $\phi(v, y) = e^{v(x)(1+y^2)}$ would satisfy the condition of preferential independence of X on Y but not utility independence.

4. Two Attributes: Ordinal One-Switch Independence

We have emphasized the equivalence of preference independence and the ordinal zero-switch property to suggest that the ordinal one-switch property that follows is a natural generalization of preference independence.

Definition 4.1: *X is <u>ordinal one-switch independent</u> of Y, written X 1S Y, if for any pair of consequences in X either one is preferred to the other for all y, or they are indifferent for all y, or there exists a unique level of y (which may be infinite) above which one of the consequences is preferred, below which the other is preferred.*

This definition is unchanged even if X is multidimensional. It also extends readily to "n-switch independence" but we believe that zero-switch and one-switch are the most useful cases. Note that a function that satisfies ordinal zero-switch independence (preferential independence) of X from Y also satisfies the condition of ordinal one-switch independence of X from Y.

In correspondence with the single-attribute case, our definition of ordinal one-switch independence excludes the case where a pair of consequences is equally preferred only on an interval: if two X values are ever indifferent at two different values of Y then they must be indifferent for all values of Y.

The remainder of this paper will provide tests to establish whether or not a particular function $u(x, y)$ is one-switch and will provide families that, subject to given conditions, satisfy the one-switch rule. The following examples illustrate the brute-force method of testing for ordinal one-switch independence for some simple functions.

Example 4.1 Consider the ordinal utility function $u(x, y) = (x + y) - (x - y)^2$.

For any two values of X, say x_1, x_2, the difference

$$\Delta(y) = u(x_1, y) - u(x_2, y) = (x_1 - x_2) - (x_1 - x_2)(x_1 + x_2 - 2y).$$

This function satisfies X 1S Y, because the difference switches sign only once, when $y = (x_1 + x_2)/2$.

Example 4.2 Consider the function $u(x, y) = x^2 y - xy^2 + x$

The difference

$$\Delta(y) = u(x_1, y) - u(x_2, y) = (x_1 - x_2)(x_1 + x_2)y - (x_1 - x_2)y^2 + (x_1 - x_2)$$

is quadratic in y, and therefore it need not satisfy X 1S Y. For example, for $x_1 = 1$ and $x_2 = 0$, there are two switches as y varies.

5. Sufficient Conditions for Ordinal One-Switch Independence

In this section we consider how to test whether a given function $u(x, y)$ satisfies the one-switch condition. Of course, we can always try the brute-force method for particular values of X for simple functions, but more complex functions require additional tools.

Test 1: Monotone Differences :

X 1S Y if $u(x_1, y) - u(x_2, y)$ is strictly monotonic in y $\forall x_1 \neq x_2$.

If $\Delta(y) = u(x_1, y) - u(x_2, y)$ is strictly monotonic then it can cross the x-axis at most once, thus satisfying the one-switch condition. The Δ function in Example 4.1 is a linear function of y and therefore (strictly) monotonic. The Δ function in Example 4.2 is quadratic and thus not monotonic.

Monotonicity of Δ is a sufficient but not a necessary condition for satisfying the ordinal one-switch condition. To illustrate, the function $u(x, y) = e^{xy}$ satisfies the ordinal one-switch property. However, if we pick two values of X, say $x_1 = 2$ and $x_2 = 1$, the difference

$\Delta(y) = e^{2y} - e^{y}$ is not monotone because the derivative $\Delta'(y)$ is positive when $e^{y} > 0.5$ and negative when $e^{y} < 0.5$.

Test 2: Constant Sign of the Cross-Derivative (for differentiable functions) :

$$X \; 1S \; Y \; \text{if the cross derivative} \; \frac{\partial^2 u(x, y)}{\partial x \partial y} \; \text{does not change sign.}$$

If the cross derivative has a constant sign (either positive or negative), then for any $x_1 > x_2, y_1 > y_2$, the difference

$$[u(x_1, y_1) - u(x_1, y_2)] - [u(x_2, y_1) - u(x_2, y_2)] = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{\partial^2 u(x, y)}{\partial x \partial y} dx dy$$

does not change sign. This implies that the difference $\Delta(y) = u(x_1, y) - u(x_2, y)$ is strictly monotone if the cross-derivative is either always positive or always negative. This condition is sufficient but not necessary. For example $u(x, y) = x^2 y$ satisfies $X \; 1S \; Y$ but the cross derivative is $2x$ which changes sign.

We note that the condition on the sign of the cross-derivative is a symmetric condition. Therefore, it implies not only that $X \; 1S \; Y$, but also that $Y \; 1S \; X$.

Example 5.1 Consider the function $u(x, y) = \frac{x + y}{2} - k(x - y)^2$. The cross derivative,

$$\frac{\partial^2 u}{\partial x \partial y} = 2k$$

has constant sign. This implies that u satisfies $X \; 1S \; Y$ (and that $Y \; 1S \; X$).

Example 5.2 Consider the function $u(x, y) = 2xy + \sin(x + y)$. The cross-derivative

$$\frac{\partial^2 u}{\partial x \partial y} = 2 - \sin(x + y) > 0 \text{ implying that } X \; 1S \; Y \text{ (and } Y \; 1S \; X \text{)}.$$

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Test 3: Monotone Ratio:

$$X \ 1S \ Y \text{ if the ratio } \frac{u(x_1, y)}{u(x_2, y)} \text{ is strictly monotone in } y \text{ for } \forall x_1 \neq x_2.$$

If $u(x_1, y)$ and $u(x_2, y)$ are equal at both y_1 and y_2 then their ratio is 1 at those two values and thus the ratio cannot be strictly monotone in y. To illustrate, if $u(x, y) = e^{xy}$, then the ratio $\frac{u(x_1, y)}{u(x_2, y)} = e^{(x_1 - x_2)y}$ is strictly monotone and therefore this function satisfies the ordinal one-switch condition.

Test 4: Local and Boundary Double Switches

If $u(x, y)$ does not satisfy X 1S Y then it must have a pair, x_1 and x_2, that switches twice, i.e.

$$x_1 \succ x_2 \text{ " } y < y_1; \ x_2 \succ x_1 \text{ " } y_1 < y < y_2; \text{ and } x_1 \succ x_2 \text{ " } y > y_2$$

But finding such pairs x_1 and x_2 might be difficult. To help identify such pairs, we introduce a test based on two new definitions: local and boundary double-switches.

A local double switch is one where x_1 and x_2 lie in a local neighborhood of each other and switch twice.

Definition 5.1 $u(x, y)$ has a <u>local double switch</u> at x_1 if, for all small enough values of $d > 0$, we have $x_1 \sim x_1 + d$ at two values of y, but not for all y.

A local double switch may be found quite easily by identifying any x such that $\frac{\partial u(x, y)}{\partial x}$ changes sign twice as y varies.

Definition 5.2. X has a <u>boundary double switch</u> in X if $\exists x_1, x_2$ s.t. $x_1 \sim x_2$ at both boundary values of Y on which u is to be defined but there exists at least one value of y within the boundary for which x_1, x_2 are not indifferent.

This kind of double switch is also relatively easy to test for. If y_1 and y_2 are any two values of y, in particular those at the boundaries, then we can assess, plot, or calculate $u(x, y_1)$ and $u(x, y_2)$ and then check directly whether these two curves cross twice. If they do, we then verify strict inequality of the curves $u(x_1, y)$ and $u(x_2, y)$ for at least one value of y.

Clearly for u to be one-switch it is <u>necessary</u> that it have neither a local nor a boundary double switch. But when is the lack of local or boundary double switches <u>sufficient</u> to imply X 1S Y?

Theorem 5.1 *If $u(x, y)$ has neither a local double switch nor a boundary double switch in X, and if for all fixed values of Y, $u(x, y)$ is unimodal in x, then X 1S Y.*

The proof is lengthy, but the intuition is simple. If $u(x, y)$ has a double switch with $x_1 \sim x_2$ at both y_1 and y_2, then we look for a "nearby" double switch involving a pair of x's that are closer than x_1 and x_2. In general this is not always possible (as in Example 5.3 below). However if the curves $u(x, y)$ are unimodal in x, then a sequence of ever closer x's can be constructed until either the x's are arbitrarily close (a local double switch) or the y's reach the boundary (a boundary double switch). The proof, along with others, is in an appendix.

The following example shows that a general u can have a double switch but neither a local-double switch nor a boundary double switch.

Example 5.3   Consider the function

$$u(x, y) = \frac{1}{3}x^3 - \frac{(2m(y)+1)}{2}x^2 + m(y)(m(y)+1)x,$$

where $m(y) = \dfrac{e^y}{e^y + e^{-y}}$.

To test for local double-switches we take the partial derivative

$$\frac{\partial u(x, y)}{\partial x} = (x - m(y))(x - m(y) - 1).$$

The partial derivative is zero if and only if $x = m(y)$ or $x = m(y) + 1$. Therefore, $u(x, y)$ has no local double switches.

To test for boundary double-switches, we have

$$u(x, -\infty) = x\left(\frac{x^2}{3} - \frac{x}{2}\right) \quad \text{and} \quad u(x, \infty) = x\left(\frac{x^2}{3} - \frac{3x}{2} + 2\right).$$

Two values x_1 and x_2 are indifferent at $y = -\infty$ if and only if $2\left(x_1^2 + x_1 x_2 + x_2^2\right) = 3\left(x_1 + x_2\right)$ and at $y = +\infty$ if and only if $2\left(x_1^2 + x_1 x_2 + x_2^2\right) + 12 = 9\left(x_1 + x_2\right)$. Comparing these two expressions we see that any solution must satisfy $x_1 + x_2 = 2$ and $\left(x_1^2 + x_1 x_2 + x_2^2\right) = 3$, which are not simultaneously possible for different x's. Hence there are no boundary double switches.

However u does have a double switch in this example, and thus X is not 1S Y. For example $x_1 = .25$, $x_2 = 1.75$ are indifferent at $m = .25$ and $.75$, corresponding to $y_1 = -.55$ and $y_2 = .75$ respectively. Note that $u(x, y)$ is not unimodal in x as it is a cubic function.

Based on our difficulty in constructing Example 5.3, we believe that as a matter of practice the absence of local and boundary double switches can be an important indication that X 1S Y even when the marginals are not unimodal.

Milgrom and Shannon (1994) developed a single crossing test for three dimensional utility functions that will be useful to us later. Similar to the Debreu result, their result does not apply to the case $n=2$.

6. Families that Satisfy One-Switch Independence

In our prior work, we applied the one switch independence idea to cardinal utility functions (Abbas and Bell 2011) and showed that the most general such function is $u(x, y) = g_0(y) + g_1(y)[f_1(x) + f_2(x)w(y)]$, where $g_1(y)$ does not change sign and $w(y)$ is a monotone function. This function necessarily also satisfies ordinal one switch (because a sure thing is a special case of a gamble). We have

$$\Delta(y) = u(x_1, y) - u(x_2, y) = g_1(y)\left[f_1(x_1) - f_1(x_2) + w(y)[f_2(x_1) - f_2(x_2)]\right],$$

which, since $g_1(y) > 0$, changes sign at most once when $w(y) = \dfrac{f_1(x_1) - f_1(x_2)}{f_2(x_2) - f_2(x_1)}$.

In Abbas and Bell (2011), we illustrated how to assess such functions and in Abbas and Bell (2012) we provided a variety of special cases that the analyst may wish to choose from since they also satisfy the ordinal condition. Note that this cardinal form does not necessarily satisfy

Tests 1, 2, 3 or 4. For example, $\Delta(y)$ need not be strictly monotone because $g_1(y)$ can be any non-negative function, such as 2+sin(y).

As we have seen, ordinal functions satisfying preferences over consequences are more general than equivalent cardinal functions and so it is not surprising that there are even more general functions satisfying this ordinal one switch property. The following theorem provides a family of such functions.

Theorem 6.1 The sum of functions $u(x, y) = v_0(x) + \sum_{i=1}^{\infty} k_i \phi_i(v(x)) \psi_i(w(y))$

satisfies X 1S Y if each derivative term $k_i \phi_i' \psi_i'$ has constant sign for all i and if w is strictly monotonic.

Note that $\dfrac{\partial^2 u}{\partial x \partial y} = v'(x) w'(y) \, \Sigma k_i \phi_i' \psi_i'$ has constant sign if $v(x)$ is monotonic but not otherwise, a further demonstration that the cross derivative condition is a sufficient but not necessary condition.

Example 6.1 Consider the function $u(x, y) = v_0(x) + xy + e^x e^y + e^{-x} e^{-y}$

This function satisfies the format of Theorem 6.1 because the term "$k_i \phi_i' \; \psi_i'$" is respectively $1, e^x e^y$ and $e^{-x} e^{-y}$ which all have the same sign and $w'(y) = 1 > 0$.

Example 6.2 Consider the function $u(x, y) = v_1(x) e^{w_1(y)} - v_2(x) e^{-w_2(y)}$. The difference $\Delta(y) = ae^{w_1} - be^{-w_2}$ is zero if $e^{(w_1 + w_2)} = b/a$. This always has at most one solution if $e^{w_1 + w_2}$ is monotone, that is, if $w_1'(y) + w_2'(y) > 0$. Note that w_1 and w_2 do not have to be monotone individually. So $u(x, y) = xe^{2y} - x^2 e^{-y}$ has $w_1' + w_2' = 2 - 1 > 0$ but $w_1' > 0$ and $w_2' < 0$. $\Delta(y) = (x_1 - x_2)e^{2y} - (x_1^2 - x_2^2)e^{-y}$, which changes sign uniquely when $e^{3y} = x_1 + x_2$

but $\dfrac{\partial^2 u}{\partial x \partial y} = 2e^{2y} + 2xe^{-y}$ changes sign when $x = -e^{3y}$.

7. Ordinal Utility Functions Satisfying One-Switch Independence

It will be convenient to adopt the notation $v_1(x) = u(x, y_1)$, $v_2(x) = u(x, y_2)$ where y_1, y_2 are the boundaries of an interval of interest.

Theorem 7.1   *If X 1S Y then there exists a function ϕ such that*

$$u(x, y) = \phi\big(v_1(x), v_2(x), y\big) \ for \ y_1 \leq y \leq y_2.$$

where $v_1(x) = u(x, y_1), v_2(x) = u(x, y_2)$, ϕ is strictly monotone in v_1 and in v_2, and the function ϕ satisfies the boundary conditions

$$\phi\big(v_1, v_2, y_1\big) = v_1, \quad \phi\big(v_1, v_2, y_2\big) = v_2.$$

Theorem 7.1 provides a necessary condition for ordinal one-switch functions. Compare this formulation to that of preferential independence in Theorem 3.1, where only one univariate function of x was required. Theorem 7.1 shows that ordinal one-switch independence is a generalization of preferential independence requiring two functions $v_1(x), v_2(x)$.

Example 7.1   Consider the cardinal one-switch form

$$u(x, y) = g_0(y) + g_1(y)\big(v_1(x) + w(y)v_2(x)\big)$$

with $g_1(y) > 0$ and $w(y)$ monotone.

As we have seen, this function satisfies *X* 1S *Y* because it satisfies the cardinal condition, but note that it can also be written as $u(x, y) = \phi\big(v_1(x), v_2(x), y\big)$, where ϕ is linear in both $v_1(x), v_2(x)$. Therefore it satisfies the necessary condition for ordinal one-switch independence, as expected.

Example 7.2 Consider the function $u(x, y) = x^2 y - xy^2 + x = x(1 - y^2) + x^2 y$. This function can be expressed in terms of the representation $u(x, y) = \phi\big(v_1(x), v_2(x), y\big)$. If we define

$v_1(x) = u(x, 0) = x$ and $v_2(x) = u(x, 1) = x^2$ then

$u(x, y) = \phi\big(v_1(x), v_2(x), y\big) = v_1(x)(1 - y^2) + v_2(x)y.$

However, as we showed in Example 4.2, this function does not satisfy the one-switch rule.

Our focus will now be on twice continuously differentiable functions ϕ. For expositional purposes it will be convenient to define $DR(v_1, v_2, y)$ as the ratio of partial derivatives $\partial\phi(v_1, v_2, y)/\partial v_1$ to $\partial\phi(v_1, v_2, y)/\partial v_2$, i.e.

$$DR(v_1, v_2, y) = \frac{\partial\phi(v_1, v_2, y)/\partial v_1}{\partial\phi(v_1, v_2, y)/\partial v_2}$$

Theorem 7.2 Monotone Tradeoffs: *Given $u(x,y) = \phi(v_1(x), v_2(x), y)$ where ϕ is strictly monotone in v_1, v_2 and satisfies the boundary conditions of Theorem 7.1 then if $\phi(v_1(x), v_2(x), y)$ has a local double switch then $DR(v_1(x), v_2(x), y)$ is not strictly monotone in y.*

Put the other way, this result shows that if $DR(v_1, v_2, y)$ is strictly monotone in y then the corresponding $u(x, y)$ cannot have a local double switch. Note that the structure of ϕ ensures that it also cannot have a boundary double switch. For if $x_1 \sim x_2$ at both y_1 and y_2, then ϕ automatically makes them indifferent for all y, which is not a double switch.

Any function $\phi(v_1, v_2, y)$ can be used to generate utility functions $u(x, y)$ by replacing v_1 by $v_1(x) = u(x, y_1)$ and v_2 by $v_2(x) = u(x, y_2)$. Milgrom and Shannon (1994) proved an important result for three dimensional ordinal utility functions that we can apply directly to ϕ to guarantee that the corresponding $u(x, y)$ satisfies *X* 1S *Y*.

Theorem 7.3 *The function $\phi(v_1, v_2, y)$ satisfies X 1S Y over any interval (y_1, y_2) for any choice of functions $v_1 = u(x, y_1)$ and $v_2 = u(x, y_2)$ iff ϕ is strictly monotone in v_1 and v_2 and $DR(v_1, v_2, y)$ is strictly monotone in y.*

Example 7.4 Consider the case $u(x, y) = f_1(x) + f_2(x)y$. This always satisfies *X* 1S *Y*. We may construct the corresponding ϕ as follows, with, for example, $y_1 = -1$ and $y_2 = 1$. We have

$v_1(x) = u(x, -1) = f_1(x) - f_2(x)$, $v_2(x) = u(x,1) = f_1(x) + f_2(x)$, so that $f_1 = \dfrac{v_1 + v_2}{2}$ and

$f_1 = \dfrac{v_1 - v_2}{2}$ yielding $\phi = u = \dfrac{v_1 + v_2}{2} + y\dfrac{v_1 - v_2}{2}$.

Checking the conditions of Theorem 7.3 we note this is strictly monotonic in v_1 and v_2 over the

range (-1,1) and $DR(v_1, v_2, y) = \dfrac{1-y}{1+y}$ which is strictly monotonic over the same range.

While we know from Theorem 7.1 that every $u(x,y)$ satisfying X 1S Y can be expressed as $\phi(v_1(x), v_2(x), y)$ and we see from Theorem 7.3 that every $\phi(v_1(x), v_2(x), y)$ obeying its conditions satisfies X 1S Y, we see from the following example that not every one-switch $\phi(v_1(x), v_2(x), y)$ satisfies the conditions of Theorem 7.3.

Example 7.5 The function $u(x, y) = ye^{x+y} + e^{-x}$ satisfies X 1S Y but the corresponding $\phi(v_1(x), v_2(x), y)$ does not satisfy the conditions in Theorem 7.3.

To see that u is one-switch note that if a>b are two values of X then they cross when

$ye^y = \dfrac{e^{-b} - e^{-a}}{e^b - e^a}$, which has a unique positive solution in y because ye^y is monotone for y >0.

However ye^y is not monotonic when y<0. The ϕ corresponding to u satisfies the conditions of Theorem 7.3 when y>0, but not always when y<0. The problem arises due to the fact that although u does allow switching, it does so only when y>0, but the violations of ϕ occur when y<0 (ϕ does not violate the conditions if y is restricted to the positive range).

Corollary 7.4 *The function* $\phi(v_1, v_2, y)$ *satisfies X 1S Y over the interval* (y_1, y_2) *for any choice of functions* $v_1 = u(x, y_1)$ *and* $v_2 = u(x, y_2)$ *if* ϕ *is strictly monotonic in* v_1 *and* v_2 *and if*

$\dfrac{\partial^2 \phi}{\partial v_1 \partial y} < 0$ *and* $\dfrac{\partial^2 \phi}{\partial v_2 \partial y} > 0$.

The cross-derivative conditions imply that $DR(v_1, v_2, y)$ is strictly monotone decreasing so that Theorem 7.3 applies immediately. We highlight this result because the conditions on ϕ in this result seem highly intuitive and transparent. Generating ordinal utility functions using the family $\phi(v_1, v_2, y)$ with appropriate restrictions seems to us to be a good way to generate a wide range of one-switch utility functions.

8. Conclusions

The concept of zero-switch preference, expressed ordinally as preference independence and cardinally as utility independence, has a long pedigree in the literature. The ordinal property in the univariate case corresponds to a monotone function of wealth, and to the notion of preferential independence for multiple attributes. Cardinally, this zero-switching property corresponds to the more specific linear and exponential utility functions and to utility independence for the case of multiple attributes.

While the concept of zero-switching preferences leads to simple functional forms, it is hard to proceed if they do not apply. We propose that the natural extension of these concepts is that preferences can switch but at most once. In the ordinal case, we have shown that this property corresponds to the notion of a unimodal function, a natural extension of monotone functions in terms of switching preferences. The extension of the ordinal property of preferential independence to one-switch preferences has been the focus of this work.

We defined the notion of ordinal one-switch utility independence and focused on two main problems. The first was whether a given utility function satisfies this ordinal property. We defined easily-discoverable local and boundary double switches and showed when testing for them was sufficient to prove X 1S Y. Though the absence of local and boundary switches is not, in general, a definitive indication that there are no double switches, we believe that it is a very strong indication.

The second focus was on how to generate one-switch functions. We showed that ordinal one-switch utility functions may be represented in the form $u(x, y) = \phi(v_1(x), v_2(x), y)$ and, subject to

reasonable restrictions on ϕ, that any function of the form $\phi\big(v_1(x), v_2(x), y\big)$ will be a one-switch utility function. We believe that, despite exceptions (Example 7.5), most one-switch functions can be generated in this way. Consequently we propose the functional form $u(x,y) = \phi\big(v_1(x), v_2(x), y\big)$ as the ordinal generalization of our recent work on one-switch independence (Abbas and Bell 2011, 2012).

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Appendix

Proof of Theorem 5.1

For a given $u(x, y)$ that does not satisfy X 1S Y, let S be the set of all (x_1, x_2) that switch more than once, where $x_1 < x_2$. We will say that a pair $(x_1, x_2) \in S$ is <u>dominated</u> if $\exists (x_1', x_2') \in S$ with $x_2' - x_1' < x_2 - x_1$. We will say that a function "changes sign twice" if as y increases, it is strictly positive, then strictly negative, then strictly positive, or the same statement with positive and negative reversed.

<u>Lemma A1</u>   If $(x_1, x_2) \in S$ and if $u(x_1, y) - u(x_2, y)$ changes sign at least twice, then (x_1, x_2) is dominated.

<u>Proof</u>   If $\Delta(y) = u(x_1, y) - u(x_2, y)$ changes sign twice then, relying on the continuity of u, so does any function of the form $u(x_1 + \delta_1, y) - u(x_2 - \delta_2, y)$ if δ_1 and δ_2 are sufficiently small. By picking $\delta_1 > 0, \delta_2 > 0$ we see that (x_1, x_2) is dominated.   QED.

Suppose $(x_1, x_2) \in S$ has n zeros; i.e. $y_1 < y_2 < \cdots < y_n$ where $\Delta(y_i) = 0$. We know from Lemma A1 that an undominated u cannot change sign twice. Therefore $\Delta'(y_i) = 0$ for at least $n - 1$ of the $y_i's$. If all of the y_i's are finite then $n - 1$ of them involve $\Delta(y_i)$ being u-shaped at y_i, that is, they touch the axis at y_i without changing sign. For Δ to have no explicit u-shaped tangent to the axis the y's would have to be infinite.

<u>Lemma A2</u>   If $(x_1, x_2) \in S$ and $\Delta(y)$ forms a u-shaped tangent at y^*, then (x_1, x_2) is dominated if all marginals are unimodal.

<u>Proof</u> It suffices to show that for some $\delta_1 > 0, \delta_2 > 0$ the equation $u(x_1 + \delta_1, y^* + \varepsilon) = u(x_2 - \delta_2, y^* + \varepsilon)$ has two distinct solutions for some ε. As a first-order approximation we may write this as

$$u(x_1 + \delta_1, y^*) + u(x_1, \ y^* + \varepsilon) = u(x_2 - \delta_2, y^*) + u(x_2, \ y^* + \varepsilon)$$

or $u(x_1 + \delta_1, y^*) - u(x_2 - \delta_2, y^*) = -[u(x_1, \ y^* + \varepsilon) - u(x_2, \ y^* + \varepsilon)].$

We know, by assumption, that the RHS is u-shaped in ε, touching the axis at $\varepsilon = 0$. In addition, if u is unimodal, then there exist infinitely many pairs $\delta_1^* > 0, \delta_2^* > 0$ such that $u(x_1 + \delta_1^*, y^*) = u(x_2 - \delta_2^*, y^*)$. Consider the two possible pairings $\delta_1 = \delta_1^*/2, \delta_2 = \delta_2^*$ and $\delta_1 = \delta_1^*, \delta_2 = \delta_2^*/2$. One of these makes the LHS positive, the other makes it negative. Since the RHS is u-shaped, one or the other of these choices has two solutions for ε. Since δ_1^* and δ_2^* may be chosen to be arbitrarily small the approximation is inconsequential.   QED.

So, any undominated element of S must have a $\Delta(y)$ that changes sign at most once, and makes a tangent at either $-\infty$ or $+\infty$, perhaps both, but not between.

<u>Lemma A3</u>   If $(x_1, x_2) \in S$ has a $\Delta(y)$ that changes sign exactly once, and has no other zeros except at infinity, then it is dominated if all marginals are unimodal.

<u>Proof</u> If $\Delta(-\infty) = 0$, consider $u(x, -\infty)$. Again, because the marginals for fixed y are unimodal, we know $\exists \ \delta_1 > 0 \ \delta_2 > 0$ s.t. $u(x_1 + \delta_1, -\infty) = u(x_2 - \delta_2, -\infty)$. But for δ_1, δ_2 sufficiently small, $u(x_1 + \delta_1, y) - u(x_2 - \delta_2, y)$ must also cross zero. Hence $(x_1 + \delta_1, x_2 - \delta_2) \in S$. Hence (x_1, x_2) is dominated.   QED.

What remains is that if $(x_1, x_2) \in S$ is undominated, it must have a Δ that is strictly non-zero except at $+\infty$ and $-\infty$ where it is zero. This is a boundary double switch.

To outline the remainder of the proof, starting with any double switch having $x_1 < x_2$ we may construct a sequence of double switches by replacing each sequence member with another pair that dominates it. It is apparent from our derivation that each step in the sequence can be assumed to be arbitrarily small. Either this sequence converges to a local double switch, or it has a subsequence that converges to a local double switch, in which case we are done, or it

converges prior to becoming a local double switch. We will show that the sequence may always be restarted from such a situation.

<u>Lemma A4</u> If $x_1 - \delta_1$ and $x_2 + \delta_2$ switch twice where δ_1, δ_2 are small, and $u(x_1, y) = u(x_2, y)$ for all y, then $x_1 + \delta_1$ and $x_2 - \delta_2$ switch twice.

<u>Proof</u> We know $u(x_1 - \delta_1, y) - u(x_2 + \delta_2, y)$ is double switch, but for small δ this equals $u(x_1, y) - u(x_2, y) - \delta_1 \frac{\partial u(x_1, y)}{\partial x} - \delta_2 \frac{\partial u(x_2, y)}{\partial x}$ which means that $-\delta_1 \frac{\partial u(x_1, y)}{\partial x} - \delta_2 \frac{\partial u(x_2, y)}{\partial x}$ switches twice, since $u(x_1, y) \equiv u(x_2, y)$. But then $u(x_1 + \delta_1, y) - u(x_2 - \delta_2, y) = u(x_1, y) - u(x_2, y) + \delta_1 \frac{\partial u(x_1, y)}{\partial x} + \delta_2 \frac{\partial u(x_2, y)}{\partial x}$ also switches twice. QED.

We note that if (x_1^k, x_2^k) is a sequence of double switches converging to (x_1^*, x_2^*) where $(x_1^* < x_2^*)$ then if $u(x_1^*, y) \equiv u(x_2^*, y)$ then by selecting some large enough k, we know by Lemma A4 that $(2x_1^* - x_1^k, 2x_2^* - x_2^k)$ is double switch and, moreover, that $(2x_1^* - x_1^k, 2x_2^* - x_2^k)$ dominates x_1^k, x_2^k since $x_2^* - x_1^* < x_2^k - x_1^k$.

<u>Proof of Theorem 5.1</u>

Since u is not one-switch, then S has at least one element say (x_1^0, x_2^0). Construct a sequence $(x_1^k, x_2^k) \in S$ by selecting for x_1^k, x_2^k any element of S that dominates x_1^{k-1}, x_2^{k-1}. If (x_1^k, x_2^k) converges to a limit x_1^∞, x_2^∞ (or a subsequence does) where $x_2^\infty - x_1^\infty > 0$ then there are two cases to consider. Either x_1^∞, x_2^∞ is in S or it is not. If it is, simply restart the sequence with (x_1^∞, x_2^∞) in place of (x_1^0, x_2^0). If it is not, then Lemma A4 must apply in which case the sequence may again be restarted. Since the sequence so constructed cannot end finitely, $x_2^k - x_1^k$ converges to 0, and the sequence (x_1^k, x_2^k) must contain a subsequence in which $x_1^k \to x^*$ and $x_2^k \to x^*$. This point is a local double switch. QED.

<u>Proof of Theorem 6.1.</u>

$$\Delta(y) = \alpha + \sum_{i=1}^{\infty} k_i \beta_i \psi_i (w(y)) \text{ where } \alpha = v_0(x_1) - v_0(x_2), \text{ and } \beta_i = \phi_i(v(x_1)) - \phi_i(v(x_2)).$$

The derivative with respect to Y gives $\Delta'(y) = w'(y) \Sigma k_i \beta_i \psi_i'$.

This derivative has constant sign for all y because each $k_i \beta_i \psi_i'$ has the same sign as $k_i \phi_i' \, \psi_i'$. Therefore, $\Delta(y)$ is monotone. Hence X $1S$ Y. QED.

<u>Proof of Theorem 7.1</u>

We start with the following Lemma, where we define $v_1(x) = u(x, y_1)$, $v_2(x) = u(x, y_2)$.

Lemma 7.1 *If X 1S Y and if there exist any two distinct values x_1, x_2 such that*

$$v_1(x_1) = v_1(x_2) \ \text{ and } \ v_2(x_1) = v_2(x_2)$$

then $u(x_1, y) = u(x_2, y)$ for all y.

Proof of Lemma. If there exist x_1, x_2 that satisfy $v_1(x_1) = v_1(x_2)$ and $v_2(x_1) = v_2(x_2)$, then this means indifference of x_1, x_2 at two points y_1, y_2, which (by our definition) is a violation of X $1S$ Y unless $u(x_1, y) = u(x_2, y)$ for all y.

Geometrically speaking, Lemma 7.1 implies that if we pick any two curves $u(x, y_1)$ and $u(x, y_2)$ on the surface of a function $u(x, y)$ that satisfies X 1S Y, then there cannot be any <u>two distinct</u> x_1, x_2 whose utility values are equal on these curves unless they are equal on the entire domain $u(x_1, y) = u(x_2, y)$. A corollary of this result implies that there cannot be any regions of $v_1(x)$ and $v_2(x)$ that are flat for the same values of x. QED.

We now prove the Theorem.

Proof of Theorem 7.1 <u>Define</u> ϕ by the relation $\phi(v_1(x), v_2(x), y) = u(x, y)$. This assignment converts a two dimensional function $u(x, y)$ into a three-dimensional function $\phi(v_1(x), v_2(x), y)$. To provide this mapping, every x must correspond to a unique pair $(v_1(x), v_2(x))$, i.e., this representation fails only if for some x_1 and x_2, we have $v_1(x_1) = v_1(x_2), v_2(x_1) = v_2(x_2)$ but $u(x_1, y) \neq u(x_2, y)$ for some y. But this cannot happen if X 1S Y as we have discussed in Lemma 7.1. The condition on the boundary conditions follows from consistency of both sides of

the equation $\phi(v_1(x), v_2(x), y) = u(x, y)$. Lemmas 7.2 and 7.3 below show the necessity of strict monotonicity with v_1 and v_2 .

Lemma 7.2 *If X 1S Y, and if* $u(x_2, y_1) > u(x_1, y_1)$ *and* $u(x_2, y_2) > u(x_1, y_2)$ *then necessarily* $u(x_2, y) > u(x_1, y) \; \forall y_1 < y < y_2$.

Proof of Lemma 7.2 Note that if $u(x_2, y_1) > u(x_1, y_1)$ and $u(x_2, y_2) > u(x_1, y_2)$ but there exists $y_3 \in (y_1, y_2)$ such that $u(x_2, y_3) < u(x_1, y_3),$ then this would imply that preferences for x_2 and x_1 have switched more than once, which is a violation of ordinal one-switch independence. The case of equality at y_3 is also excluded because it would imply that the difference changes sign from positive to zero and back to positive which is a violation of the one-switch condition. QED.

Lemma 7.3 *If X 1S Y, and if* $u(x, y) = \phi(v_1(x), v_2(x), y)$ *, where* $v_1(x) = u(x, y_1)$ *and* $v_2(x) = u(x, y_2)$ *then* ϕ *must be strictly monotonic with respect to* v_1 *and* v_2 *where defined* .

Proof of Lemma 7.3 This Lemma translates the results of Lemma 7.2 into the formulation of $\phi(v_1, v_2, y)$: If $v_1(x_2) > v_1(x_1)$ and $v_2(x_2) > v_2(x_1)$, then x_2 is preferred to x_1 at both y_1 and y_2 , and so it is preferred for all $y_1 < y < y_2$. Hence $\phi(v_1 + d_1, v_2 + d_2, y) > \phi(v_1, v_2, y) \; \forall y \in (y_1, y_2)$ if $d_1 \geq 0, d_2 \geq 0$ and $d_1 + d_2 > 0$. QED

Proof of Theorem 7.2 If there is a local double switch then $x_1 \sim x_1 + d$ at y_1 and y_2 say. This means that for the corresponding $\phi(v_1(x), v_2(x), y)$ we have $\dfrac{\partial \phi}{\partial v_1} \dfrac{dv_1(x)}{dx} + \dfrac{\partial \phi}{\partial v_2} \dfrac{dv_2(x)}{dx} = 0$ at the two values of y. Thus $DR(v_1, v_2, y)$ equals $-v_2'/v_1'$ at two values of y and is therefore not strictly monotonic. QED.

Proof of Theorem 7.3 In our notation, Milgrom and Shannon (1994) showed that attributes v_1 and v_2 will cross at most once as y varies if $\dfrac{\partial \phi}{\partial v_2}$ is never zero, if $DR(v_1, v_2, y)$ is strictly monotonic in y, and if any two indifferent pairs of v values at a given y, say v^* and w^* at $y = y_1$, are always connected by a path of indifferent points also at y_1. Though their result concerned three attributes we can apply their reasoning to ϕ, which has three attributes, and thereby obtain a result for two attributes. As our conditions are slightly different we exhibit the proof.

Since ϕ is strictly monotonic in v_1 and v_2 it is clear from the geometry that, for any fixed y, its isopreference curves are connected. More specifically however, suppose v^* and w^* are two values of (v_1, v_2) that are indifferent for some y_3, where $y_1 \le y_3 < y_2$. Note that because of strict monotonicity in v_1 and v_2 we must have $(v_1^*-w_1^*)(v_2^*-w_2^*)<0$ and this is true for any pair of points that is indifferent for a fixed y.

Suppose that $v_1^*>w_1^*$ and $v_2^*<w_2^*$ and consider a new point $(v_1^*-d_1(v_1^*-w_1^*), v_2^*-d_2(v_2^*-w_2^*))$ where d_1, $d_2 >0$ are to be chosen. We can always find d_1 and d_2 so that this new point is indifferent to v^* (and to w^*) at y_3. This is because $\phi(v_1^*-d_1(v_1^*-w_1^*), v_2^*)$ is strictly decreasing in d_1 and $\phi(v_1^*, v_2^*-d_2(v_2^*-w_2^*))$ is strictly increasing in d_2. So we can always find d_1, $d_2>0$ to ensure $\phi(v_1^*-d_1(v_1^*-w_1^*), v_2^*-d_2(v_2^*-w_2^*))=\phi(v^*,w^*)$. In this way we can construct a series of points v(k) such that v(1)=v*, v(n)=w* and $v_1(k)>v_1(k+1)$, $v_2(k)<v_2(k+1)$.

Now suppose that DR(v*,y) is strictly monotone decreasing in y. Note that DR cannot be infinite or zero, except perhaps at the boundaries of Y. (If it were to be infinite at an interior point it could not then be decreasing before that point, if zero it could not be decreasing afterwards).

Following Milgrom and Shannon we now observe that $d_1(v_1^*-w_1^*)\dfrac{\partial \phi}{\partial v_1} +d_2(v_2^*-w_2^*)\dfrac{\partial \phi}{\partial v_2} =0$ at y_3 so that $d_1(v_1^*-w_1^*)DR(v^*,y_3) +d_2(v_2^*-w_2^*)=0$ at y_3 and therefore by strict monotonicity of DR we have $d_1(v_1^*-w_1^*)DR(v^*,y_4) +d_2(v_2^*-w_2^*)<0$ at any larger value of y, say y_4.

Therefore $d_1(v_1{*}-w_1{*})\dfrac{\partial\phi}{\partial v_1}+d_2(v_2{*}-w_2{*})\dfrac{\partial\phi}{\partial v_2}$ <0 at y_4 or $v(1)<v(2)$.

But since $DR(v_1,v_2,y)$ is non-negative and cannot be zero except at the boundaries then for every $v(k)\sim v(k+1)$ at y_3 we have $v(k)<v(k+1)$ at y_4. Hence we cannot have $v{*}\sim w{*}$ at y_4.

To show the other direction of the proof we will suppose a violation occurs at some point $(v_1{*}$, $v_2{*}$, $y{*})$ and then select functions $v_1(x)$ and $v_2(x)$ to exploit the violation. If ϕ is not strictly monotone in v_1 at this point, select $v_1(x)$ and $v_2(x)$ so that $v_1(x^*)=v_1^*$, $v_2(x^*)=v_2^*$, $v_1'(x^*)>0,$ and $v_2'(x^*)=0$. Then for small d, we have x*+d preferred to x* at the lower boundary (where v_1 is defined), x*+d<=x* at $y*$ (because ϕ is not monotonic in v_1 at that point) and x*+d~x* at the upper boundary (where v_2 is defined). If DR(v_1*, v_2*, y) is equal at y_1 and y_2 then as before, select functions $v_1(x)$ and $v_2(x)$ so that $v_1(x^*)=v_1^*$, $v_2(x^*)=v_2^*$, and

$$-\dfrac{v_2'(x^*)}{v_1'(x^*)}=DR(v_1^*,v_2^*,y_1). \quad \text{QED}$$