Limits on
Interest Rate Rules
in the IS Model

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Many central banks have long used a short-term nominal interest rate as the main instrument through which monetary policy actions are implemented. Some monetary authorities have even viewed their main job as managing nominal interest rates, by using an interest rate rule for monetary policy. It is therefore important to understand the consequences of such monetary policies for the behavior of aggregate economic activity.

Over the past several decades, accordingly, there has been a substantial amount of research on interest rate rules.[1] This literature finds that the feasibility and desirability of interest rate rules depends on the structure of the model used to approximate macroeconomic reality. In the standard textbook Keynesian macroeconomic model, there are few limits: almost any interest rate

[1] This literature is voluminous, but may be usefully divided into four main groups. First, there is work with small analytical models with an "IS-LM" structure, including Sargent and Wallace (1975), McCallum (1981), Goodfriend (1987), and Boyd and Dotsey (1994). Second, there are simulation studies of econometric models, including the Henderson and McKibbin (1993) and Taylor (1993) work with larger models and the Fuhrer and Moore (1995) work with a smaller one. Third, there are theoretical analyses of dynamic optimizing models, including work by Leeper (1991), Sims (1994), and Woodford (1994). Finally, there are also some simulation studies of dynamic optimizing models, including work by Kim (1996).

policy can be used, including some that make the interest rate exogenously determined by the monetary authority. In fully articulated macroeconomic models in which agents have dynamic choice problems and rational expectations, there are much more stringent limits on interest rate rules. Most basically, if it is assumed that the monetary policy authority attempts to set the nominal interest rate without reference to the state of the economy, then it may be impossible for a researcher to determine a unique macroeconomic equilibrium within his model.

Why are such sharply different answers about the limits to interest rate rules given by these two model-building approaches? It is hard to reach an answer to this question in part because the modeling strategies are themselves so sharply different. The standard textbook model contains a small number of behavioral relations—an IS schedule, an LM schedule, a Phillips curve or aggregate supply schedule, etc.—that are directly specified. The standard fully articulated model contains a much larger number of relations—efficiency conditions of firms and households, resource constraints, etc.—that implicitly restrict the economy's equilibrium. Thus, for example, in a fully articulated model, the IS schedule is not directly specified. Rather, it is an outcome of the consumption-savings decisions of households, the investment decisions of firms, and the aggregate constraint on sources and uses of output.

Accordingly, in this article, we employ a series of macroeconomic models to shed light on how aspects of model structure influence the limits on interest rate rules. In particular, we show that a simple respecification of the IS schedule, which we call the expectational IS schedule, makes the textbook model generate the same limits on interest rate rules as the fully articulated models. We then use this simple model to study the design of interest rate rules with nominal anchors.[2] If the monetary authority adjusts the interest rate in response to deviations of the price level from a target path, then there is a unique equilibrium under a wide range of parameter choices: all that is required is that the authority raise the nominal rate when the price level is above the target path and lower it when the price level is below the target path. By contrast, if the monetary authority responds to deviations of the inflation rate from a target path, then a much more aggressive pattern is needed: the monetary authority must make the nominal rate rise by more than one-for-one with the inflation rate.[3] Our results on interest rate rules with nominal anchors are preserved when we further extend the model to include the influence of expectations on aggregate supply.

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[2] An important recent strain of literature concerns the interaction of monetary policy and fiscal policy when the central bank is following an interest rate rule, including work by Leeper (1991), Sims (1994) and Woodford (1994). The current article abstracts from consideration of fiscal policy.

[3] Our results are broadly in accord with those of Leeper (1991) in a fully articulated model.

1.   INTEREST RATE RULES IN THE TEXTBOOK MODEL

In the textbook IS-LM model with a fixed price level, it is easy to implement monetary policy by use of an interest rate instrument and, indeed, with a pure interest rate rule which specifies the actions of the monetary authority entirely in terms of the interest rate. Under such a rule, the monetary sector simply serves to determine the quantity of nominal money, given the interest rate determined by the monetary authority and the level of output determined by macroeconomic equilibrium. Accordingly, as in the title of this article, one may describe the analysis as being conducted within the "IS model" rather than in the "IS-LM model."

   In this section, we first study the fixed-price IS model's operation under a simple interest rate rule and rederive the familiar result discussed above. We then extend the IS model to consider sustained inflation by adding a Phillips curve and a Fisher equation. Our main finding carries over to the extended model: in versions of the textbook model, pure interest rate rules are admissible descriptions of monetary policy.

Specification of a Pure Interest Rate Rule

We assume that the "pure interest rate rule" for monetary policy takes the form

$$R_t = \underline{R} + x_t, \tag{1}$$

where the nominal interest rate R_t contains a constant average level $\underline{R}$. (Throughout the article, we use a subscript t to denote the level of the variable at date t of our discrete time analysis and an underbar to denote the level of the variable in the initial stationary position). There are also exogenous stochastic components to interest rate policy, x_t, that evolve according to

$$x_t = \rho x_{t-1} + \varepsilon_t, \tag{2}$$

with ε_t being a series of independently and identically distributed random variables and ρ being a parameter that governs the persistence of the stochastic components of monetary policy. Such pure interest rate rules contrast with alternative interest rate rules in which the level of the nominal interest rate depends on the current state of the economy, as considered, for example, by Poole (1970) and McCallum (1981).

The Standard IS Curve and the Determination of Output

In many discussions concerning the influence of monetary disturbances on real activity, particularly over short periods, it is conventional to view output as determined by aggregate demand and the price level as predetermined. In such discussions, aggregate demand is governed by specifications closely related to the standard IS function used in this article,

$$y_t - \underline{y} = -s[r_t - \underline{r}], \tag{3}$$

where y denotes the log-level of output and r denotes the real rate of interest. The parameter s governs the slope of the IS schedule as conventionally drawn in (y, r) space: the slope is s^{-1} so that a larger value of s corresponds to a flatter IS curve. It is conventional to view the IS curve as fairly steep (small s), so that large changes in real interest rates are necessary to produce relatively small changes in real output.

With fixed prices, as in the famous model of Hicks (1937), nominal and real interest rates are the same ($R_t = r_t$). Thus, one can use the interest rate rule and the IS curve to determine real activity. Algebraically, the result is

$$y_t - \underline{y} = -s[(\underline{R} - \underline{r}) + x_t]. \tag{4}$$

A higher rate of interest leads to a decline in the level of output with an "interest rate multiplier" of s.[4]

Poole (1970) studies the optimal choice of the monetary policy instrument in an IS-LM framework with a fixed price level; he finds that it is optimal for the monetary authority to use an interest rate instrument if there are predominant shocks to money demand. Given that many central bankers perceive great instability in money demand, Poole's analytical result is frequently used to buttress arguments for casting monetary policy in terms of pure interest rate rules. From this standpoint it is notable that in the model of this section—which we view as an abstraction of a way in which monetary policy is frequently discussed—the monetary sector is an afterthought to monetary policy analysis. The familiar "LM" schedule, which we have not as yet specified, serves only to determine the quantity of money given the price level, real income, and the nominal interest rate.

Inflation and Inflationary Expectations

During the 1950s and 1960s, the simple IS model proved inappropriate for thinking about sustained inflation, so the modern textbook presentation now includes additional features. First, a Phillips curve (or aggregate supply schedule) is introduced that makes inflation depend on the gap between actual and capacity output. We write this specification as

$$\pi_t = \psi(y_t - \underline{y}), \tag{5}$$

where the inflation rate π is defined as the change in log price level, $\pi_t \equiv P_t - P_{t-1}$. The parameter ψ governs the amount of inflation (π) that arises from a given level of excess demand. Second, the Fisher equation is used to describe the relationship between the real interest rate (r_t) and the nominal interest rate (R_t),

$$R_t = r_t + E_t \pi_{t+1}, \tag{6}$$

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[4] Many macroeconomists would prefer a long-term interest rate in the IS curve, rather than a short-term one, but we are concentrating on developing the textbook model in which this distinction is seldom made explicit.

where the expected rate of inflation is $E_t\pi_{t+1}$. Throughout the article, we use the notation $E_t z_{t+s}$ to denote the date t expectation of any variable z at date $t + s$.

To study the effects of these two modifications for the determination of output, we must solve for a reduced form (general equilibrium) equation that describes the links between output, expected future output, and the nominal interest rate. Closely related to the standard IS schedule, this specification is

$$y_t - \underline{y} = -s[(\underline{R} - \underline{r}) + x_t] + s\psi[E_t y_{t+1} - \underline{y}]. \tag{7}$$

This general equilibrium locus implies that there is a difference between temporary and permanent variations in interest rates. Holding $E_t y_{t+1}$ constant at $\underline{y}$, as is appropriate for temporary variations, we have the standard IS curve determination of output as above. With $E_t y_{t+1} = y_t$, which is appropriate for permanent disturbances, an alternative general equilibrium schedule arises which is "flatter" in (y, R) space than the conventional specification. This "flattening" reflects the following chain of effects. When variations in output are expected to occur in the future, they will be accompanied by inflation because of the positive Phillips curve link between inflation and output. With the consequent higher expected inflation at date t, the real interest rate will be lower and aggregate demand will be higher at a particular nominal interest rate.

Thus, "policy multipliers" depend on what one assumes about the adjustment of inflation expectations. If expectations do not adjust, the effects of increasing the nominal interest rate are given by $\frac{\Delta y}{\Delta R} = -s$ and $\frac{\Delta \pi}{\Delta R} = -s\psi$, whereas the effects if expectations do adjust are $\frac{\Delta y}{\Delta R} = -s/[1 - s\psi]$ and $\frac{\Delta \pi}{\Delta R} = -s\psi/[1 - s\psi]$. At the short-run horizons that the IS model is usually thought of as describing best, the conventional view is that there is a steep IS curve (small s) and a flat Phillips curve (small ψ) so that the denominator of the preceding expressions is positive. Notably, then, the output and inflation effects of a change in the interest rate are of larger magnitude if there is an adjustment of expectations than if there is not. For example, a rise in the nominal interest rate reduces output and inflation directly. If the interest rate change is permanent (or at least highly persistent), the resulting deflation will come to be expected, which in turn further raises the real interest rate and reduces the level of output.

There are two additional points that are worth making about this extended model. First, when the Phillips curve and Fisher equations are added to the basic Keynesian setup, one continues to have a model in which the monetary sector is an afterthought. Under an interest rate policy, one can use the LM equation to determine the effects of policy changes on the stock of money, but one need not employ it for any other purpose. Second, higher nominal interest rates lead to higher real interest rates, even in the long run. In fact, because there is expected deflation which arises from a permanent increase in

the nominal interest rate, the real interest rate rises by more than one-for-one with the nominal rate.[5]

Rational Expectations in the Textbook Model

There has been much controversy surrounding the introduction of rational expectations into macroeconomic models. However, in this section, we find that there are relatively minor qualitative implications within the model that has been developed so far. In particular, a monetary authority can conduct an unrestricted pure interest rate policy so long as we have the conventional parameter values implying $s\psi <$ 1. In the rational expectations solution, output and inflation depend on the entire expected future path of the policy-determined nominal interest rate, but there is a "discounting" of sorts which makes far-future values less important than near-future ones.

To determine the rational expectations solution for the standard Keynesian model that incorporates an IS curve (3), a Phillips curve (5), and the Fisher equation (6), we solve these three equations to produce an expectational difference equation in the inflation rate,

$$\pi_t = -s\psi[(R_t - \underline{r}) - E_t\pi_{t+1}], \tag{8}$$

which links the current inflation rate π_t to the current nominal interest rate and the expected future inflation rate.[6] Substituting out for π_{t+1} using an updated version of this expression, we are led to a forward-looking description of current inflation as related to the expected future path of interest rates and a future value of the inflation rate,

$$\pi_t = -s\psi(R_t - \underline{r}) - (s\psi)^2 E_t(R_{t+1} - \underline{r}) \ldots$$
$$-(s\psi)^n E_t(R_{t+n-1} - \underline{r}) + (s\psi)^n E_t\pi_{t+n}. \tag{9}$$

For short-run analysis, the conventional assumption is that there is a steep IS curve (small s) because goods demand is not too sensitive to interest rates and a flat Phillips curve (small ψ) because prices are not too responsive to aggregate demand. Taken together, these conditions imply that $s\psi <$ 1 and that there is substantial "discounting" of future interest rate variations and of the "terminal inflation rate" $E_t\pi_{t+n}$: the values of the exogenous variable R and endogenous variable π that are far away matter much less than those nearby. In particular, as we look further and further out into the future, the value of long-term inflation, $E_t\pi_{t+n}$, exerts a less and less important influence on current inflation.

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[5] This implication is not a particularly desirable one empirically, and it is one of the factors that leads us to develop the models in subsequent sections.

[6] Alternatively, we could have worked with the difference equation in output (7), since the Phillips curve links output and inflation, but (8) will be more useful to us later when we modify our models to include price level and inflation targets.

   Using this conventional set of parameter values and making the standard rational expectations solution assumption that the inflation process does not contain explosive "bubble components," the monetary authority can employ any pure nominal interest rate rule.[7] Using the assumed form of the pure interest rate policy rule, (1) and (2), the inflation rate is

$$\pi_t = -s\psi\left[\frac{1}{1-s\psi}(\underline{R}-\underline{r}) + \frac{1}{1-s\psi}x_t\right]. \tag{10}$$

Thus, a solution exists for a wide range of persistence parameters in the policy rule (all $\rho < (s\psi)^{-1}$). Notably, it exists for $\rho = 1$, in which variations in the random component of interest rates are permanent and the "policy multipliers" are equal to those discussed in the previous subsection.[8]

2. EXPECTATIONS AND THE IS SCHEDULE

Developments in macroeconomics over the last two decades suggest the importance of modifying the IS schedule to include a dependence of current output on expected future output. In this section, we introduce such an "expectational IS schedule" into the model and find that there are important limits on interest rate rules. We conclude that one *cannot* or *should not* use a pure interest rate rule, i.e., one without a response to the state of the economy.

Modifying the IS Schedule

Recent work on consumption and investment choices by purposeful firms and households suggests that forecasts of the future enter importantly into these decisions. These theories suggest that the conventional IS schedule (3) should be replaced by an alternative, expectational IS schedule (EIS schedule) of the form

$$y_t - E_t y_{t+1} = -s(r_t - \underline{r}). \tag{11}$$

Figure 1 draws this schedule in (y, r) space, i.e., we graph

$$r_t = \underline{r} - \frac{1}{s}(y_t - E_t y_{t+1}).$$

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[7] More precisely, we require that the policy rule must result in a finite inflation rate, i.e., $|\pi_t| = |s\psi\left[\sum_{j=0}^{\infty}(s\psi)^j E_t(R_{t+j}-\underline{r})\right]| < \infty$. Since $s\psi < 1$, this requirement is consistent with a wide class of driving processes as discussed in the appendix.

[8] With $s\psi \geq 1$, there is a very different situation, as we can see from looking at (9): future interest rates are more important than the current interest rate, and the terminal rate of inflation exerts a major influence on current inflation. Long-term expectations hence play a very important role in the determination of current inflation. In this situation, there is substantial controversy about the existence and uniqueness of a rational expectations equilibrium, which we survey in the appendix and discuss further in the next section of the article.

Figure 1   The Expectational IS Schedule

log of output (y)
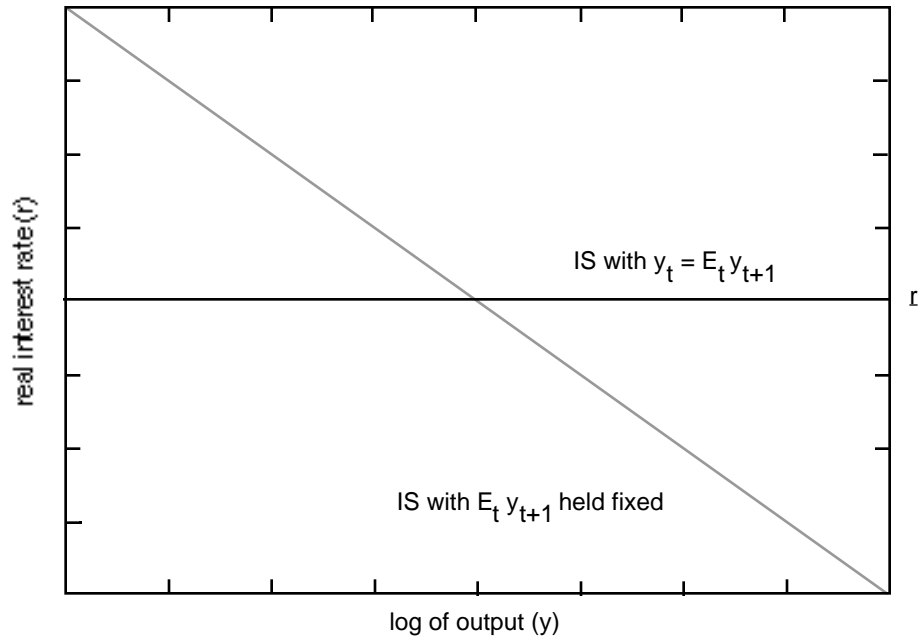
In this figure, expectations about future output are an important shift factor in the position of the conventionally defined IS schedule.

The expectational IS schedule thus emphasizes the distinction between temporary and permanent movements in real output for the level of the real interest rate. If a disturbance is temporary (so that we hold expected future output constant, say at $E_t y_{t+1} = \underline{y}$), then the linkage between the real rate and output is identical to that indicated by the conventional IS schedule of the previous section. However, if variations in output are expected to be permanent, with $E_t y_{t+1} = y_t$, then the IS schedule is effectively horizontal, i.e., $r_t = \underline{r}$ is compatible with any level of output. Thus, the EIS schedule is compatible with the traditional view that there is little long-run relationship between the level of the real interest rate and the level of real activity. It is also consistent with Friedman's (1968a) suggestion that there is a natural real rate of interest ($\underline{r}$) which places constraints on the policies that a monetary authority may pursue.[9]

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[9] In this sense, it is consistent with the long-run restrictions frequently built into real business cycle models and other modern, quantitative business cycle models that have temporary monetary nonneutralities (as surveyed in King and Watson [1996]).

To think about why this specification is a plausible one, let us begin with consumption, which is the major component of aggregate demand (roughly two-thirds in the United States). The modern literature on consumption derives from Friedman's (1957) construction of the "permanent income" model, which stresses the role of expected future income in consumption decisions. More specifically, modern consumption theory employs an Euler equation which may be written as

$$\sigma\left(E_t c_{t+1} - c_t\right) = \left(r_t - \underline{r}\right), \tag{12}$$

where c is the logarithm of consumption at date t, and σ is the elasticity of marginal utility of a representative consumer.[10] Thus, for the consumption part of aggregate demand, modern macroeconomic theory suggests a specification that links the *change* in consumption to the real interest rate, not one that links the level of consumption to the real interest rate. McCallum (1995) suggests that (12) rationalizes the use of (11). He also indicates that the incorporation of government purchases of goods and services would simply involve a shift-term in this expression.

Investment is another major component of aggregate demand, which can also lead to an expectational IS specification in the following way.[11] For example, consider a firm with a constant-returns-to-scale production function, whose level of output is thus determined by the demand for its product. If the desired capital-output ratio is relatively constant over time, then variations in investment are also governed by anticipated changes in output. Thus, consumption and investment theory suggest the importance of including expected future output as a positive determinant of aggregate demand. We will consequently employ the expectational IS function as a stand-in for a more complete specification of dynamic consumption and investment choice.

Implications for Pure Interest Rate Rules

There are striking implications of this modification for the nature of output and interest rate linkages or, equivalently, inflation and interest rate linkages. Combining the expectational IS schedule (11), the Phillips curve (5), and the Fisher equation (6), we obtain

$$y_t - \underline{y} = -s[(\underline{R} - \underline{r}) + x_t] + (1 + s\psi)(E_t y_{t+1} - \underline{y}). \tag{13}$$

The key point is that expected future output has a greater than one-for-one effect on current output independent of the values of the parameters s and ψ.

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[10] See the surveys by Hall (1989) and Abel (1990) for overviews of the modern approach to consumption. In these settings, the natural real interest rate, $\underline{r}$, would be determined by the rate of time preference, the real growth rate of the economy, and the extent of intertemporal substitutions.

[11] In critiquing the traditional IS-LM model, King (1993) argues that a forward-looking rational expectations investment accelerator is a major feature of modern quantitative macroeconomic models that is left out of the traditional IS specification.

This restriction to a greater than one-for-one effect is sharply different from that which derives from the traditional IS model and the Fisher equation, i.e., from the less than one-for-one effect found in (7) above.

One way of summarizing this change is by saying that the general equilibrium locus governing *permanent variations* in output and the real interest rate becomes upward-sloping in (y, R) space, not downward-sloping. Thus, when we assume that $E_t y_{t+1} = \underline{y}$, we have the conventional linkage from the nominal rate to output. However, when we assume that $E_t y_{t+1} = y_t$, then we find that there is a positive, rather than negative, linkage. Interpreted in this manner, (13) indicates that a permanent lowering of the nominal interest rate will give rise to a permanent decline in the level of output. This reversal of sign involves two structural elements: (i) the horizontal "long-run" IS specification of Figure 1 and (ii) the positive dependence on expected future output that derives from the combination of the Phillips curve and the Fisher equation.

The central challenge for our analysis is that this model's version of the general equilibrium under an interest rate rule obeys the unconventional case for rational expectations theory that we described in the previous section, irrespective of our stance on parameter values. The reduced-form inflation equation for our economy, which is similar to (8), may be readily derived as[12]

$$(1 + s\psi)E_t \pi_{t+1} - \pi_t = s\psi(R_t - \underline{r}) = s\psi[(\underline{R} - \underline{r}) + x_t]. \qquad (14)$$

Based on our earlier discussion and the internal logic of rational expectations models, it is natural to iterate this expression forward. When we do so, we find that

$$\pi_t = -s\psi[(R_t - \underline{r}) + (1 + s\psi)E_t(R_{t+1} - \underline{r}) + \ldots$$

$$+ (1 + s\psi)^n E_t(R_{t+n} - \underline{r})] + (1 + s\psi)^{n+1} E_t \pi_{t+n+1}. \qquad (15)$$

As we look further and further out into the future, the value of long-term inflation, $E_t \pi_{t+n+1}$, exerts a more and more important influence on current inflation. With the EIS function, therefore, it is always the case that there is an important dependence of current outcomes on long-term expectations. One interpretation of this is that public confidence about the long-run path of inflation is very important for the short-run behavior of inflation.

Macroeconomic theorists who have considered the solution of rational expectations models in this situation have not reached a consensus on how to proceed. One direction is provided by McCallum (1983), who recommends

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[12] The ingredients of this derivation are as follows. The Phillips curve specification of our economy states that $\pi_t = \psi(y_t - \underline{y})$. Updating this expression and taking additional expectations, we find that $E_t \pi_{t+1} = \psi(E_t y_{t+1} - \underline{y})$. Combining these two expressions with the expectational IS function (11), we find that $E_t \pi_{t+1} - \pi_t = \psi(E_t y_{t+1} - y_t) = s\psi(r_t - \underline{r})$. Using the Fisher equation together with this result, we find the result reported in the text.

forward-looking solutions which emphasize fundamentals in ways that are similar to the standard solution of the previous section. Another direction is provided by the work of Farmer (1991) and Woodford (1986), which recommends the use of a backward-looking form. These authors stress that such solutions may also include the influences of nonfundamental shocks. In the appendix, we discuss the technical aspects of these alternative approaches in more detail, but we focus here on the key features that are relevant to thinking about limits on interest rate rules. We find that the forward-looking approach suggests that no stable equilibrium exists if the interest rate is held fixed at an arbitrary value or governed by a pure rule. We also find that the backward-looking approach suggests that many stable equilibria exist, including some in which nonfundamental sources of uncertainty influence macroeconomic activity.

Forward-Looking Equilibria

One important class of rational expectations equilibrium solutions stresses the forward-looking nature of expectations, so that it can be viewed as an extension of the solutions considered in the previous section. These solutions depend on the "fundamental" driving processes, which in our case come from the interest rate rule. McCallum (1983) has proposed that macroeconomists focus on such solutions; he also explains that these are "minimum state variable" or "bubble free" solutions to (14) and provides an algorithm for finding these solutions in a class of macroeconomic models.

In this case, the inflation solution depends only on the current interest rate under the policy rule (1) and (2). To obtain an empirically useful solution using this method, we must circumscribe the interest rate rule so that the limiting sum in the solution for the inflation rate in (15) is finite as we look further and further ahead.[13] In the current context, this means that the monetary authority must (i) equate the nominal and real interest rate on average (setting $\underline{R} - \underline{r} = 0$ in (10) and (ii) substantially restrict the amount of persistence (requiring $\rho < (1 + s\psi)^{-1}$). These two conditions can be understood if we return to (15), which requires that $\pi_t = -s\psi[(R_t - \underline{r}) + \ldots + (1 + s\psi)^n E_t(R_{t+n} - \underline{r})] + (1 + s\psi)^{n+1} E_t \pi_{t+n+1}$. First, the average long-run value of inflation must be zero or otherwise the terms like $(1 + s\psi)^{n+1} E_t \pi_{t+n+1}$ will cause the current inflation rate to be positive or negative infinity. Second, the stochastic variations in the interest rate must be sufficiently temporary that there is a finite sum $(R_t - \underline{r}) + (1 + s\psi)E_t(R_{t+1} - \underline{r}) + \ldots + (1 + s\psi)^n E_t(R_{t+n} - \underline{r}) = x_t + (1 + s\psi)\rho x_t + \ldots (1 + s\psi)^n \rho^n x_t$ as n is made arbitrarily large.

How do these requirements translate into restrictions on interest rate rules in practice? Our view is that the second of these requirements is not too important, since there will always be finite inflation rate equilibria for any finite-order

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[13] Flood and Garber (1980) call this condition "process consistency."

moving-average process. (As explained further in the appendix, such solutions always exist because the limiting sum is always finite if one looks only a finite number of periods ahead). However, we think that the first requirement (that $\underline{R} - \underline{r} = 0$) is much more problematic: it means that the average expected inflation rate must be zero. This requirement constitutes a strong limitation on pure interest rate rules. Further, it is implausible to us that a monetary authority could actually satisfy this condition, given the uncertainty that is attached to the level of $\underline{r}$.[14] If the condition is not satisfied, however, there does not exist a rational expectations equilibrium under an interest rate rule if one restricts attention to minimum state variable equilibria.

Backward-Looking Equilibria

Other macroeconomists like Farmer (1991) and Woodford (1986) have argued that (14) leads to empirically interesting solutions in which inflation depends on nonfundamental factors, such as sunspots, but does so in a stationary manner. In particular, working along the lines of these authors, we find that any inflation process of the form

$$\pi_t = \left(\frac{1}{1 + s\psi}\right)\pi_{t-1} + \left(\frac{s\psi}{1 + s\psi}\right)(R_{t-1} - \underline{r}) + \zeta_t \qquad (16)$$

is a rational expectations equilibrium consistent with (14).[15] In this expression, ζ_t is an arbitrary random variable that is unpredictable using date $t - 1$ information. Such a "backward-looking" solution is generally nonexplosive, and interest rates are a stationary stochastic process.[16]

There are three points to be made about such equilibria. First, there may be a very different linkage from interest rates to inflation and output in such equilibria than suggested by the standard IS model of Section 1. A change in the nominal interest rate at date t will have no effect on inflation and output at date t if it does not alter ζ_t: inflation may be predetermined relative to interest rate policy rather than responding immediately to it. Second, a permanent increase in the nominal interest rate at date t will lead ultimately to a permanent *increase* in inflation and output, rather than to the decrease described in the

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[14] One measure of this uncertainty is provided by the controversy over Fama's (1975) test of the link between inflation and nominal interest rates, which assumed that the ex ante real interest rate was constant. In a critique of Fama's analysis, Nelson and Schwert (1977) argued compellingly that there was sufficient unforecastable variability in inflation that it was impossible to tell from a lengthy data set whether the real rate was constant or evolved according to a random walk.

[15] It can be confirmed that this is a rational expectations solution by simply updating it one period and taking conditional expectations, a process which results in (8).

[16] By *generally,* we mean that it is stationary as long as we assume that $s\psi >$ 0, as used throughout this paper.

previous section of the article.[17] Third, if there are effects of interest rate changes on output and inflation within a period, then these may be completely unpredictable to the monetary authority since ζ_t is arbitrary: ζ_t can therefore depend on $R_t - E_{t-1}R_t$. We could, for example, see outcomes which took the form

$$\pi_t = \left(\frac{1}{1 + s\psi} \right)\pi_{t-1} + \left(\frac{s\psi}{1 + s\psi} \right)(R_{t-1} - \underline{r}) + \zeta_t(R_t - E_{t-1}R_t),$$

so that the short-term relationship between inflation (output) and interest rate shocks was random in magnitude and sign.

Combining the Cases: Limits on Pure Interest Rate Rules

Thus, depending on what one admits as a rational expectations equilibrium in this case, there may be very different outcomes; but either case suggests important limits on pure interest rate rules.

   With forward-looking equilibria that depend entirely on fundamentals, there may well be no equilibrium for pure interest rate rules, since it is implausible that the monetary authority can exactly maintain a zero gap between the average nominal rate and the average real rate ($\underline{R} - \underline{r} = 0$) due to uncertainty about $\underline{r}$. However, if one can maintain this zero gap, there are some additional limits on the driving processes for autonomous interest rate movements. Thus, for the autoregressive case in (2), interest rate policies cannot be "too persistent" in the sense that we must require $\rho(1 + s\psi) < 1$.

   With backward-looking equilibria, there is a bewildering array of possible outcomes. In some of these, inflation depends only on fundamentals, but the short-term relationship between inflation and interest rates is essentially arbitrary. In others, nonfundamental sources of uncertainty are important determinants of macroeconomic activity. If such an equilibrium were observed in an actual economy, then there would be a very firm basis for the monetarist claim that interest rate rules lead to excess volatility in macroeconomic activity, even though there would be a very different mechanism than the one that typically has been suggested. That is, the sequence of random shocks ζ_t amounts to an entirely avoidable set of shocks to real macroeconomic activity (since, via the Phillips curve, inflation and output are tightly linked, $\pi_t = \psi(y_t - \underline{y})$).[18] While feasible, pure interest rate rules appear very undesirable in this situation.

   Under either description of equilibrium, the limits on the feasibility and desirability of interest rate rules arise because individuals' beliefs about

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[17] That is, there is a sense in which this Keynesian model produces neoclassical conclusions in response to interest rate shocks with a backward-looking equilibrium.

[18] This policy effect is formally similar to one that Schmitt-Grohe and Uribe (1995) describe for balanced budget financing. Perhaps these changes in expectations could be the "inflation scares" that Goodfriend (1993) suggests are important determinants of macroeconomic activity during certain subperiods of the post-war interval.

long-term inflation receive very large weight in determination of the current price level. Inflation psychology exerts a dominant influence on actual inflation if a pure interest rate rule is used.

3.   INTEREST RATE RULES WITH NOMINAL ANCHORS

In this section, building on the prior analyses of Parkin (1978) and McCallum (1981), we study the effects of appending a "nominal anchor" to the model of the previous section, which was comprised of the expectational IS specification, the Phillips curve, and the Fisher equation. Such policies can work to stabilize long-term expectations, eliminating the difficulties that we encountered above. We look at two rules that are policy-relevant alternatives in the United States and other countries.

The first of these rules, which we call *price-level targeting,* specifies that the monetary authority sets the interest rate so as to partially respond to deviations of the current price level from a target path $\underline{P}_t$, while retaining some independent variation in the interest rate x_t. We view the target price level path as having the form $\underline{P}_t = \underline{P}_0 + \underline{\pi}_t$, but more complicated stochastic versions are also possible. In this section, we shall view x_t as an arbitrary sequence of numbers and in later sections we will view it as a zero mean stochastic process. The interest rate rule therefore is written as

$$R_t = \underline{R} + f(P_t - \underline{P}_t) + x_t, \tag{17}$$

where the parameter f governs the extent to which the interest rate varies in response to deviations of the current price level from its target path.

The second of these rules, which we call *inflation targeting,* specifies that the monetary authority sets the interest rate so as to partially respond to deviations of the inflation rate from a target path $\underline{\pi}_t$, while retaining some independent variation in the interest rate. Algebraically, the rule is

$$R_t = \underline{R} + g(\pi_t - \underline{\pi}) + x_t. \tag{18}$$

We explore these target schemes for two reasons. First, they are relevant to current policy debate in the United States and other countries. Second, they each can be implemented without knowledge of the money demand function, just as pure interest rate rules could in the basic IS model. [19]

The difference between these two policies involves the extent of "base drift" in the nominal anchor, i.e., they differ in terms of whether the central

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[19] This latter rule is related to proposals by Taylor (1993). It is also close to (but not exactly equal to) the widely held view that the Federal Reserve must raise the real rate of interest in response to increases in inflation to maintain the target rate of inflation (such an alternative rule would be written as $R_t = \underline{R} + g(E_t\pi_{t+1} - \underline{\pi}) + x_t$).

bank is presumed to eliminate the effects of past gaps between the actual and the target price level.[20] In each case, for analytical simplicity, we assume that the central bank can observe the current price level without error at the time it sets the interest rate.

Inflation Targets with an Interest Rate Rule

It is relatively easy to use (14) to characterize the conditions under which an interest rate rule can implement an inflation target without introducing a multiplicity of equilibria. To analyze this case, we replace R_t in (14) with its value under the interest rate rule, which is $R_t = \underline{R} + g(\pi_t - \underline{\pi}) + x_t$. The result is

$$(1 + s\psi)E_t(\pi_{t+1} - \underline{\pi}) - (1 + s\psi g)(\pi_t - \underline{\pi}) = s\psi[x_t + (\underline{R} - \underline{\pi} - \underline{r})].$$

It is clear that there is a unique solution of the standard form if and only if $g > 1$. This solution is

$$\pi_t - \underline{\pi} = -\left(\frac{s\psi}{1 + s\psi g}\right)\left\{\sum_{j=0}^{\infty}\left(\frac{1 + s\psi}{1 + s\psi g}\right)^j [E_t x_{t+j} + (\underline{R} - \underline{\pi} - \underline{r})]\right\}. \quad (19)$$

Thus, to have the inflation rate average to $\underline{\pi}$ we must impose $(\underline{R} - \underline{\pi} - \underline{r}) = 0$ and use the fact that the unconditional expected value of each of the terms $E_t x_{t+j}$ is zero. However, if the equilibrium real interest rate were unknown by the monetary authority, as is plausibly the case, then there would simply be an average rate of inflation that differed from the target level persistently. In particular and in contrast to the analysis of "pure" interest rate rules above, there would not be any difficulty with the existence of rational expectations equilibrium. That is, the form of the interest rate rule means that there is a "discounted" influence of future inflation in (19); the central bank has assured that the exact state of long-term inflation expectations is unimportant for current inflation by the form of its interest rate rule.[21]

Price-Level Targets with an Interest Rate Rule

There is a somewhat more complicated solution when an interest rate rule is used to target the price level. However, this solution embodies the very intuitive result that an interest rate rule leads to a conventional, unique, forward-looking

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[20] In both of these policy rules, to make the solutions algebraically simple, we assume that $\underline{R} = \underline{r} + \underline{\pi}$. This does not correspond to an assumption that the central bank knows the real interest rate—it is only a normalization that serves to make the average and target inflation rates or price level paths coincide.

[21] Interestingly, if one modifies the rule so that it is the expected rate of inflation that is targeted, $R_t = \underline{R} + g(E_t\pi_{t+1} - \underline{\pi}) + x_t$, then the same condition for a standard rational expectations equilibrium emerges, $g > 1$. It is also the case that $g > 1$ is the relevant condition for a model with flexible prices, which may be verified by combining the Fisher equation and the policy rule.

equilibrium so long as $f > 0$. More specifically, imposing $(\underline{R} - \underline{\pi} - \underline{r}) = 0$, we can show that the unique stable solution takes the form

$$P_t = \mu_1 P_{t-1} + \left(\frac{s\psi}{1+s\psi}\right)\left\{\sum_{j=0}^{\infty}\left(\frac{1}{\mu_2}\right)^{j+1}(f\underline{P}_{t+j} - E_t x_{t+j} - \underline{\pi})\right\}, \quad (20)$$

where the μ parameters satisfy $\mu_1 < \frac{1}{(1+s\psi)}$ and $\mu_2 > 1$ if $f > 0$.[22] The form of this solution is plausible, given the structure of the model. The past price level is important because this is a model with a Phillips curve, i.e., it is a sticky price solution. Expectations of a higher target price level path raise the current price level. Increases in the current or future autonomous component of the interest rate lower the current price level.

This simple and intuitive condition for price level determinacy prevails in all of the models studied analytically in this article and in many other simulation models that we have constructed. (For example, it is also the case that $f > 0$ is the relevant condition for a model with flexible prices, which may be verified by combining the Fisher equation and the policy rule as in Boyd and Dotsey [1994]). All the monetary authority needs to do to provide an anchor for expectations is to follow a policy of raising the nominal interest rate when the price level exceeds a target path. [23]

4.   EXPECTATIONS AND AGGREGATE SUPPLY

In this section, we consider the introduction of expectations into the aggregate supply side (or Phillips curve) of the model economy. Given the emphasis that macroeconomics has placed on the role of expectations on the aggregate supply side (or the "expectations adjustment" of the Phillips curve), this placement may seem curious. However, we have chosen it deliberately for two reasons, one historical and one expositional.

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[22] To reach this conclusion, we write the basic dynamic equation for the model (14) as

$$s\psi R_t + (1 + s\psi)\underline{\pi} = [(1 + s\psi)\mathbf{F} - 1][\mathbf{F} - 1]E_t P_{t-1}, \quad (21)$$

using the lead operator $\mathbf{F}$, defined so that $\mathbf{F}^n E_t x_{t+j} = E_t x_{t+j+n}$. Inspecting this expression, we see that the two roots of the polynomial $H(\mathbf{z}) = (1 + s\psi)[\mathbf{z} - \frac{1}{(1+s\psi)}][\mathbf{z} - 1]$ are 1 and $\frac{1}{(1+s\psi)}$. More generally, for any second order polynomial $H(\mathbf{z}) = A[\mathbf{z}^2 - S\mathbf{z} + P] = A(\mathbf{z} - \mu_1)(\mathbf{z} - \mu_1)$, the sum of the roots is S and the product of the roots is P. If there is a price level target in place, then we require $R_t = \underline{R} + f(P_t - \underline{P}_t) + x_t$, which alters the polynomial to $(1 + s\psi)[\mathbf{z} - \frac{1}{(1+s\psi)}][\mathbf{z} - 1] - f\mathbf{z}$, i.e., we perturb the sum, but not the product, of the roots. Accordingly, one root satisfies $\mu_1 < \frac{1}{(1+s\psi)}$ and the other satisfies $\mu_2 > 1$.

[23] This difference between price level and inflation rules is very suggestive. That is, by binding itself to a long-run path for the price level, the monetary authority appears to give itself a wider range of short-run policy options than if it seeks to target the inflation rate. We are currently using the models of this article and related fully articulated models to explore these connections in more detail.

We started our analysis of interest rate rules by studying the textbook IS-LM-PC model that became the workhorse of Keynesian macroeconomics during the early 1960s.[24] In the late 1960s, a series of studies by Milton Friedman suggested an alternative set of linkages to the IS-LM-PC model. First, Friedman (1968a) suggested that there was a "natural" real rate of interest that monetary policy cannot affect in the long run. He used this natural rate of interest to argue that the long-run effect of a sustained inflation due to a monetary expansion could not be that suggested by the Keynesian model discussed in Section 1 above, which associated a lower interest rate with higher inflation. Instead, he argued that the nominal interest rate had to rise one-for-one with sustained inflation and monetary expansion due to the natural real rate of interest. Friedman thus suggested that this natural rate of interest placed important limits on monetary policies. In Section 2 of the article, using a model with a natural rate of interest but with a long-run Phillips curve, we found such limits on interest rate rules. By focusing first on the role of expectations in aggregate demand (the IS curve), we made clear that the crucial ingredient to our case for limits on interest rate rules is the existence of a natural real rate of interest rather than information on the long-run slope of the Phillips curve.

Friedman (1968b) argued that a similar invariance of real economic activity to sustained inflation should hold, i.e., that there should be no long-run slope to the Phillips curve. He suggested this invariance resulted from the one-for-one long-run expected inflation on the wage and price determination that underlay the Phillips curve. We now discuss adding expectations in aggregate supply, working first with flexible price models and then with sticky price models.

Flexible Price Aggregate Supply Theory

In an influential study, Sargent and Wallace (1975) developed a log-linear model that embodied Friedman's ideas and followed Lucas (1972) in assuming rational expectations. Essentially, Sargent and Wallace took the IS schedule and Fisher equation from the Keynesian model of Section 1, but introduced the following expectational Phillips curve:

$$\pi_t = \psi(y_t - \underline{y}) + E_{t-1}\pi_t. \tag{22}$$

Initial interest in the Sargent and Wallace (1975) study focused on a "policy irrelevance" implication of their work, which was that systematic monetary policy—cast in terms of rules governing the evolution of the stock of money—had no effect on the distribution of output. That conclusion is now understood

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[24] Our model was somewhat simplified relative to the more elaborate dynamic versions of these models, in which lags of inflation were entered on the right-hand side of the inflation equation (5), perhaps as proxies for expected inflation.

to depend in delicate ways on the specification of the IS curve (3) and the Phillips curve (22), but it is not our focus here.

Another important aspect of the Sargent and Wallace study was their finding that there was nominal indeterminacy under a pure interest rate rule. To exposit this result, it is necessary to introduce a money demand function of the form used by Sargent and Wallace,

$$M_t^d - P_t = \delta y_t - \gamma R_t,$$

where M_t^d is the demand for nominal money, M_t.

Since nominal indeterminacy in the Sargent-Wallace model arises even if real output is constant, we may proceed as follows to determine the conditions under which such indeterminacy arises. First, we may take expectations at $t - 1$ of (22), yielding $E_{t-1}y_t = \underline{y}$. Second, using the standard IS function (3), we learn that this output neutrality result implies $E_{t-1}r_t = \underline{r}$, i.e., that the real interest rate is invariant to expected monetary policy. Third, the Fisher equation then implies that $E_{t-1}R_t = \underline{r} + E_{t-1}\pi_{t+1}$. Fourth, the pure interest rate rule implies that $E_{t-1}R_t = \underline{R} + E_{t-1}x_t$. Combining these last two equations, we find that expected inflation is well determined under an interest rate rule, $E_{t-1}\pi_{t+1} = (\underline{R} - \underline{r}) + E_{t-1}x_t$, but that there is nothing that determines the levels of money and prices, i.e., the money demand function determines the expected level of real balances, $E_{t-1}(M_t - P_t) = \delta\underline{y} - \gamma E_{t-1}R_t$, not the level of nominal money or prices.

It turns out that our two policy rules resolve this nominal indeterminacy under exactly the same parameter restrictions as are required to yield a determinate equilibrium in Section 3 above. For example, it is easy to see that the inflation rule, which implies that $E_{t-1}R_t = \underline{R} + g(E_{t-1}\pi_t - \underline{\pi}) + E_{t-1}x_t$, requires $g > 1$ if the implied dynamics of inflation $E_{t-1}\pi_{t+1} = (\underline{R} - \underline{r}) + g(E_{t-1}\pi_t - \underline{\pi}) + E_{t-1}x_t$ are to be determinate, which leads to a determinate price level. A similar line of argument may be used to show that $f > 0$ is the condition for determinacy with a price-level target.

Practical macroeconomists have frequently dismissed the Sargent and Wallace (1975) analysis of limits on interest rate rules because of its underlying assumption of complete price flexibility. However, as we have seen, conclusions concerning indeterminacy similar to those arising from the Sargent-Wallace model occur in natural rate models without price flexibility.[25]

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[25] From this perspective, the Sargent-Wallace analysis is of interest because there is a natural real rate of interest without an expectational IS schedule. Instead, the natural rate arises due to general equilibrium conditions. Limits to interest rate rules thus appear to arise in natural rate models, irrespective of whether these originate in the IS specification or as part of a complete general equilibrium model.

Sticky Price Aggregate Supply Theory

An alternative view of aggregate supply has been provided by New Keynesian macroeconomists. One of the most attractive and tractable representations is due to Calvo (1983) and Rotemberg (1982), who each derive the same aggregate price adjustment equation from different underlying assumptions about the costs of adjusting prices.[26] To summarize the results of this approach, we use the alternative expectations-augmented Phillips curve,

$$\pi_t = \beta E_t \pi_{t+1} + \psi(y_t - \underline{y}), \tag{23}$$

which is a suitable approximation for small average inflation rates. This relationship has a long-run trade-off between inflation and real activity, $\psi/(1 - \beta)$. Since the parameter β has the dimension of a real discount factor in this model, β is necessarily smaller than unity but not too much so, and the long-run inflation cost of greater output is very high. Thus, while the Calvo and Rotemberg specification is not quite as classical as that of Sargent and Wallace, in the long run it is still *very* classical relative to the naive Phillips curve that we employed above.

With the Calvo and Rotemberg specification of the expectations-augmented Phillips curve (23), the expectational IS function (11) and the Fisher equation (6), we can again show that there are limits to interest rate rules of exactly the form discussed earlier. Further, we can also show that the necessary structure of nominal anchors is $g > 1$ for inflation targets and $f > 0$ for price level targets.[27] That is, we again find that the monetary authority can anchor the economy by responding weakly to the deviations of the price level from a target path, but that much more aggressive responses to deviations of inflation from target are required.

5. SUMMARY AND CONCLUSIONS

In this article, we have studied limits on interest rate rules within a simple macroeconomic model that builds rational expectations into the IS schedule and the Phillips curve in ways suggested by recent developments in macroeconomics.

We began with a version of the standard fixed-price textbook model. Working within this setup in Section 1, we replicated two results found by many prior researchers. First, almost any interest rate rule can feasibly be employed:

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[26] Calvo (1983) obtains this result for the aggregate price level in a setting where individual firms have an exogenous probablility of being permitted to change their price in a given period. Rotemberg (1982) derives it for a setting in which the representative firm has quadratic costs of adjusting prices. Rotemberg (1987) discusses the observational equivalence of the two setups.

[27] The derivations are somewhat more tedious than those of the main text and are available on request from the authors.

there are essentially no limits on interest rate rules. In particular, we found that a central bank can even follow a "pure interest rate rule" in which there is no dependence of the interest rate on aggregate economic activity. Second, under this policy specification, the monetary equilibrium condition—the LM schedule of the traditional IS-LM structure—is unimportant for the behavior of the economy because an interest rate rule makes the quantity of money demand-determined. Accordingly, as suggested in the title of this article, we showed why many central bank and academic researchers have regarded the traditional framework essentially as an "IS model" when an interest rate rule is assumed to be used.

We then undertook two standard modifications of the textbook model so as to consider the consequences of sustained inflation. One was the addition of a Phillips curve mechanism, which specified a dependence of inflation on real activity. The other was the introduction of the distinction between real and nominal interest rates, i.e., a Fisher equation. Within such an extended model, we showed that there continued to be few limits on interest rate rules, even with rational expectations, as long as prices were assumed to adjust gradually and output was assumed to be demand-determined.

Our attention then shifted in Section 2 to alterations of the IS schedule, incorporating an influence of expectations of future output. To rationalize this "aggregate demand" modification, we appealed to modern consumption and investment theories—the permanent income hypothesis and the rational expectations accelerator model—which suggest that the standard IS schedule is badly misspecified. These theories predict a relationship between the expected growth rate of output (or aggregate demand) and the real interest rate, rather than a connection between the level of output and the real interest rate. (That is, the standard IS schedule will give the correct conclusions only if expected future output is unaffected by the shocks that impinge on the economy, which is a case of limited empirical relevance). We showed that such an "expectational IS schedule" places substantial limits on interest rate rules under rational expectations. These limits derive from a major influence of expected future policies on the present level of inflation and real activity. Analysis of this model consequently required us to discuss alternative solution methods for rational expectations models in some detail. We focused on the conditions under which such equilibria exist and are unique.

Depending on the equilibrium concept that one employs, pure interest rate rules are either infeasible or undesirable when there is an expectational IS schedule. If one follows McCallum (1983) in restricting attention to minimum state variable equilibria, in which only fundamentals drive inflation and real activity, then there is likely to be no equilibrium under a pure interest rate rule. Equilibria are unlikely to exist because existence requires that the pure interest rate make the (unconditional) expected value of the nominal rate and the expected value of the real rate coincide, i.e., that it make the unconditional

expected inflation rate zero. We find it implausible that any central bank could exactly satisfy this condition in practice. Alternatively, if one follows Farmer (1991) and Woodford (1986) in allowing a richer class of monetary equilibria, in which fundamental and nonfundamental sources of shocks can be relevant to inflation and real activity, then there are also major limits or, perhaps more accurately, drawbacks to conducting monetary policy via a pure interest rate rule. The short-term effects of changes in interest rates on macroeconomic activity were found to be of arbitrary sign (or zero); the longer term effects are of opposite sign to the predictions of the standard IS model.

In Section 3, we followed prior work by Parkin (1978), McCallum (1981), and others in studying interest rate rules that have a nominal anchor. First, we showed that a policy of targeting the price level can readily provide the nominal anchor that leads to a unique real equilibrium: there need only be modest increases in the nominal rate when the price level is above its target path. Second, we also showed that a policy of inflation targeting requires a much more aggressive response of nominal interest rates: a unique equilibrium requires that the nominal interest rate must increase by more than one percent when inflation exceeds the target path by one percent. Our focus on these two policy targeting schemes was motivated by their current policy relevance.

In Section 4, we added expectations to the aggregate supply side of the economy, proceeding according to two popular strategies. First, we considered the flexible price aggregate supply specification that Sargent and Wallace (1975) used to study interest rate rules. Second, we considered the sticky price model of Calvo (1983) and Rotemberg (1982). Both of these extended models required the same parameter restrictions on policy rules with nominal anchors as in the simpler model of Section 3, thus suggesting a robustness of our basic results on the limits to interest rate rules and on the admissible form of nominal anchors in the IS model.

Having learned about the limits on interest rate rules in some standard macroeconomic models, we are now working to learn more about the positive and normative implications of alternative feasible interest rate rules in small-scale rational expectations models. We are especially interested in contrasting the implications of rules that require a return to a long-run path for the price level (as with our simple price level targeting specification) with rules that allow the long-run price level to vary through time (as with our simple inflation targeting specifications).

APPENDIX

This appendix discusses issues that arise in the solution of linear rational expectations models, using as an example the first model studied in the main text. That model is comprised of a Phillips curve ($\pi_t = P_t - P_{t-1} = \psi(y_t - \underline{y})$), an IS function ($y_t - \underline{y} = -s(r_t - \underline{r})$), the Fisher equation ($r_t = R_t - E_t\pi_{t+1}$) and a pure interest rate role for monetary policy ($R_t = \underline{R} + x_t$). Combining the expressions we find a basic expectational difference equation that governs the inflation rate,

$$\pi_t = \theta E_t\pi_{t+1} - \theta(\underline{R} - \underline{r} + x_t), \tag{24}$$

where we define $\theta = s\psi$ so as to simplify notation in this discussion. Iterating this expression forward, we find that

$$\pi_t = -\left\{ \sum_{j=0}^{J-1} \theta^{j+1} E_t\left[\underline{R} - \underline{r} + x_{t+j}\right] \right\} + \theta^J E_t\pi_{t+J}. \tag{25}$$

Our analysis will focus on the important special case in which

$$x_t = \rho x_{t-1} + \varepsilon_t, \tag{26}$$

where ε is a serially uncorrelated random variable, but we will also discuss some additional specifications.[28]
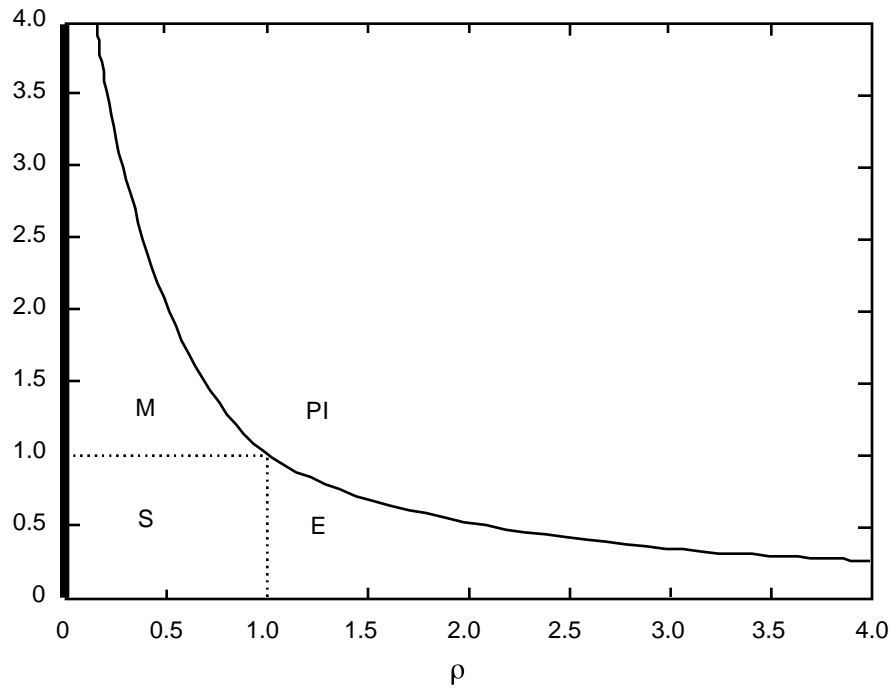
The Standard Case

The standard case explored in the literature involves the assumption that $\theta < 1$ and $\rho < 1$. Then, the policy rule implies that the interest rate is a stationary stochastic process and it is natural to look for inflation solutions that are also stationary stochastic processes. It is also natural to take the limit as $J \to \infty$ in (25), drop the last term, and write the result as

$$\pi_t = -\left\{ \sum_{j=0}^{\infty} \theta^{j+1} E_t\left[\underline{R} - \underline{r} + x_{t+j}\right] \right\}. \tag{27}$$

Figure A1 indicates the region that is covered by this standard case. Under the driving process (26), it follows that the stationary solution is one reported many times in the literature:

$$\pi_t = -\left\{ \frac{\theta}{1-\theta}\left[\underline{R} - \underline{r}\right] + \frac{\theta}{1-\theta\rho}x_t \right\}. \tag{28}$$

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[28] If we write a general autoregressive driving process as $x_t = qv_t$ and $v_t = \sum_{j=0}^{J} \rho_j v_{t-j} + \varepsilon_t$, then one can always (i) cast this in first-order autoregressive form and (ii) undertake a canonical variables decomposition of the resulting first-order system. Then, each of the canonical variables will evolve according to specifications like those in (26) so that the issues considered in this appendix arise for each canonical variable.

Figure A1 Alternative Solution Regions

This solution will be a reference case for us throughout the remainder of the discussion: it can be derived via the method of undetermined coefficients as in McCallum (1981) or simply by using the fact that $E_t x_{t+j} = \rho^j x_t$ together with the standard formula for a geometric sum.

In Figure A1, the region $\rho = 0$ is drawn in more darkly to remind us that it implicitly covers all driving processes of the finite moving average form,

$$x_t = \sum_{h=0}^{H} \delta_h \varepsilon_{t-h},$$

some of which will get more attention later.

Extension to $\rho \geq 1$

There are a number of economic contexts which mandate that one consider larger ρ. Notably, the studies of hyperinflation by Sargent and Wallace (1973) and Flood and Garber (1980), which link money rather than interest rates to prices, necessitate thinking about driving processes with large ρ so as to fit the explosive growth in money over these episodes.

It turns out that (28) continues to give intuitive economic answers when $\rho = 1$ even though its use can no longer be justified on the grounds that it involves a "stationary solution arising from stationary driving processes" as in Whiteman (1983). Most basically, if $\rho = 1$, then shifts in x_t are expected to be permanent in the sense that $E_t x_{t+j} = x_t$. The coefficient on x_t is therefore equal to the coefficient on $\underline{R} - \underline{r}$, which is natural since each is a way of representing variation that is expected to be permanent.

In Figure A1, the entire region E, as defined by $\rho \geq 1$ and $\theta\rho \leq 1$, can be viewed as a natural extension of the standard case. This latter condition is important for two reasons. First, it requires that the geometric sum defined in (27) be finite. Sargent (1979) refers to this as requiring that the driving process has exponential order less than $\frac{1}{\theta}$. Second, it requires that a solution of the form (28) has the property that

$$\lim_{J \to \infty} \theta^J E_t \pi_{t+J} = -\lim_{J \to \infty} \theta^J E_t \left\{ \frac{\theta}{1 - \theta}\left[\underline{R} - \underline{r}\right] + \frac{\theta}{1 - \theta\rho} x_{t+J} \right\} = 0,$$

so that it is consistent with the procedure of moving from (25) to (27). Violation of either the driving process constraint or the limiting stock price constraint implies that defined in (25) is infinite when $J \to \infty$. Parametrically, these two situations each occur when $\theta\rho \geq 1$ in Figure A1. Following the terminology of Flood and Garber (1980) these outcomes may be called process inconsistent, so that this region—in which equilibria do not exist—is labelled PI.

Extension to $\theta \geq 1$

There are also a number of models that require one to consider larger θ than in the standard case. In this case, McCallum (1981) has shown that there is typically a unique forward-looking equilibrium based solely on exogenous fundamentals. There may also be other "bubble" equilibria: these are considered further below but are ignored at present.

To understand the logic of McCallum's argument, it is best to start with the case in which $\rho = 0$ and $\underline{R} - \underline{r} = 0$. In this case, (24) becomes

$$\pi_t = \theta E_t \pi_{t+1} - \theta \varepsilon_t.$$

Since interest rate shocks are serially uncorrelated and mean zero, it is natural to treat $E_t \pi_{t+1} = 0$ for all t and thus to write the solution as

$$\pi_t = -\theta \varepsilon_t.$$

Thus, there is no difficulty with the finiteness of $\sum_{j=0}^{\infty} \theta^{j+1} E_t[x_{t+j}]$ in this case since $E_t[x_{t+j}] = 0$ for all $j > 0$. There is also no difficulty with $\lim_{J \to \infty} \theta^J E_t \pi_{t+J}$ since $E_t \pi_{t+J} = 0$ for all $J > 0$.

There are two direct extensions of this "white noise" case. First, with any finite order moving average process ($x_t = \sum_{h=0}^{H} \delta_h \varepsilon_{t-h}$), it is clear that similar solutions can be constructed that depend only on the shocks in the

moving average.[29] In this case, it is also clear that $\sum_{j=0}^{\infty} \theta^{j+1} E_t[x_{t+j}] < \infty$ since $E_t[x_{t+J}] = 0$ for all $J > H$. Likewise, it is clear that $\lim_{J \to \infty} \theta^J E_t \pi_{t+J} = 0$ since $E_t \pi_{t+J} = 0$ for all $J > H$. Second, for any $\rho \leq \frac{1}{\theta}$, it follows that the stationary solution (28), which is $\pi_t = -\frac{\theta}{1-\theta\rho} x_t$ in this case, is a rational expectations equilibrium for which the conditions $\sum_{j=0}^{\infty} \theta^{j+1} E_t[x_{t+j}] < \infty$ and $\lim_{J \to \infty} \theta^J E_t \pi_{t+J} = 0$ are fulfilled since $\rho\theta < 1$. The full range of equilibria studied by McCallum is displayed in the area of Figure A1.

As stressed in the main text, there is also a central limitation associated with this region—there cannot be a constant term in the "fundamentals" that enter in equations like (24), which implies that in this context that $\underline{R} = \underline{r}$. The reason that this constant term is inadmissable when $\theta \geq 1$ is direct from (25): if it is present when $\theta \geq 1$, then it follows that the limiting value of the fundamentals component is infinite. While potentially surprising at first glance, this requirement is consistent with the general logic of McCallum's solution region—as indicated by Figure A1, it is obtained by requiring driving processes that have exponential order less than $\frac{1}{\theta}$, so that a constant term is generally ruled out along with $\rho = 1$ since, as discussed above, each is a way of representing permanent changes.

Bubbles

To this point, we have considered only solutions based on fundamentals. Let us call these solutions f_t and write the inflation rate as the sum of these and a bubble component b_t:

$$\pi_t = f_t + b_t.$$

In view of (24), the bubble solution must satisfy

$$b_t = \theta E_t b_{t+1}$$

or equivalently

$$b_{t+1} = \frac{1}{\theta} b_t + \zeta_{t+1},$$

where ζ_{t+1} is a sequence of unpredictable zero mean random variables (technically, a martingale difference sequence). Thus, in the standard case of $\theta < 1$, the bubble must be explosive—this sometimes permits one to rule out bubbles on empirical or other grounds (such as the transversality condition in certain optimizing contexts). By contrast, in the situation where $\theta > 1$ then the bubble component will be stationary.

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[29] The form of this solution is $\pi_t = \sum_{h=0}^{H} \omega_h \varepsilon_{t-h}$, where the ω coefficients satisfy $\omega_h = \sum_{j=0}^{H-h} \theta^{j+1} \delta_{h+j}$.

These conditions arise because the bubble enters only in the term in (25) with the "exponential coefficient" θ^J. If $\theta < 1$, the future is discounted: we require that very large changes in expectations about the future must take place to produce a bubble of a given size today. By contrast, with $\theta > 1$, a very small change in long-term expectation can induce a bubble of a given size today because it is "emphasized" rather than discounted by the term θ^J.

Bubble solutions are sometimes written as

$$\pi_t = \frac{1}{\theta}\pi_{t-1} + R_{t-1} + \xi_t, \tag{29}$$

where ζ_{t+1} is a sequence of unpredictable zero mean random variables as in Farmer (1991). In this solution, the lagged inflation rate appears as a "state variable" and there is no evident effect of shocks to R_t on π_t. This latter implication is apparently inconsistent with the $\pi_t = f_t + b_t$ decomposition that we used earlier. However, upon substitution, we find that

$$\pi_t = f_t + b_t = \frac{1}{\theta}(f_{t-1} + b_{t-1}) + R_{t-1} + \xi_t,$$

and using $\theta E_{t-1} f_t = b_{t-1} + \theta R_{t-1}$, we find that

$$(f_t - E_{t-1}f_t) + (b_t - E_{t-1}b_t) = \xi_t,$$

where $E_{t-1}b_t = \frac{1}{\theta}b_t$. Thus, in the representation (29), ξ_t could depend on shocks to R_t since it is arbitrary. Alternatively, $(b_t - E_{t-1}b_t)$ could "offset" shocks to $(f_t - E_{t-1}f_t)$, leaving no effects of changes in the interest rate within period t.

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