A THEORY OF BARGAINING
WITH MONETARY TRANSFERS

by

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I. Introduction

The outcome of a bargaining situation often involves monetary transfers among the participants. These transfers serve to modify the payoffs associated with an underlying agreement that has been reached. They may represent the resolution of a conflict between achieving equity among the players and implementing an efficient decision. Some of the players may justifiably claim that they should be compensated for acceding to a collective choice in which they are worse off than they would be in another less efficient, feasible outcome.

In this paper I present a normative theory for a class of bargaining problems in which monetary transfers are possible. The data of each problem are the payoff vectors attainable by various underlying agreements. A resolution of a bargaining problem consists of two parts: the underlying agreement to be implemented, and the monetary transfers that will be made so as to modify the payoffs inherent in this agreement.

A solution is a rule that specifies how each of these two components of the outcome depends upon the set of possible underlying agreements. I derive solutions from a set of axioms that express an ethical basis for collective choice in this type of situation.

There is an important difference between this model and bargaining theories in the Nash tradition. Nash-type bargaining models, having their basis in decision theory, determine the solution as a function only of the feasible set of utility allocations. They do not allow any dependence of the outcome upon how the final allocation is realized. In this paper, on the other hand, a distinction is drawn between that part of the payoff that is inherent in the underlying agreement and that part represented by the monetary transfer. Two bargaining situations with quite different sets of underlying possibilities can still have the same feasible set of final utility allocations because of the ability to implement transfers. I will not require the outcomes to be identical in such instances.

This feature is responsible for the special structure of this model and its solution. It hampers a direct comparison with Nash-type theories of bargaining. However, the axioms I impose have the same ethical basis as those often utilized in Nash-type models. It is interesting to observe that although in Nash-type models the axioms I use would be in conflict, the analogous principles in the model of this paper can be satisfied simultaneously.

The remainder of the paper is organized as follows: The model of bargaining is set out in Section II. Section III describes the axioms and gives a brief discussion. Section IV characterizes a family of solutions in bargaining problems involving no equity-efficiency tradeoff. In Section V an attempt is made to select among these solutions based on an additional postulate. In Section VI these solutions are extended to general bargaining problems in which equity-efficiency tradeoffs exist. Section VII gives various examples of the solution. Some of these are used as counter-examples to support statements made earlier in the text. Section VIII discusses various aspects of the model and the axioms in a more extended fashion than could be accomplished earlier. An Appendix contains the proofs of the main Theorems.

II. The Bargaining Model

We consider bargaining problems involving n players, identified with the elements of $\{1, \ldots, n\}$. The basic structure of the bargaining situations studied in this paper is given by a set of points in R^n; each member of this set describes the outcome of a bargaining or negotiation process. Generally, the symbol S will denote this set and lower case Roman letters, x, y, z, ..., will denote points in R^n. If $x \in S$, we understand that there is some outcome which gives player i the payoff x_i, for $i = 1, \ldots, n$. The set S will be called the set of "underlying possibilities," or "real outcomes." By this we mean that there are feasible bargaining arrangements that lead to the payoffs in S without resorting to monetary transfers.

The payoffs in S can be combined with monetary transfers among the agents. The actual result of the bargaining process specifies an underlying outcome, yielding x_i to each player, and a vector of transfers $t = (t_i)$ such that $\Sigma t_i = 0$. Player i's evaluation of this result is assumed to be $x_i + t_i$.

Thus, the set of payoffs that are achievable from an underlying possibility set S is

$$\bar{L}(S) = \{z \in R^n \mid z = x + t,\ x \in S,\ t \in R^n,\ \Sigma t_i = 0\}.$$

The principal difference between the theory described in this paper and the Nash-type bargaining models is that the solutions we study can, and do, select different points in $\bar{L}(S)$ as a function of S, even though the set $\bar{L}(S)$ itself may be invariant.

The linearity of utility in the monetary transfer is, of course, a very particular specification. Were all players risk neutral, it would be satisfied. In general, however, it is not. For bargains among large players, such as firms or labor unions, it may be a good approximation. Moreover, one must recognize that the risk attitudes of agents are their own private information. In Nash-type bargaining models, agents would always feign risk-neutrality in order to manipulate the solution to their advantage. There may be additional merit, therefore, in focusing on the specification of utility described above.

The following properties of S are assumed:

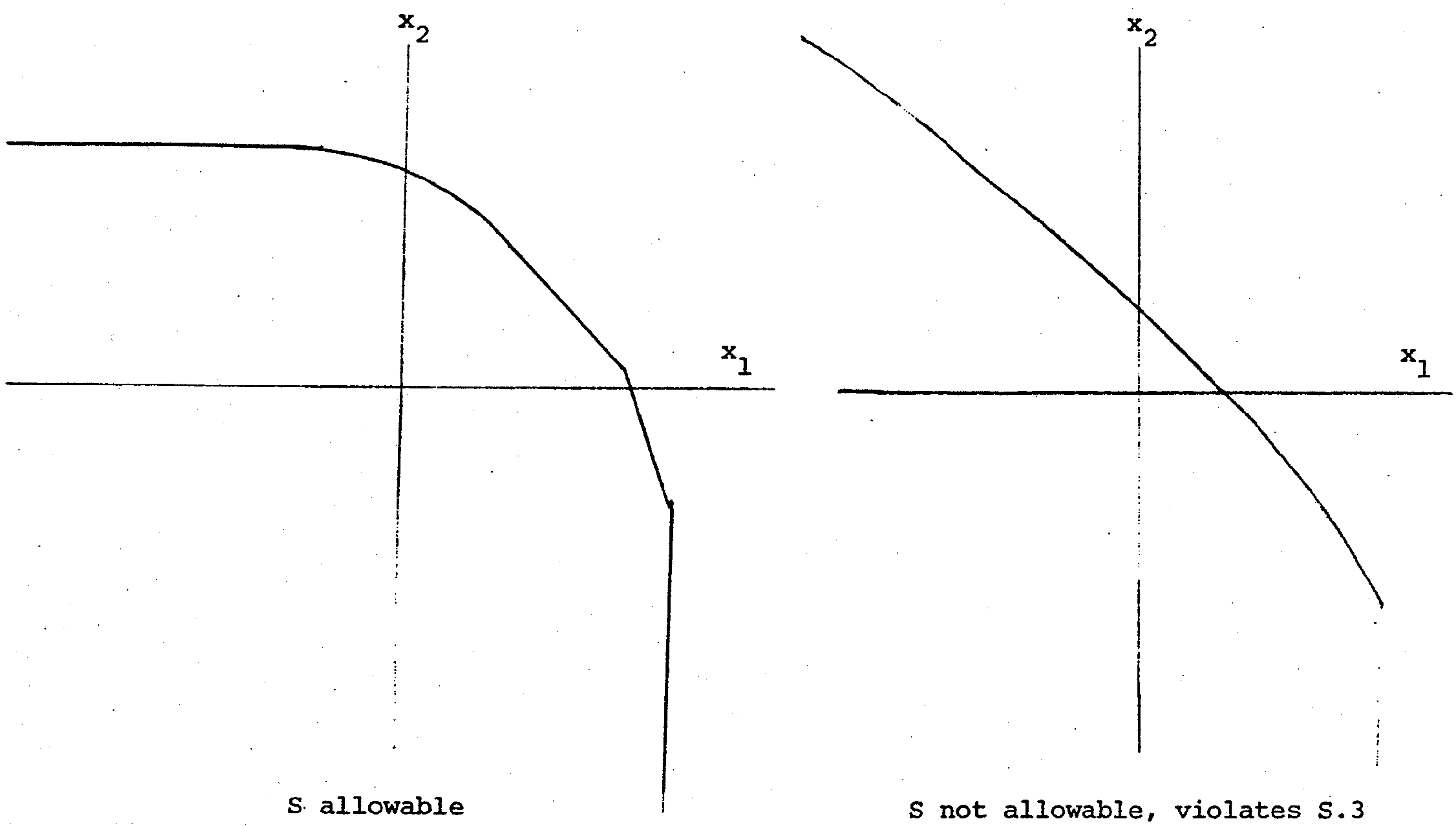
S.1. S is nonempty, closed and convex.

S.2. S is comprehensive: $x \in S$ and $y < x$ implies $y \in S$.

S.3. S is majorized: there exists x_{max} such that $x \in S$ implies $x \leqslant x_{max}$.

Assumptions S.1 and S.2 are standard. S.1 entails the idea of randomizing among a set of basic agreements to create any convex combination of their payoffs. S.3 is of a technical nature and is imposed so as to rule out cases where an infinite number of agreements exist that successively benefit one player at another's unbounded expense. Allowable and nonallowable sets of underlying possibilities are shown in Figure 2.1.

There are three distinct roles played by assumption S.3 in our theory. First, it insures the existence of a point, $x^* \in S$, at which $x_1 + x_2$ is maximized. Of course, there could be many maximizers. Second, it insures that the set of such maximizers is compact. Finally,



S allowable S not allowable, violates S.3

Figure 2.1

by virtue of S.2, we are assured the existence of a <u>least majorizing point</u> $\bar{x}$ given by the properties:

$$x \in S \text{ implies } x \leqslant \bar{x}$$

and if x_o is such that for all $x \in S$, $x \leqslant x_o$, then $x_o \geqslant \bar{x}$.

An <u>outcome</u> of the bargaining problem consists of a point $x \in S$ and a vector of monetary transfers $t \in R^n$ such that $\Sigma t = 0$. The outcome (x, t) generates the <u>payoffs</u> $x + t \in R^n$.

Let $\mathscr{S}$ be the collection of all sets S satisfying S.1, S.2 and S.3. For any $S \in \mathscr{S}$ we define the <u>efficient payoff plane</u>,

$$L(S) = \{z \in R^n \mid z = x + t, \Sigma t_i = 0, x \in S \text{ and } \Sigma x_i \geqslant \Sigma x'_i \text{ for all } x' \in S\}.$$

A <u>solution</u> is a function

$$f : \mathscr{S} \rightarrow R^n$$

satisfying $f(S) \in L(S)$ for all $S \in \mathscr{S}$. Thus, solutions always result in efficient outcomes, by definition.

The interpretation of a solution is clear. One of the maximizers of Σx_i over S is chosen. Then, transfers are made from this point so as to achieve the payoffs f(S).

Throughout this paper we will deal with members of $\mathscr{S}$, that is to say, with comprehensive, closed, convex, sets of underlying possibilities. Two sets of underlying possible bargaining outcomes generating the same comprehensive convex hull lead, by assumption, to the same payoffs. It will ease our notation somewhat if we identify every set in R^n with its comprehensive convex hull. In effect, we are defining f on the equivalence classes of subsets of R^n obtained by the property of having the same comprehensive hull, whenever that comprehensive hull is in the domain $\mathscr{S}$.

III. Axioms

In this section I will describe the axioms to be used. The main theorem, which follows in Section IV, does not require all these axioms in the case of two players. However, all are used in the general case.

Most readers will find axioms A.1 and A.2 innocuous, and A.3 and A.4 somewhat restrictive but standard in nature. These four axioms are analogous to the Maschler-Perles axioms, adapted to the present transferable utility context.

It is interesting to observe that without transferable utility they are inconsistent for any $n > 2$, and for $n = 2$ they define a unique solution. In the model of this paper, they are consistent for all n. For $n = 2$ they define a unique solution, and for $n > 2$ there are many solutions.

In order to narrow down the set of solutions, hopefully obtaining a unique member of this family, I postulate a new axiom, A.5, in Section V. A.5 is unusual in several respects, and it is also somewhat problematical as we shall see below. Nevertheless, I will try to make a compelling case for it later in this section. Axiom A.5 is not independent of A.1 - A.4; rather it characterizes "the" solution in terms of an invariance property with respect to the set of all solutions satisfying these hypotheses.

Axioms A.1 - A.4 are all used to obtain solutions on the family of sets in $\mathcal{S}$ that are the comprehensive hulls of compact convex subsets of hyperplanes $L_\alpha = \{x \in R^n | \Sigma x_i = \alpha, \alpha \in R\}$. Axiom A.6, introduced in Section VII, is used to extend the solution to all of $\mathcal{S}$ by associating, to each $S \in \mathcal{S}$, another member of $\mathcal{S}$ that is the comprehensive hull of some set in $L(S)$. Axiom A.6 is not compelling, in my view. There may well be equally defensible methods of extending the solution.

A technical discussion of some aspects of axioms A.1 - A.4 is postponed to Section VIII. In the present section I will confine my attention to their basic properties.

Let $\mathscr{K}$ be the family of sets in $\mathscr{S}$ that are the comprehensive hulls of their own intersections with L(K). That is, $K \in \mathscr{K}$ if and only if

$$K = \{z \in R^n \mid z \leqslant x, \text{ for } x \in K \cap L(K)\} .$$

Bargaining problems in $\mathscr{K}$ can be thought of as those involving no equity-efficiency tradeoff. The entire Pareto boundary of $K \in \mathscr{K}$ is efficient even in the presence of monetary transfers. This motivates Axiom 1, which is stated as follows:

A.1. <u>Selection</u>

For all $K \in \mathscr{K}$, $f(K) \in K$.

This axiom states that in those bargaining problems which do not involve equity-efficiency tradeoffs, there is no reason to use monetary transfers to reach an outcome that could not be achieved by randomizing directly over the set of efficient choices. The role of transfers in this model is to compensate players who would have been favored in an underlying outcome that was rejected in favor of a more efficient alternative. Transfers are the vehicle through which an equity-efficiency tradeoff is effected.

Axiom 1 also entails the fact that $f(\{x\}) = x$ for all $x \in R^n$. This is a type of unbiasedness principle: If there is only one possible outcome, transfers are not relevant and should not be used. This property, together with linearity (to be assumed as A.3), imply that solutions are translation-invariant — a natural property since risk-neutral utilities are defined only up to such transformations as well.

Let π be a permutation of the set of players $\{1, \ldots, n\}$. Let $\pi(x)$ be the vector in which the coordinates of x are permuted according to π. If $S \in \mathscr{S}$ let $\pi(S)$ be the set in $\mathscr{S}$ consisting of $\pi(x)$ for all $x \in S$.

A.2. <u>Anonymity</u>

The solution f commutes with all permutations π.

This axiom is standard.

A.3. <u>Additivity</u>

For $S_1, S_2 \in \mathscr{S}$ and $\lambda \in [0, 1]$

$$\lambda f(S_1) + (1-\lambda)f(S_2) = f(\lambda S_1 + (1-\lambda)S_2).$$

The additivity axiom has been extensively discussed by Maschler and Perles (1981) under the idea of superadditivity: $\lambda f(S_1) + (1-\lambda)f(S_2) \leqslant f(\lambda S_1 + (1-\lambda)S_2)$. Their justification for it is as follows: Imagine a bargaining situation in which one of two feasible sets S_1 or S_2 will arise with probabilities λ and $1-\lambda$, respectively. If the players cannot bargain before the resolution of the uncertainty, and f is the solution, they get $\lambda f(S_1) + (1-\lambda)f(S_2)$. If they can bargain before, and can make their agreements contingent on the resolution, then the feasible set is $\lambda S_1 + (1-\lambda)S_2$. The axiom can be interpreted as saying that the ability to make such contracts should not be detrimental to any player. If it were, he should refuse to agree, preferring the expected bargaining outcome to be reached at a later point in time.

In the case with transfers, the linearity of the efficient plane $L(S)$ with respect to $S \in \mathscr{S}$ precludes the case of strong inequality in these relations. Therefore A.3 and superadditivity are the same.

A.4 Continuity

The solution f is continuous in the Hausdorff topology on $\mathcal{S}$.

The Hausdorff topology defines a concept of convergence for sets. A sequence $\langle S_i \rangle$ converges to S if every point of S can be approximated by a sequence selected from $\langle S_i \rangle$ and every point not in S can be bounded away from $\langle S_i \rangle$. This is the standard concept of convergence for closed sets. However, there are other possibilities, defining both stronger and weaker continuity axioms.

A more detailed discussion must be postponed until Section VII where examples will be given. For the present, it must suffice to say that weaker convergence concepts, implying therefore a stronger continuity axiom, are incompatible with the other axioms. And stronger convergence concepts are too vulnerable to arbitrary errors of measurement and description.

Restricted to $\mathcal{K}$, these alternative topologies all coincide with the Hausdorff topology. We will now show how the four axioms presented thus far delineate a family of solutions on $\mathcal{K}$. Later sections will extend these solutions to $\mathcal{S}$ and will attempt to select among them on the basis of other criteria.

IV. Solutions on $\mathcal{K}$

In this section we define a family of solutions on $\mathcal{K}$. To understand the nature of their construction, it is useful to know about the Steiner point of a convex body. A brief digression on this topic follows:

Let $\mathscr{C}$ be the class of compact, convex subsets of R^n and let S^{n-1} be the surface of the sphere in R^n of unit radius and centered at zero. Let μ be the measure on S^{n-1} proportional to Lebesgue measure and with mass 1. Finally, let $\phi(C,u) = \max_{y \in C} y \cdot u$ be defined for each $C \in \mathscr{C}$ and each $u \in S^{n-1}$. It is called the support function of C at u.

The Steiner point of C, written s(C), is defined by

$$s(C) = n \int_{S^{n-1}} u\phi(C,u)\, d\mu \, .$$

If σ is a similarity transformation carrying C_1 into C_2, then $\sigma(s(C_1)) = s(C_2)$. That is, s commutes with all similarities. Moreover, s satisfies linearity, because $\phi(\cdot, u)$ is linear, and s is continuous.

The following theorem is due to Schneider (1971).

Steiner Point Characterization Theorem

The Steiner point $s: \mathscr{C} \to R^n$ is the unique function satisfying

i) commutation with all similarity transformations

ii) linearity

iii) continuity.

Now let us return to the space $\mathcal{K}$. Let $\mathcal{K}_o \subseteq \mathcal{K}$ be the set of bargaining situations $K \in \mathcal{K}$ for which $L(K) = \{x \mid \sum_{i=1}^{n} x_i = 0\}$. Because the Selection axiom (A.1) entails $f(\{x\}) = x$, the hypothesis of Additivity gives us translation invariance. It suffices therefore to find a solution on $\mathcal{K}_o$. This solution can then immediately be extended to all of $\mathcal{K}$.

Obviously (A.2) - (A.4) are very close to the three properties used in Schneider's theorem. The difference is that (A.2) requires commutativity only with respect to permutations of coordinates, which are very special cases of similarities. We characterize the larger family of functions, of which the Steiner point is a member, that are only required to commute with permutations.

We will see that (A.1) also plays a crucial role in this characterization. It is a consequence of Schneider's theorem, as it is satisfied by the Steiner point. But under the weakening of the commutativity requirement, it is needed as an independent hypothesis. (Examples to this effect are given in Section VIII.) Therefore the theorem given below cannot be regarded, strictly speaking, as a generalization of Schneider's theorem.

Let $\hat{S}^{n-1} = S^n \cap L_o$. That is, $\hat{S}^{n-1}$ is an n-1 sphere in R^n, centered at zero, and oriented in such a way that the algebraic sum of the coordinates of each of its points is zero. Let $\hat{M}$ be the set of atomless unit measures on $\hat{S}^{n-1}$ which are invariant with respect to permutation of coordinates.

<u>Theorem 1</u>

Let $n \geqslant 3$.

Let $f: \mathcal{K}_o \to R^n$ satisfy (A.1) - (A.4), then

$$f(K) = \int_{\hat{S}^{n-1}} \operatorname*{argmax}_{x \in K} x \cdot u \, d\hat{\mu}(u)$$

for some $\hat{\mu} \in \hat{M}$.

For $n = 2$, if $f: \mathscr{K}_0 \to R^n$ satisfies (A.2) and (A.3), then $f(K)$ is the midpoint of K.

To demonstrate this theorem, we will first drop the Anonymity requirement and show

<u>Proposition 1</u>

Let $n \geq 3$.

Let $f: \mathscr{K}_o \to R^n$ satisfy (A.1), (A.3) and (A.4), then

$$f(K) = \int_{\hat{S}^{n-1}} \underset{x \in K}{\operatorname{argmax}}\, x \cdot u \; d\mu(u)$$

for some μ, an atomless measure on $\hat{S}^{n-1}$.

<u>Proof</u> in appendix.

One should note that the formula in Theorem 1 is well-defined for all $K \in \mathscr{K}_o$ because the set of all $u \in \hat{S}^{n-1}$ for which $\underset{x \in K}{\operatorname{argmax}}\, x \cdot u$ is not a singleton is closed and of measure zero with respect to any atomless measure. It is immediate that Proposition 1 implies Theorem 1, via the Anonymity axiom (A.2). (The case of $n = 2$ is trivial, and is included only for completeness.)

V. Limiting the Solutions on $\mathscr{K}$: The Recursive Solution

In the last section we characterized the solutions on $\mathscr{K}$ that are compatible with (A.1) - (A.4). In this section we explore the possibility of limiting these solutions on the basis of another axiom. The goal would be, ideally, to find a unique solution, but we are only partially successful in this quest.

Let G be the set of all solutions satisfying A.1 - A.4. Given a bargaining situation S, the effective range of disagreement is limited to those points that could be obtained as the result of some solution $f \in G$. Any such point could be defended, in that an "ethical" procedure selected it. Other points cannot be defended in this way.

Let Γ be the union of the graphs of solutions in G. That is $\Gamma \subseteq \mathscr{S} \times R^n$ defined by

$$(S, x) \in \Gamma \quad \text{if and only if}$$
$$x = f(S) \quad \text{for some} \quad f \in G.$$

Let $\Gamma_S = \{x \mid (S,x) \in \Gamma\} = \{x \mid x = f(S) \text{ for some } f \in G\}$.

Viewing bargaining as a process of compromise reached by making ethical appeals to the arbitrator, the bargain over points in S is reduced to the bargain over solutions in G. It is natural, therefore, to assume that if the point $x \in \Gamma_S$ would be the outcome of the bargain over Γ_S, then the solution $f \in G$ such that $x = f(S)$ would be the outcome of the bargaining about which solution to use.

This idea leads to a recursive method to define the solution, which is embodied in the next axiom. Before it can be stated, however, a technical problem has to be faced. We would like to write simply,

$f(\Gamma_S) = f(S)$. Unfortunately, as we shall see, the comprehensive hull of Γ_S may not be in $\mathcal{S}$. It is typically not closed. Taking its closure, $\bar{\Gamma}_S$, and imposing the condition $f(\bar{\Gamma}_S) = f(S)$ seems to be a natural way around this difficulty. However, this procedure would be ill behaved, as we shall see in example 4. The problem is that the correspondence defined by $S \longmapsto \bar{\Gamma}_S$ is not upper hemicontinuous.

We therefore adopt the following construction: Let $\Gamma^* \subseteq \mathcal{S} \times R^n$ be the closure of Γ. Let $\Gamma^*_S = \{x | (S,x) \in \Gamma^*\}$. The comprehensive hull of Γ^*_S will always be in $\mathcal{S}$.

Our next axiom can now be stated.

A.5. <u>Principal of Recursivity</u>

The solution f satisfies $f(S) = f(\Gamma^*_S)$ for all $S \in \mathcal{S}$.

The justification for taking the closure of Γ in the axiom is based on the idea of errors of measurement in the assessment of the bargaining outcomes. If x_k is justifiable as an outcome of S_k, for $k = 1, 2, \ldots$, that is if there exists a sequence $\langle f_k \rangle$ in G such that $x_k = f_k(S_k)$, and if $x_k \to x$ and $S_k \to S$, then x should be a justifiable outcome of S. Otherwise, small disagreements among the players about the nature of the outcomes themselves could precipitate large changes in the set of ethically defensible alternatives. Using Γ^* we obtain a set of ethically defensible alternatives that is robust with respect to measurement error.

We will apply A.5 repeatedly to obtain a point-valued solution as the limit of iterates of the Γ^* correspondence: Beginning with any $S \in \mathcal{S}$, we consider Γ^*_S, $\Gamma^*_{\Gamma^*_S}$, ... , etc., and we will show that this sequence converges to a point.

It remains to be demonstrated, however, that this limit, which we will call the Recursive Solution, satisfies the original four axioms. It is here that there are some delicate technical problems with respect to continuity and linearity. These are discussed in more detail below.

Define the correspondence

$$F: \mathscr{S} \rightarrow R^n$$

by $F(S) = \Gamma_S^*$.

Theorem 2

The correspondence F is upper hemicontinuous, compact-valued, and, for each nonsingleton $K \in \mathscr{K}_o$, F(K) contains no extreme point of K.

Proof in appendix.

From Theorem 2 it is easy to see that the iterates of F converge to a point. This point is the Recursive Solution at K.

Our goal would be fulfilled if the Recursive Solution were to satisfy (A.1) - (A.4). It clearly does satisfy (A.1) and (A.2). Unfortunately, there are problems with respect to both (A.3) and (A.4).

The continuity axiom (A.4) is the less troublesome. There are some critical sets where the correspondence F fails to be continuous because of the extension of Γ to Γ^*. These induced discontinuities are hard to characterize, in general. Apparently they arise only when a face of K of dimension n-2 is parallel to a coordinate n-2 hyperplane, or when there are two faces of K which form exactly the same angle as that between two such hyperplanes. See example 5. It would seem that such situations are non-generic and that one could hope to characterize the discontinuity set more precisely, but I have not pursued this point.

It is far more disturbing, from the point of view of the theory, to note that this Recursive Solution may not obey Additivity. The problem is that the correspondence F is not additive, but rather subadditive. By virtue of Theorem 1 we know that if $x \in F(K)$ then there exists $\hat{\mu} \in \hat{M}$ such that

$$x = \int \arg\max_{z \in K} z \cdot u \, d\hat{\mu} .$$

Therefore, if $K = K_1 + K_2$, then every $x \in F(K)$ can be written in this way, and hence

$$x = \int \arg\max_{z \in K_1} z \cdot u \, d\hat{\mu} + \int \arg\max_{z \in K_2} z \cdot u \, d\hat{\mu} .$$

The terms on the right-hand side being in $F(K_1)$ and $F(K_2)$, respectively, we have

$$F(K) \subseteq F(K_1) + F(K_2) .$$

The opposite inclusion does not hold, as we show in example 6. Thus, the linearity of the Recursive Solution is in doubt. On the other hand, however, I have no counterexample. The Recursive Solution is not the same as the Steiner point. It remains possible that this is due to the violation of continuity alone — as there are many set-point functions satisfying all the other postulates. (See Sallee (1971).)

VI. Extending Solutions on $\mathscr{K}$ to Solutions on $\mathscr{S}$

The axioms given thus far, A.1 - A.5, are designed to delineate ethical outcomes for bargaining problems in $\mathscr{K}$. To extend solutions to all of $\mathscr{S}$, we need another postulate that will associate to each $S \in \mathscr{S}$ some member $K \in \mathscr{K}$ at which the solution is to be the same.

Underlying this axiom is the concept of a justified transfer, which will be made precise below. For the moment, assume that to each $S \in \mathscr{S}$ we can associate a set $J(S) \subseteq L(S)$ with the interpretation that a transfer of money among the players that realizes $x \in J(S)$ can be justified according to some criterion. The allocation x is feasible, as it lies in L(S). Suppose that a new bargaining outcome were added to S that attained x without the need to resort to monetary transfers. As x is already feasible, and is "justifiable" (in a sense yet unspecified), its addition to the underlying set of alternatives should leave the solution unaffected.

Thus the solutions are extended to all of $\mathscr{S}$ by identifying the solution on S with that on J(S). Different concepts of justifiability will result in different extensions. Before putting forward one particular concept, let us discuss some of the properties that a definition of justifiability must have to be useful within our framework.

Compatibility with the other axioms requires that the correspondence J satisfy

(J.1) $L(S) \cap S \subseteq J(S)$

(J.2) J(S) commutes with the permutation of coordinates

(J.3) J(S) is linear

(J.4) J(S) is continuous in the Hausdorff topology.

The idea of a justified transfer is that it leads to an efficient allocation by virtue of appeals based on the presence of inefficient underlying agreements that are favorable to one or more of the players. When the bargaining situation involves no equity efficiency tradeoff, no such appeals are possible. Therefore it is natural to impose the requirement

(J.5) $J(S)$ is the identity on $\mathscr{K}$.

Finally, appeals to the arbitrator should never provide to a player more than he would get in any underlying agreement. Thus we have

$$\text{(J.6)} \qquad J(S) \subseteq L(S) \cap \left(\{\bar{x}\} - R^n_+\right) .$$

Properties (J.1) - (J.6) seem to be minimal requirements for our theory. At present I do not know if they characterize a particular correspondence or class of correspondences. In this section I present one example of a correspondence that satisfies these properties. In Section VIII.5 I discuss some other possible concepts of justifiability, and I will show why they fail on one or more of these criteria.

The concept of justifiability is based on the idea that the arbitrator has chosen the point $x \in L(S)$ by making a Pareto-improving move from S with equal probability being attached to each of the players being benefited, and subject to the further provision that no player can obtain more at x than he could at the most favorable point in S. Formally, let

$$T_i(\bar{S}) = \{x \in L(S) \mid x = y + z_i,\ y \in \bar{S},\ z_{ii} \geq 0,\ z_{ij} = 0,\ j \neq i,\ x_i \leq \bar{x}\}$$

where $\bar{S}$ is the Pareto frontier of S and $\bar{x}$ is the least majorizer of S. The justified set, J(S), is given by

$$J(S) = \frac{1}{n} \sum_{i=1}^{n} T_i(\bar{S}) .$$

The axiom is

A.6. <u>Invariance to Additions of Justifiable Points</u>

$$f(S) = f(S \cup \{x\}) \qquad \text{for all} \quad x \in J(S)$$

where we recall our convention of identifying sets with their comprehensive convex hulls.

Repeated application of A.6 yields

$$f(S) = f(S \cup J(S)) ,$$

but, as S is contained in the comprehensive hull of J(S), A.6 reduces to

$$f(S) = f(J(S)) .$$

Thus, this axiom has accomplished the identification of sets in $\mathcal{S}$ with sets in $\mathcal{K}$.

The application of (A.6), Invariance to the Addition of Justified Alternatives, provides us with a direct correspondence between a solution on $\mathcal{S}$ and a solution on $\mathcal{K}_o$ defined in Section IV. The continuity and linearity of the justified set, J(S), with respect to S, together with its obvious commutativity with respect to permutations of coordinates, ensures that these solutions are well-defined and satisfy axioms (A.1)-(A.4) on $\mathcal{S}$.

VII. Examples

In the following examples we demonstrate how F(S) varies with S in a few cases. All these problems will involve three players, as the solution for two players is very simple and the solution for four or more is beyond (my) geometric intuition.

It is useful to have a picture of L_o, and to be able to describe the permutation-invariant measures on $\hat{S}^2$. This is shown in Figure 7.1.

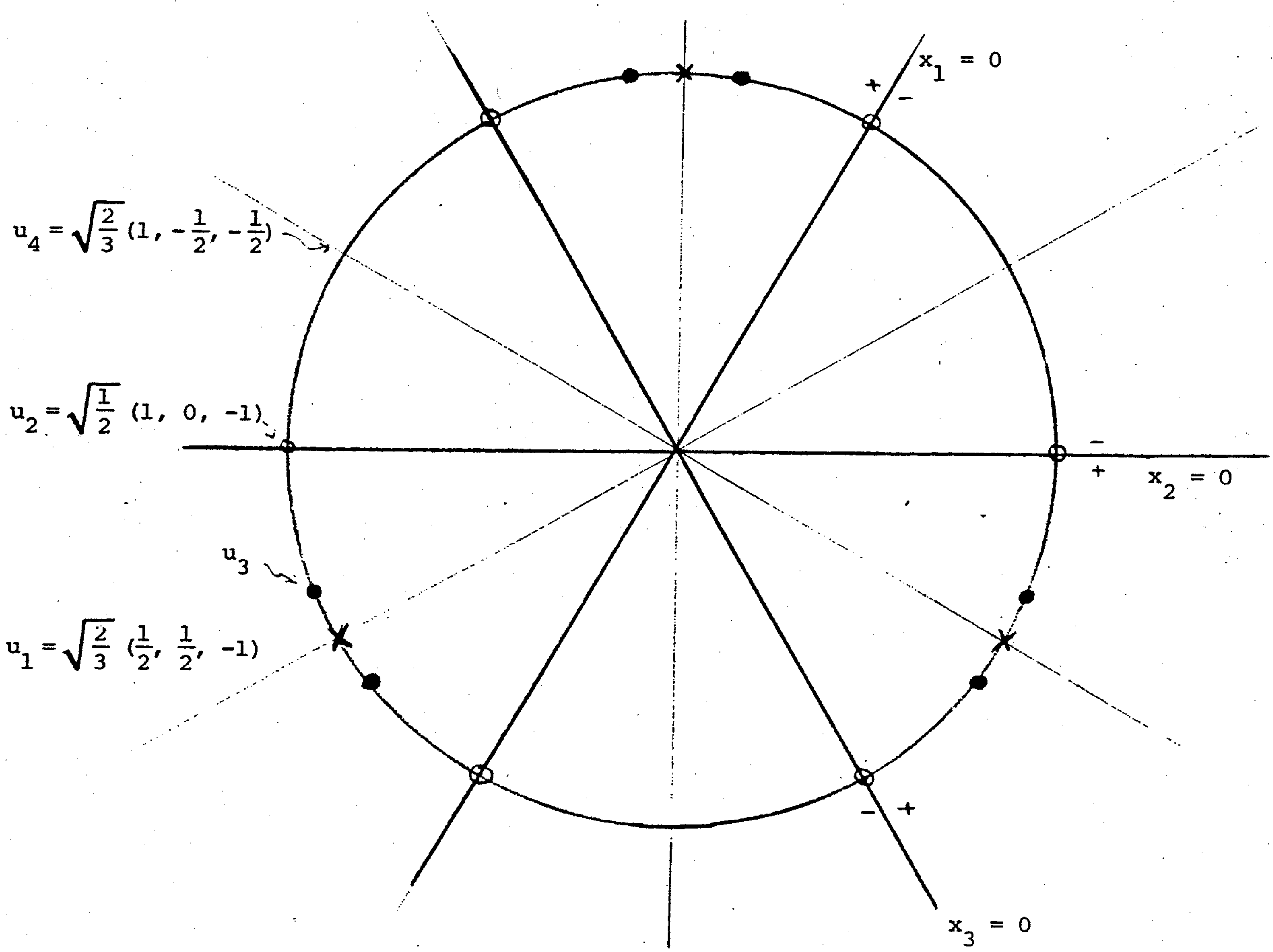


Figure 7.1. L_o, $\hat{S}^2$, and some permutation-invariant measures.

Figure 7.1 depicts the coordinate axes in the plane L_o, as indicated by the lines $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$. It also shows some other points of interest and usefulness in the construction of permutation-invariant measures.

Consider any point, u, in $\hat{S}^2$. Unless two of the coordinates of u are equal, the six permutations of u will be distinct. A permutation-invariant measure will be obtained by giving all six of these points mass $\frac{1}{6}$. When there are only three distinct permutations of u, one weights them each $\frac{1}{3}$.

It is easy to see that all permutation-invariant measures are convex combinations of measures generated in this way. Several such measures are shown in this figure. If $u_1 = \sqrt{\frac{2}{3}}\left(\frac{1}{2}, \frac{1}{2}, -1\right)$, the permutations are as shown by the points marked X. If $u_2 = \sqrt{\frac{1}{2}}\ (1, 0, -1)$, the permutations are located at the points marked o. Between u_1 and u_2, say at u_3, the permutations take values at the six points marked ●.

In this way, we can graphically depict all the permutation-invariant measures by parametrically moving u from $u_1 = \sqrt{\frac{2}{3}}\left(\frac{1}{2}, \frac{1}{2}, -1\right)$ to $u_4 = \sqrt{\frac{2}{3}}\left(1, -\frac{1}{2}, -\frac{1}{2}\right)$ and by obtaining the induced permutations of u in each instance. This is the procedure that will be followed to obtain F(K) in the examples below.

Example 1

Consider any bargaining situation with exactly two equally efficient underlying outcomes, a and b. Let S be the segment from a to b. The set of solutions satisfying (A.1) - (A.4) is the middle third of the line segment S. Consequently, the solution reached upon applying (A.5) repeatedly is the midpoint of S, which, obviously, happens to coincide with its Steiner point.

To demonstrate this conclusion, take any line in the plane L_o passing through the origin. This divides $\hat{S}^2$ into two parts. By direct inspection one can see that for any u between u_1 and u_2, the number of permutations of u contained in one of these two parts must be 2, 3, or 4. Thus the weight associated to a given endpoint of S in the characterization formula of Theorem 1 is between $\frac{1}{3}$ and $\frac{2}{3}$.

<u>Example 2</u>

In this example we will find the set F(S) for a somewhat more interesting set S, as shown in Figure 7.3. The triangle S is formed by the origin and two points labeled ① and ②. It has a right angle at the origin. At ① the angle is somewhat less than 30° and at ②, correspondingly, it is somewhat over 60°.

Now consider u varying between u_1 and u_4 as in Figure 7.3, so that the parametric variation in the extreme points of the set of permutation-invariant measures can be generated. For u in the arc A, two of the permutations are such that the value of arg max $x \cdot \pi(u)$ is at each of the extreme points of S. For $u \in B$, point ② is supported by three permutations, or $\frac{1}{2}$ of the induced permutation-invariant measure, whereas points ① and the origin receive weights $\frac{1}{3}$ and $\frac{1}{6}$, respectively.

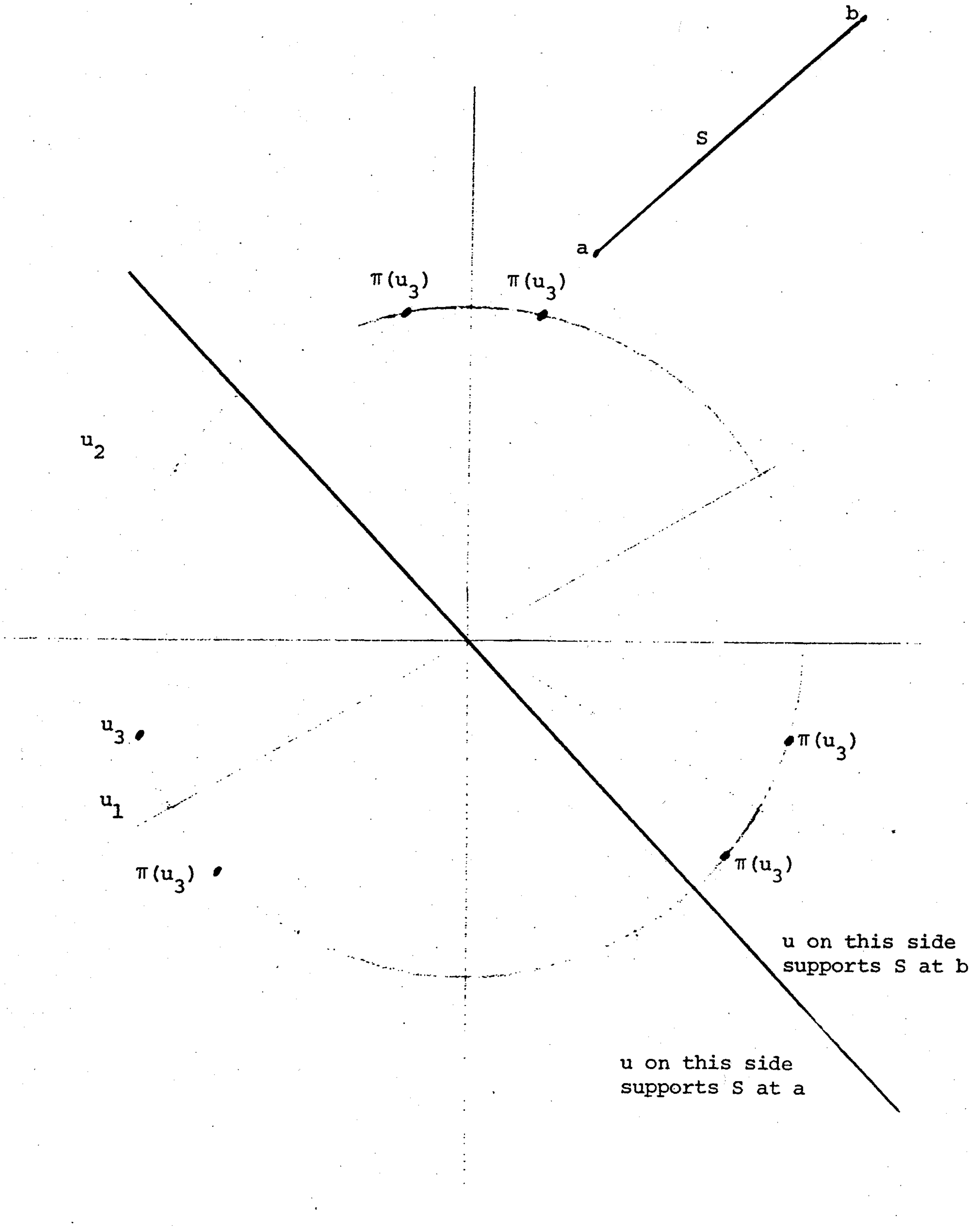


Figure 7.2. For u_3, a gets weight 1/3 and b gets weight 2/3.

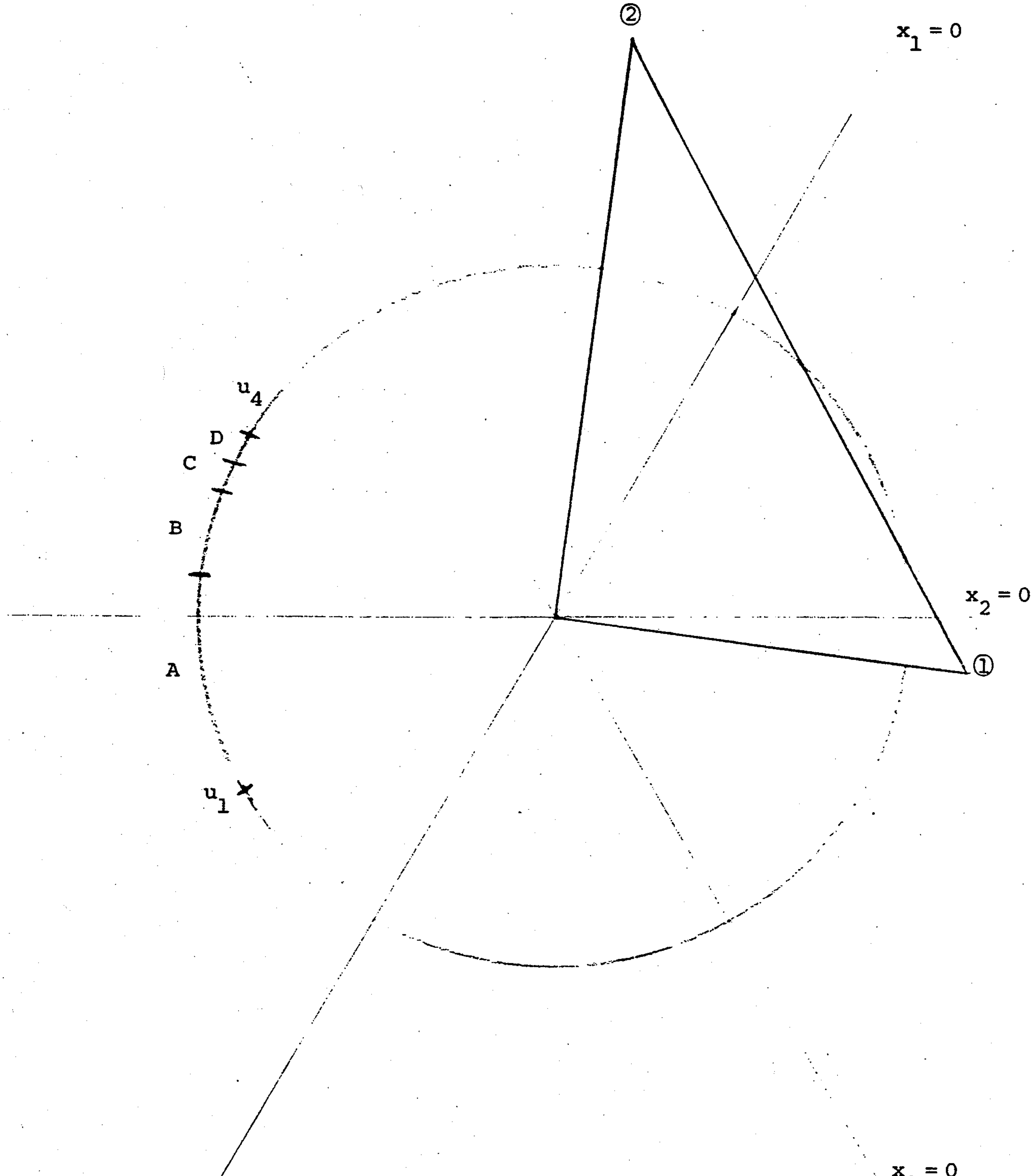


Figure 7.3. Applying the Theorem 1 to find F(S) for $S = co\{0, ①, ②\}$.

Continuing with this procedure, the following table describes the weights assigned by the measure generated by u in each arc.

	Weight placed on extreme point of S for u in indicated arc		
Arc	Origin	①	②
A	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
B	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$
C	0	$\frac{1}{3}$	$\frac{2}{3}$
D	0	$\frac{1}{2}$	$\frac{1}{2}$

The points of F(S) are simply the convex combinations of the weighted averages of the extreme points of S, for weights as given in this table. This is shown in Figure 7.4, where each point is labeled according to the arc within which it was generated. It should be noted that the Steiner point of F(S) is not the Steiner point of S. Moreover, continuing the iterative procedure suggested by A.5, we would find that the iterates of F(S) eventually exclude the Steiner point of S.

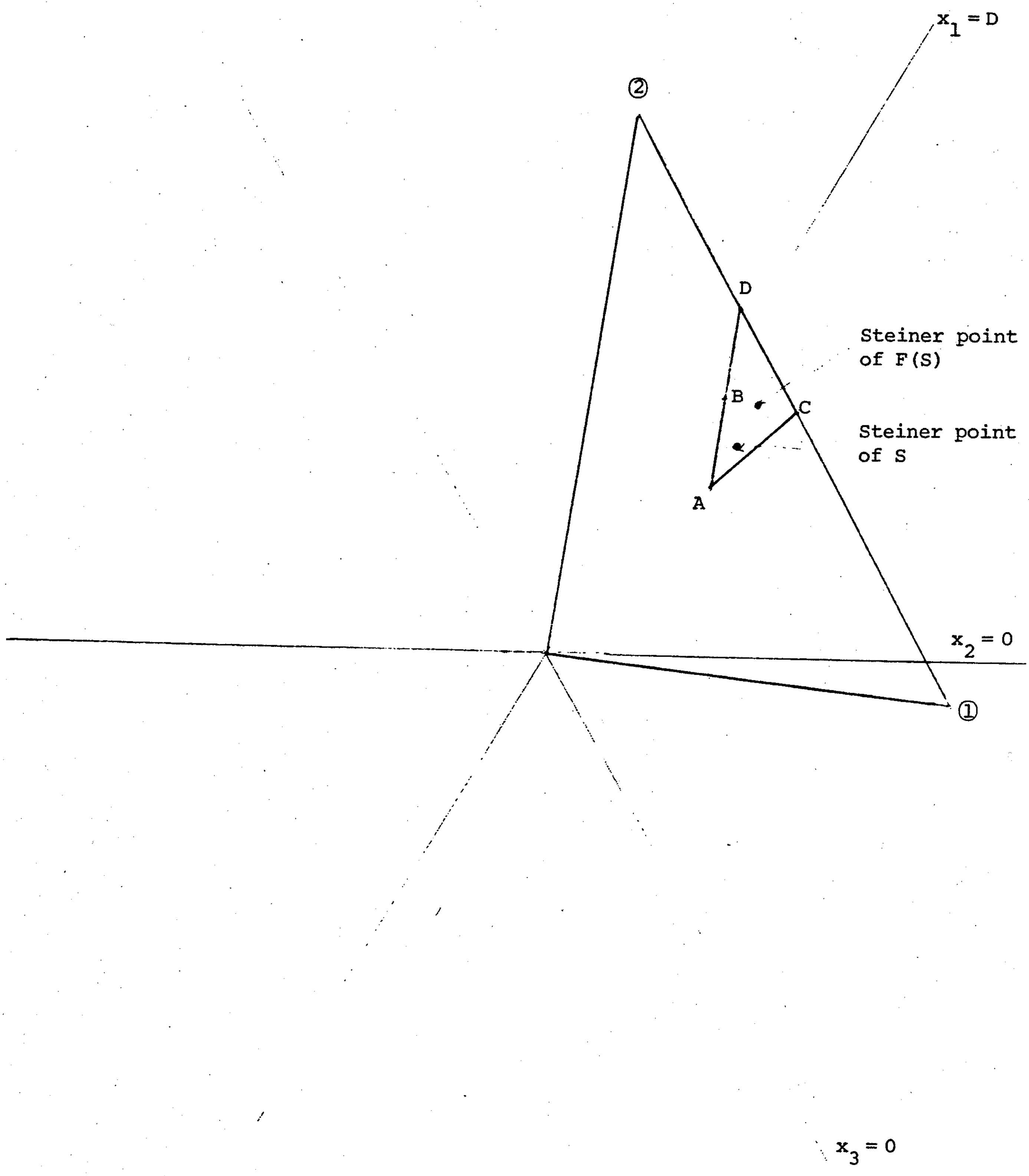


Figure 7.4. F(S) as constructed by procedure of Figure 7.3.

Example 3

In this example we show how axiom A.6 is used to extend the solution to $\mathscr{S}$. To illustrate this point, we describe a bargaining situation involving the dissolution of a contract which is not legally enforceable, and hence does not serve as a status quo. The case of a binding contract is also considered.

Consider a case in which two parties, players 1 and 2, have made a contract that will give them a surplus of 1 each. Then player 3 offers player 1 a contract that increases player 1's surplus to 2, and gives player 3 a surplus of 1. Thus S is the comprehensive hull of $\{(1, 1, 0), (2, 0, 1)\}$.

Applying the principle of invariance with respect to justified transfers, we find that the solution should be the same as that for a choice among the efficient points in J(S), given by the shaded region in Figure 7.5.

Then, applying axioms A.1 - A.4 to J(S), we obtain that F(J(S)) is the interval from (1-7/9, 4/9, 7/9) to (1-13/18, 5/9, 13/18). Applying A.5 and the result of example 1 yields the solution (1-3/4, 1/2, 3/4). Because of the symmetry of J(S), this solution is the Steiner point of J(S).

It is interesting to examine the idea of compensation of players for their accession to the efficient underlying agreement, in the context of this example. Player 2 is the one who would be better off at the inferior alternative (1, 1, 0) than at (2, 0, 1). Because both players 1 and 3 gain 1 unit of payoff from making the change to the efficient agreement, the solution requires that they contribute equally to player 2. Note also that player 2 is not fully compensated for his participation.

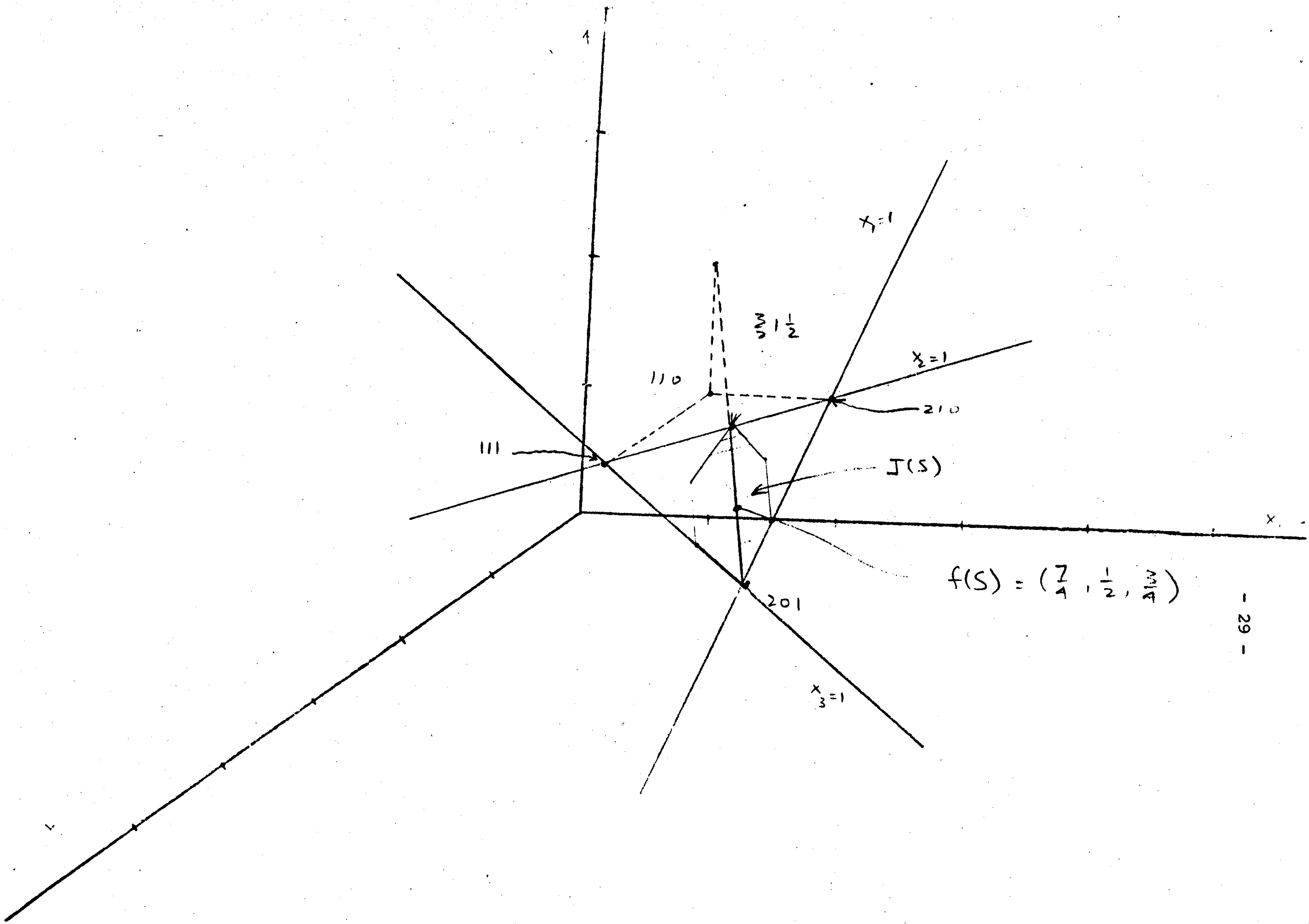


Figure 7.5. S generated by (110), (201).

This could be accomplished in the allocation (1-1/2, 1, 1/2). The solution of this paper strikes a balance between full compensation and the equal division of surplus.

If, however, (1, 1, 0) were a status quo payoff, and were accorded a special role in the computation of J(S) as suggested in Section VIII, below, then (1-1/2, 1, 1/2) would be the solution.

Example 4

This example is devoted to disproving the upper hemicontinuity of the correspondence $\bar{\Gamma}_S$. It is for this reason that we have to take the closure of the graph of the correspondence, and not simply the closure of the image F(S) for each S, when applying A.5. Were we to apply A.5 to a lower hemicontinuous correspondence, the limit might not be point-valued. For some S, the induced sequence of iterates of $\bar{\Gamma}$ could converge to a nonsingleton subset of S. Yet this limit would not necessarily be its own image under $\bar{\Gamma}$.

This example is a slight modification of example 2. Contract point ① slightly towards the origin so as to establish a 30-60-90° right triangle. Following the procedure of example 2, we can compute that the closure of the set Γ_S is as shown in Figure 7.6. Geometrically, arcs C and D of Figure 7.3 have become coincident, with the result that point C is not approachable by any solution, although it is a solution for sets arbitrarily close to S.

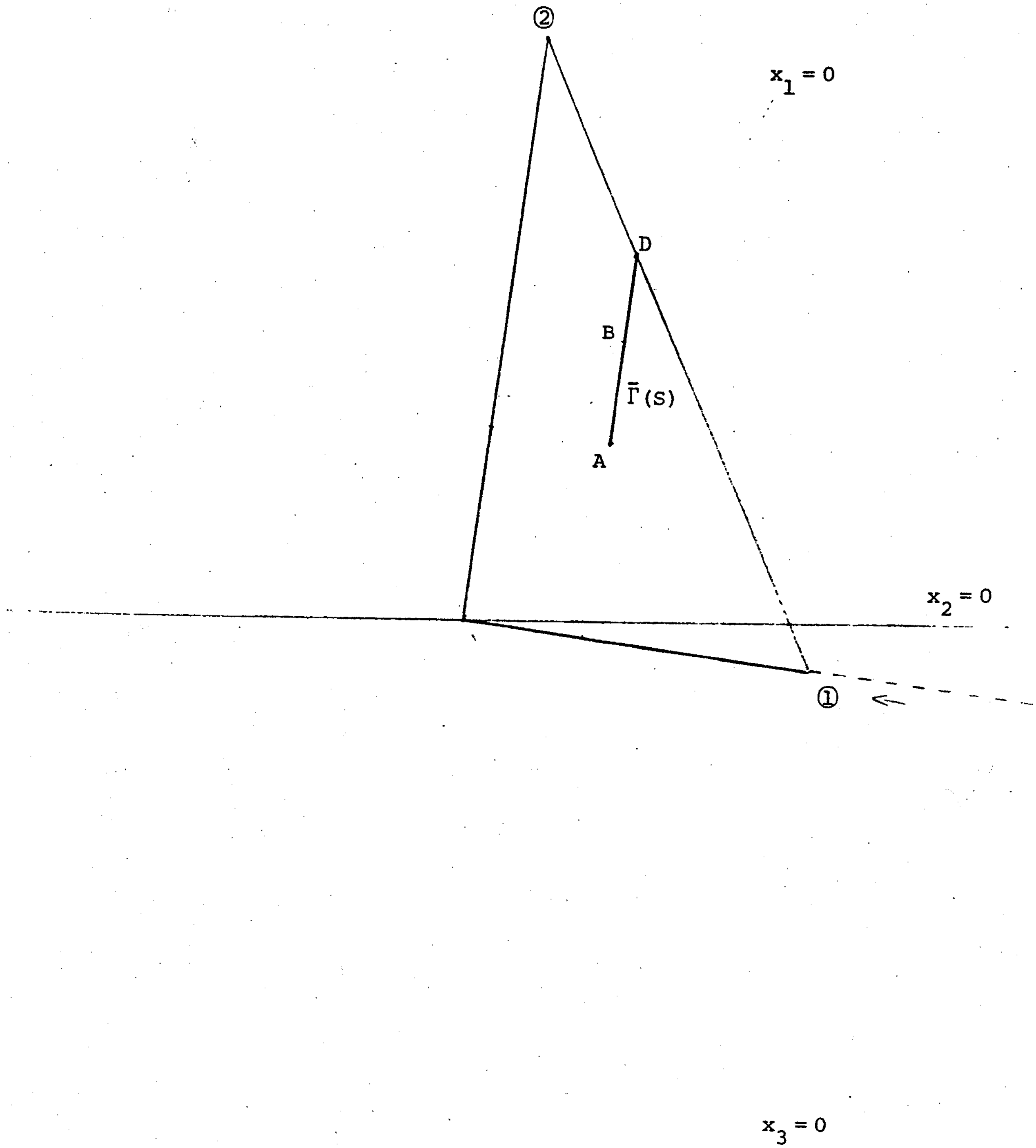


Figure 7.6. A discontinuity in $\bar{\Gamma}_s$, as point ① moves towards the origin. Compare to Figure 7.4, example 2.

Example 5

In this example we show that by taking the closure of the graph Γ, obtaining Γ^*, we do not get a continuous correspondence. The correspondence is, of course, upper hemicontinuous. Again, a modification of examples 2 and 4 is used. By taking small perturbations of points ① and ② in example 4, we find that the solutions are approximated by the indicated region in Figure 7.7. Notice that the set Γ_S^* of example 2 converges to a strict subset of this set as ① converges to its limiting value in example 4. Other points, such as E, are the limits of solutions for sequences of sets approaching that of example 4 through perturbations of point ②, in addition to point ① .

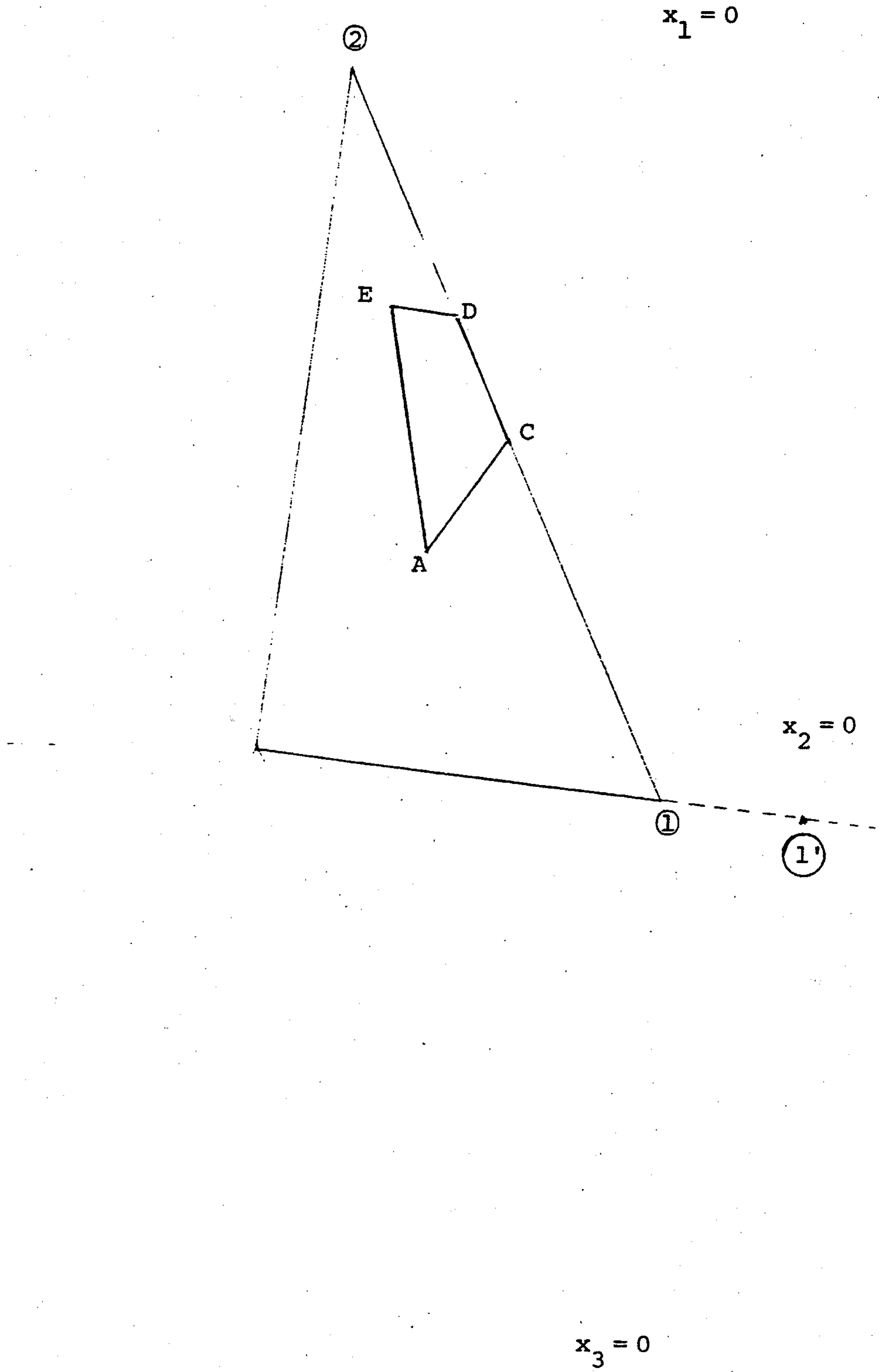


Figure 7.7. Discontinuity in $\Gamma_S^* \equiv F(S)$.

For $S' = co\{0, (1'), ②\}$, $\Gamma_{S'}^* \approx co\{A, C, D\}$.

For $S = co\{0, ①, ②\}$, $\Gamma_S^* = co\{A, C, D, E\}$.

Example 6

This example shows that F may be strictly subadditive, as mentioned in section 5.

Let K_1 be the triangle formed by the origin and points ① and ② in Figure 7.8, and let K_2 be the line segment connecting the origin to point ③. K_1 is, as in example 2, in general position. Point ③, likewise, is such that the angle of K_1 at the origin is not bisected by K_2.

Figure 7.9 shows the results of computations on K_1, K_2 and $K_1 + K_2$, following the method of the other examples in this section. We show $K_1 + K_2$ as the large pentagon. As in example 2, $F(K_1)$ is a triangle contained in K_1. $F(K_2)$, as in example 1, is the middle-third of K_2. When these are added, we have that $F(K_1) + F(K_2)$ is the pentagon ABCDE as shown.

However, when we compute $F(K_1+K_2)$, we find that its extreme points are precisely A, C and D. Thus $F(K_1+K_2)$ is strictly contained within $F(K_1) + F(K_2)$. The point B, for example, is generated as the sum of a solution based on $\hat{\mu}_1$ applied to K_1 and a different solution based on $\hat{\mu}_2$ applied to K_2.

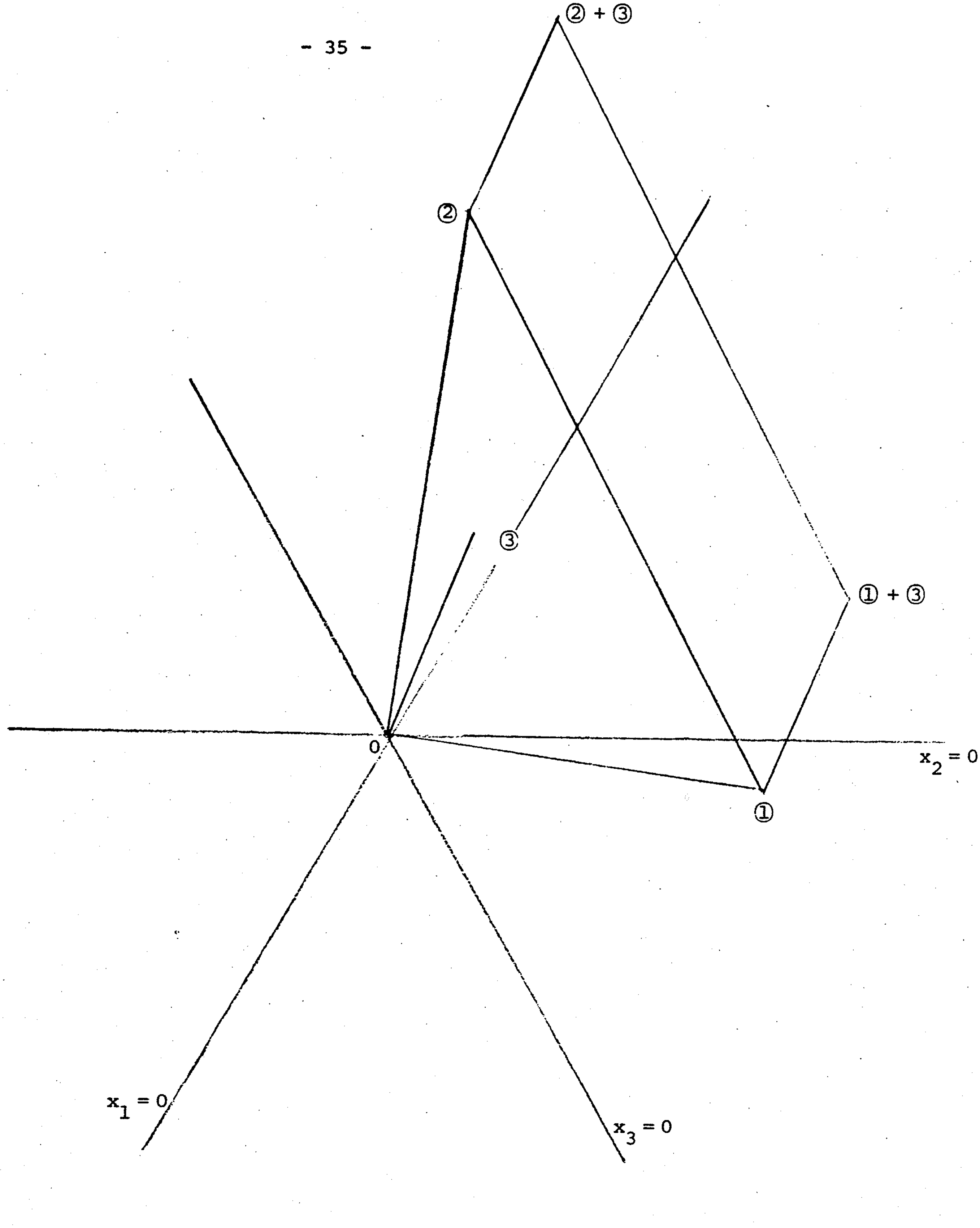


Figure 7.8. K_1 = triangle 0 ① ②

K_2 = segment 0 ③

$K_1 + K_2$ is the pentagon.

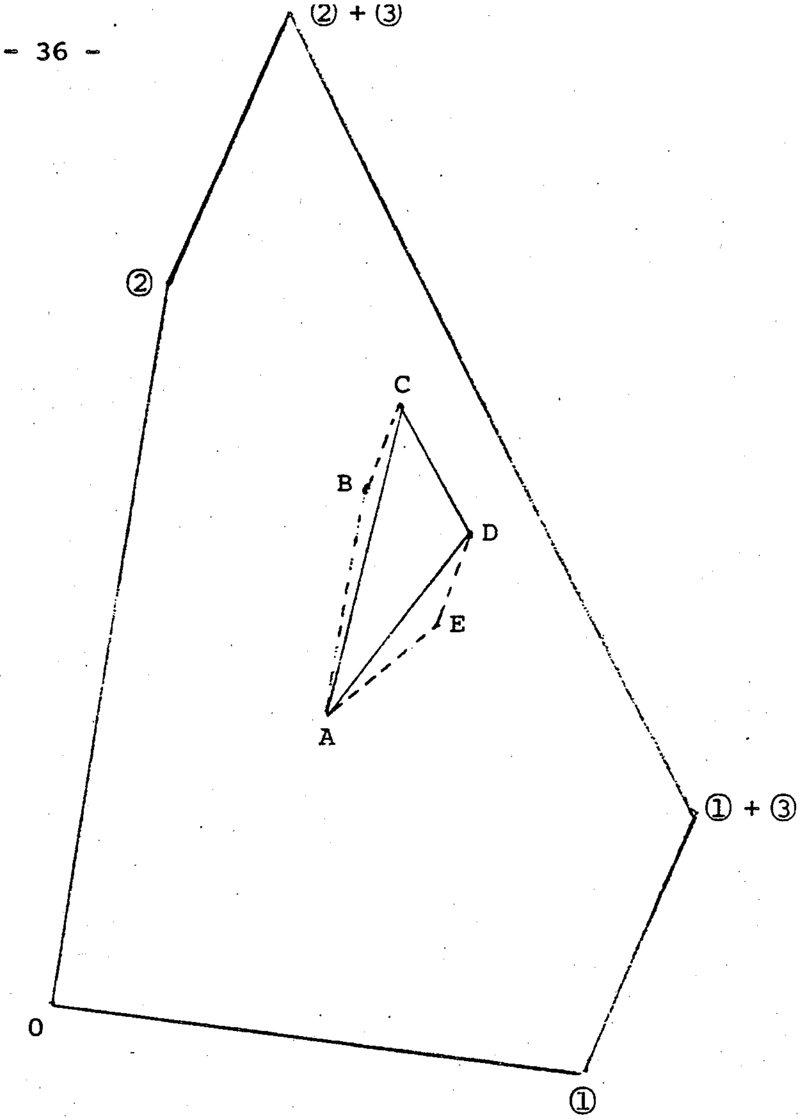


Figure 7.9. The strict subadditivity of F.

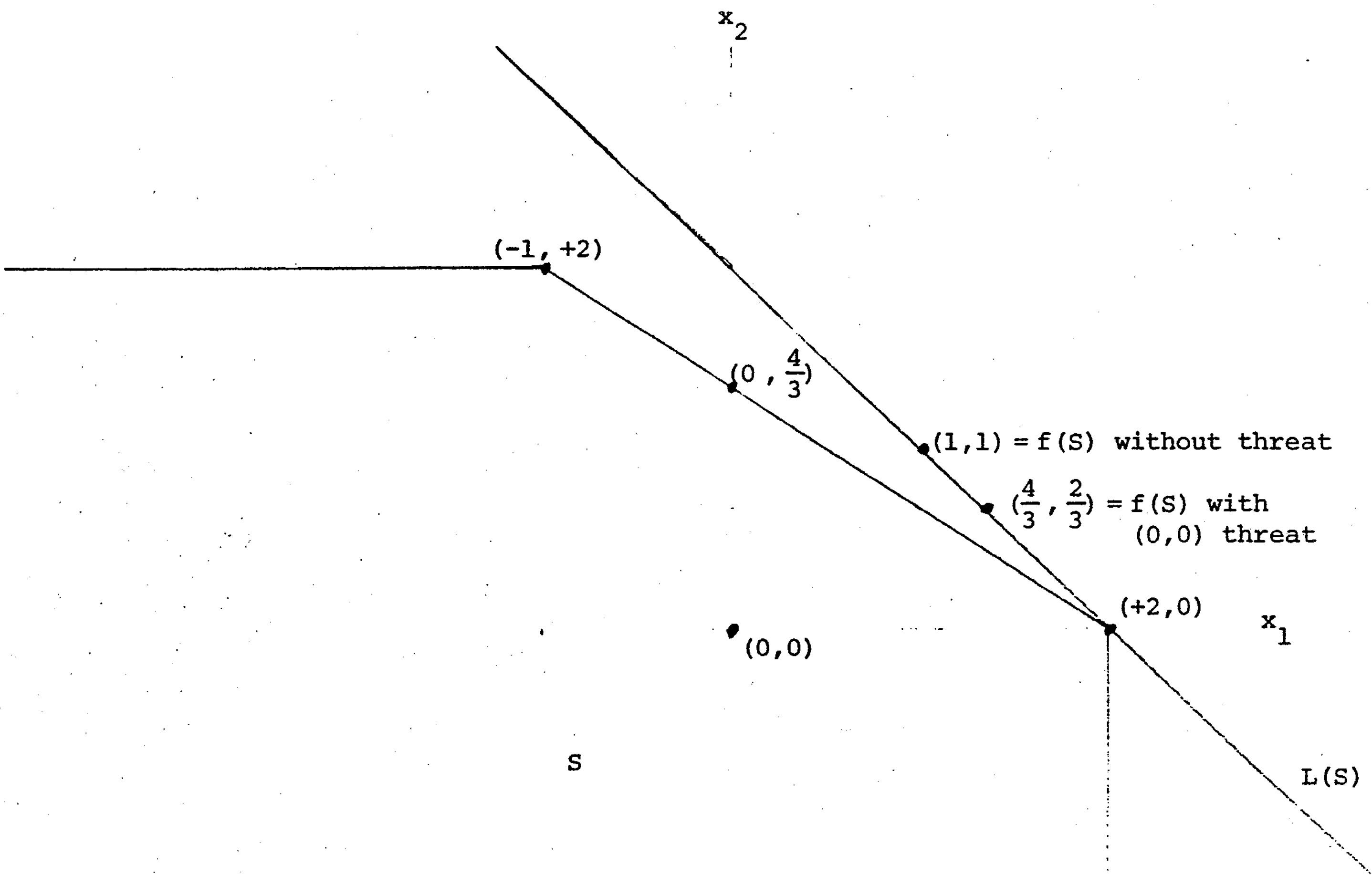
VIII. Further Remarks

In this section I discuss several issues that have arisen in the sections above.

VIII.1 Incorporating Threats Based on a *Status Quo*

Bargaining theory in the Nash tradition takes as given the presence of an outcome which can be unilaterally enforced by any agent who does not agree to the allocation specified by the solution. The threat point plays a special role in the analysis. In the theory above, all feasible allocations are on an equal footing.

To modify this theory and include a threat point, I suggest a simple construction in the normative spirit of the bargaining model as presented. No agent should be forced to accept an outcome below the threat value. Moreover, no ethical argument or appeal to the arbitrator should be allowed if it depends on the presence of such an outcome. Thus, if $S \in \mathcal{S}$ is the comprehensive hull of allocations other than the threat point x_o, then I would suggest applying the solution of this paper to the set $S \cap (\{x_o\} + R^n_+)$. For example, if $S = co\{(-1, +2), (+2, 0)\}$ and $x_o = (0, 0)$ then the solution should be $\left(\frac{4}{3}, \frac{2}{3}\right)$ instead of $(1, 1)$, because the effective feasible set should be truncated to $co\left(\left(0, \frac{4}{3}\right), (2, 0)\right)$.



VIII.2 Relaxing Axiom 1: (Selection) $f(K) \in K$

It is not possible to weaken axiom 1 to the requirement that $f(\{x\}) = x$ for all x without expanding the set of possible solutions. I give two examples of solutions on $\mathscr{K}$ that satisfy this postulate and A.2, A.3, A.4, and which fail to satisfy A.1. Under these axioms, therefore, the recursion argument used to define a unique solution may not work. This point, however, remains to be examined further.

Example 8.1

Let $K \in \mathscr{K}_o$ and let $\bar{x}(K)$ be the least majorizing point for K. Let $f(K) = \bar{x}(K) - \left(\sum_i \bar{x}_i(K)\right)(1, \ldots, 1)$. For $S \in \mathscr{S}$, employ (A.6) to obtain the justified set in $\mathscr{K}$ and then define the solution f by using translation invariance and the value of f on $\mathscr{K}_o$.

It is straightforward to show that f satisfies (A.2), (A.3) and (A.4) as well as $f(\{x\}) = x$.

Consider, however, the member of $\mathscr{K}_o$ given by

$$K = \mathrm{co}\{(0, 0, 0), (1, 0, -1), (4, 2, -6)\}.$$

We have, applying the above definition,

$$f(K) = (2, 0, -2)$$

which is not in the set K.

Example 8.2

Take a permutation-invariant measure μ on $\hat{S}^{n-1}$, other than the uniform distribution. For example, when $n = 3$, let μ be concentrated on the three points

$$u_1 = \frac{1}{\sqrt{6}} (2, -1, -1)$$

$$u_2 = \frac{1}{\sqrt{6}} (-1, 2, -1)$$

$$u_3 = \frac{1}{\sqrt{6}} (-1, -1, 2) .$$

Then $f(K) = (n-1) \int u\phi(K,u)\, d\mu$, which is a formula for the Steiner point in L_o when μ is uniform, yields a solution satisfying all the hypotheses except $f(K) \in K$.

To demonstrate this assertion, let K be the line segment in L_o joining the points (+1, 0, -1) and (0, +2, -2).

We have

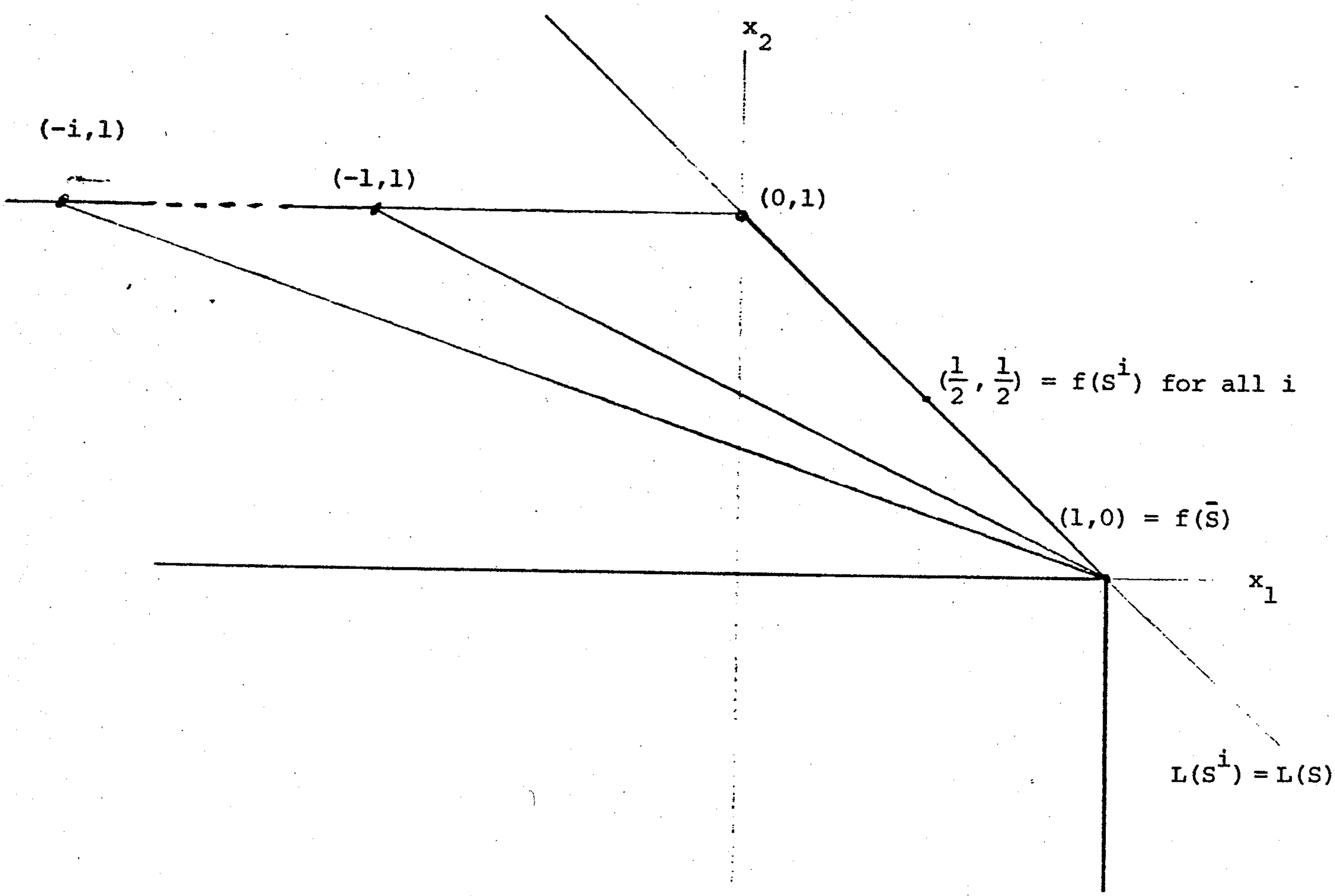
$$f(K) = \frac{1}{\sqrt{6}} (2, 8, -10) \notin K .$$

VIII.3 Strengthening the Continuity Axiom

Although the Hausdorff topology which is used to define the concept of convergence in axiom A.3 is very natural, some of its implications may not be desirable. A stronger topology may better express our ideas about the relative merits of appeals to the arbitrator for compensatory transfers in various situations. For example, consider the sequence

$$S^i = \mathrm{co}\,\{(1,\ 0),\ (-i,\ 1)\}\ .$$

Under the solution defined in this paper, we have $f(S^i) = \left(\frac{1}{2}, \frac{1}{2}\right)$ for all i. Whereas for $\bar{S} = \{(1,\ 0)\}$, clearly $f(\bar{S}) = (1,\ 0)$.



One might argue that the inefficient point in S^i, $(-i, 1)$, becomes a weaker and weaker basis upon which player 2 can appeal to the arbitrator for a transfer in his favor, to modify the efficient payoff (1, 0). To capture this idea, one wants a topology on $\mathcal{S}$ stronger than the Hausdorff topology -- strong enough that $\langle S^i \rangle$ converges to $\bar{S}$, and hence that $f(S^i)$ would be required to go to $(1, 0) = f(\bar{S})$ by the continuity postulate. The obvious topology to use is the closed-convergence topology on $\mathcal{S}$. For members of $\mathcal{S}$ this can be defined as $S^i \to \bar{S}$ if and only if, in the Hausdorff topology, $S^i \cap L_\alpha \to \bar{S} \cap L_\alpha$ for every α.

Unfortunately, this continuity axiom is incompatible with A.2 - A.4. One can see this very simply by returning to the example above. For $i = 0$, anonymity requires $f(S^o) = \left(\frac{1}{2}, \frac{1}{2}\right)$. As $i \to \infty$, continuity now requires that $f(S^i) \to (1, 0)$. Linearity requires that $\frac{1}{2} f(S^{i+2}) + \frac{1}{2} f(S^i) = f(S^{i+1})$ for all i. Thus, if $f(S^o) \neq f(S^1)$ then $f(S^i)$ must diverge, and if $f(S^o) = f(S^1)$, then $f(S^i)$ is constant, and hence not converging to (1, 0).

It is also important to note that this argument is independent of the nature of A.6. There is no way of associating the situations S^i to $K^i \in \mathcal{K}$ such that the linear solution on $\mathcal{K}$ is continuous. (Here the midpoint of an interval is the only possibility; that is why A.5 is not needed when $n = 2$.)

VIII.4 Comparison of Axioms with Those Used in Nash-Type Bargaining Models

In Nash-type bargaining models, without explicit monetary transfers, three axioms are almost universally accepted: efficiency, anonymity and invariance with respect to changes in the utility scale. To delineate a unique solution, additional axioms are used. Nash (1950) imposed the independence of irrelevant alternatives; Maschler and Perles (1981) have introduced superadditivity; and Kalai and Smorodinsky (1975) have used a form of monotonicity postulate. In this subsection I want to comment briefly on the comparison between these axioms and those used above.

First, it should be noted that the three solutions defined in the papers just cited are different. The axioms used are pairwise inconsistent. Second, the superadditivity axiom is inconsistent with the first three axioms except when $n = 2$. (An analogous solution can still be found by a monotonic dynamic procedure, but this is based on an entirely different axiomatization grounded in the concept of a "negotiation path.") Our axiom of additivity is, as mentioned, identical to this one in the model with transfers. Third, the monotonicity postulate of Kalai-Smorodinsky is not as compelling when extended to more than two players as it is for the two-person case. For $n = 2$, our axiom of invariance to the addition of justified transfers (A.6) is the same as monotonicity.

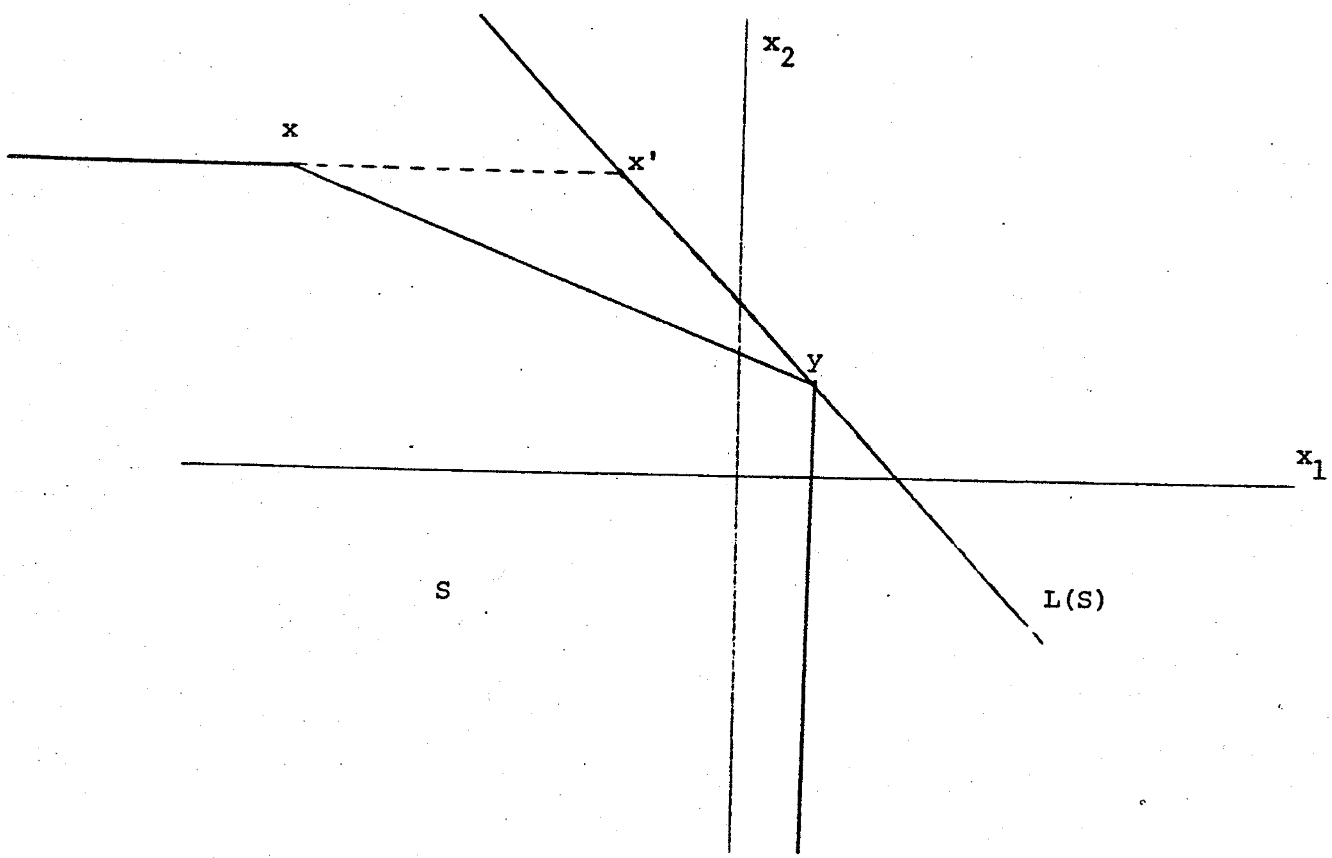
In the light of these remarks, it is interesting to note that in the presence of monetary transfers, the axioms analogous to superadditivity and monotonicity that we have employed are mutually consistent for all values of n.

The independence of irrelevant alternatives postulate has no counterpart in our theory. It is, indeed, rejected by the structure of the model itself. Our model accepts the premise that the role of monetary transfers is precisely to compensate some participants for the gain they have not received because an "irrelevant alternative" has been rejected in favor of an efficient one. Thus, all points in S except those in $L(S)$ are "irrelevant." A solution invariant to the deletion of these points would have to choose $x = \arg\max_{x \in S} \sum_i x_i$, whenever this is uniquely defined. But then, continuity and anonymity could not be satisfied, as one can see by considering the sets S^i defined in Section VIII.2 and letting $i \to 0$ continuously through positive real values.

VIII.5 Alternative Forms of Axiom A.6 — Invariance to Additions of Justified Points

Axiom (A.6) provides a way of finding a bargaining problem in $\mathcal{K}$ whose solution is the same as a given problem $S \in \mathcal{S}$. It utilizes a type of monotonicity argument. Players' appeal to the arbitrator to include a "justified" transfer in L(S) is based on the idea that they concede to the other players all of the benefits of moving from $x \in S$ to $x' \in L(S)$.

In the case of two players, we have the following:



Player 2 concedes all of the benefits of moving from x to L(S) to player 1. He says, in effect, "Imagine that x' were actually feasible, it dominates x for player 1, therefore player 1 should not do better at $S = co\{x, y\}$, than he does at $S' = co\{x', y\}$."

In the case of two players there are many equivalent ways, algebraically, to describe the mapping $S \mapsto J(S)$.

The one we have chosen is:

$$\text{a.} \quad \frac{1}{n} \sum T_i(\bar{S}) \cap L(S) \cap (\{\bar{x}\} - R^n_+)$$

Some others are:

$$\text{b.} \quad \frac{1}{n} \sum T_i(S) \cap L(S) \cap (\{\bar{x}\} - R^n_+)$$

$$\text{c.} \quad L(S) \cap (\{\bar{x}\} - R^n_+)$$

$$\text{d.} \quad \bigcap_i (T_i(S) \cap L(S))$$

$$\text{e.} \quad \text{For all } N' \subseteq \{1, \ldots, n\}$$

$$\sum_{i \in N'} x_i \leq \max_{z \in S} \sum_{i \in N'} z_i .$$

The reader can easily verify this equivalence for n = 2. The interpretation of these conditions in terms of justifiability is as follows: Condition b is similar to a, except that it allows an appeal based on the translation of an inefficient point. Condition c simply admits any alternative below the coordinate-wise maximum in S. Condition d requires that every player be able to justify the allocation as being solely in his favor, instead of the averaging procedure of a or b. Finally, condition e is a "core-like" concept. It states that no coalition should collectively be able to obtain more than it could at any underlying agreement.

One can see that for $n > 2$ these concepts are distinct. Each would lead to a different $J(S)$, in general, and hence to a different solution.

It turns out, however, definitions b., c., d. and e. violate the criteria for a concept of justifiability given in Section VII. Specifically, although all of the concepts satisfy (J.1), (J.2) and (J.6), b and c violate (J.5), d violates (J.4) and e violates (J.3). Detailed counterexamples supporting these assertions are available from the author. They are omitted here in the interest of saving space.

VIII.6 The Valuation Property

The solution on $\mathscr{K}$ has an interesting property. A function $f : \mathscr{K} \to R^n$ will be said to have the <u>valuation property</u> if, for all $K_1, K_2 \in \mathscr{K}$ such that $K_1 \cup K_2 \in \mathscr{K}$,

$$f(K_1 \cup K_2) + f(K_1 \cap K_2) = f(K_1) + f(K_2) .$$

This property is of interest in economics. It implies that the feasibility of a new underlying situation can be "valued" by the players, independent of the feasible set already available to them.

It is well known that the Steiner point has this property — see Sallee (1966) and McMullen (1977). In this subsection we will show that it is true of every solution satisfying A.1 - A.4.

If K_1, K_2 satisfy the hypotheses of the definition, then we can find a partition of the sphere $\hat{S}^{n-1}$ into $\hat{S}_1^{n-1}$ and $\hat{S}_2^{n-1}$ such that for $u \in \operatorname{int} \hat{S}_1^{n-1}$, we have $\underset{x \in K_1}{\arg\max}\; x \cdot u \notin K_2$ and $\underset{x \in K_2}{\arg\max}\; x \cdot u \in K_1$

and for $u \in \text{int}\ \hat{S}_2^{n-1}$, we have $\arg\max_{x \in K_2} x \cdot u \notin K_1$ and $\arg\max_{x \in K_1} x \cdot u \in K_2$.

Let f be a solution satisfying A.1 - A.4 and let μ be the corresponding permutation-invariant measure. Then

$$f(K_1) = \int_{\hat{S}^{n-1}} \arg\max_{K_1} x \cdot u \, d\mu = \int_{\hat{S}_1^{n-1}} \arg\max_{K_1} x \cdot u \, d\mu + \int_{\hat{S}_2^{n-1}} \arg\max_{K_1 \cap K_2} x \cdot u \, d\mu$$

$$f(K_2) = \int_{\hat{S}_1^{n-1}} \arg\max_{K_1 \cap K_2} x \cdot u \, d\mu + \int_{\hat{S}_2^{n-1}} \arg\max_{K_2} x \cdot u \, d\mu$$

$$f(K_1 \cup K_2) = \int_{\hat{S}_1^{n-1}} \arg\max_{K_1 \cup K_2} x \cdot u \, d\mu + \int_{\hat{S}_2^{n-1}} \arg\max_{K_1 \cup K_2} x \cdot u \, d\mu$$

$$= \int_{\hat{S}_1^{n-1}} \arg\max_{K_1} x \cdot u \, d\mu + \int_{\hat{S}_2^{n-1}} \arg\max_{K_2} x \cdot u \, d\mu$$

$$f(K_1 \cap K_2) = \int_{\hat{S}_1^{n-1}} \arg\max_{K_2} x \cdot u \, d\mu + \int_{\hat{S}_2^{n-1}} \arg\max_{K_1} x \cdot u \, d\mu .$$

From which the result follows directly.

Appendix

The Appendix consists of the proofs of Theorems 1 and 2. First we prove the main part of Proposition 1 which leads to Theorem 1. Proposition 1 is restated below:

Proposition 1

Let $n \geqslant 3$.

Let $f : \mathscr{K}_o \to R^n$ satisfy (A.1), (A.3) and (A.4). Then

$$f(K) = \int_{\hat{S}^{n-1}} \underset{x \in K}{\arg\max}\, x \cdot u \, d\mu(u)$$

for some μ, an atomless measure on $\hat{S}^{n-1}$.

Outline of Proof

Because of linearity, it will suffice to establish this characterization on the subset of $\mathscr{K}_o$ consisting of sets contained in the closed unit ball of L_o. Denote this family of sets by $\mathscr{K}^1$, and denote the family of all functions $f : \mathscr{K}^1 \to \mathbb{R}^n$ satisfying (A.1), (A.3) and (A.4) by F. The proof proceeds according to the following four steps:

1. We take an increasing sequence $<\mathscr{K}_j>$ of subsets of $\mathscr{K}^1$ whose union is dense in $\mathscr{K}^1$. The sequence $<\mathscr{K}_j>$ is chosen so that each $\mathscr{K}_j$ is a subset of a normed linear space of finite dimension.

2. Let $\bar{F}$ be the family of functions $f : \mathscr{K}^1 \to L_o$ that are linear and for which $f(K) \in K$ for all $K \in \mathscr{K}^1$ (that is satisfying (A.1) and (A.3) but not necessarily (A.4), on $\mathscr{K}^1$). Clearly $F \subset \bar{F}$.

For each j, let $\bar{F}_j$ be the family of functions from $\mathscr{K}_j$ to L_o, each of which is the restriction to $\mathscr{K}_j$ of some $f \in \bar{F}$.

We characterize the members $\bar{f}_j$ of $\bar{F}_j$ by

$$(*) \qquad \bar{f}_j(K) = \int \arg\max_{x \in K} x \cdot u \, d\mu(u)$$

for some measure μ on $\hat{S}^{n-1}$, not necessarily atomless.

3. Fix $f \in F$. Let M_j be the set of measures satisfying (*) for f_j. We show that M_j is compact, and, as $\langle \mathscr{K}_j \rangle$ form an increasing family of sets, that $M_j \subseteq M_{j-1}$. Therefore, $\bigcap_{j=1}^{\infty} M_j \neq \phi$. For $f \in \bar{F}$, and any $\mu \in \bigcap_{j=1}^{\infty} M_j$,

$$(**) \qquad f(K) = \int \arg\max_{x \in K} x \cdot u \, d\mu$$

for all $K \in \bigcup_{j=1}^{\infty} \mathscr{K}_j$.

4. Taking μ as in step 3, if μ were not atomless then the function defined by (**) could not be extended continuously from $\bigcup_{j=1}^{\infty} \mathscr{K}_j$ to $\mathscr{K}^1$.

Conversely, if μ is atomless, we can define the function $f : \mathscr{K}^1 \to L_o$ as

$$f(K) = \lim_j \int \arg\max_{x \in \hat{K}_j} x \cdot u \, d\mu$$

for a sequence $\hat{K}_j \in \mathscr{K}_j$ converging to K. This definition is independent of the particular sequence $\langle \hat{K}_j \rangle$, and is continuous on $\mathscr{K}^1$.

Therefore, by step 3, the functions $f \in F$ can each be characterized by $\int \arg\max_{x \in K} x \cdot u \, d\mu$ for some atomless μ, which is proposition 1.

Proof

Step 1

We associate to each compact convex set K its support function $\phi : \hat{S}^{n-1} \to R$ given by

$$\phi(K,u) = \max_{x \in K} x \cdot u \quad \text{for each } u \in \hat{S}^{n-1} .$$

Support functions are convex. Minkowski addition of sets induces pointwise addition of their support functions.

Following Schneider (1971), we consider the space of all spherical harmonic polynomials of degree j. A function $\psi : \mathbb{R}^{\ell} \to R$ is harmonic if $\sum_{i=1}^{\ell} \frac{\partial^2 \psi}{\partial x_i^2} \equiv 0$. A spherical harmonic is the restriction of a harmonic function to the unit sphere. A spherical harmonic polynomial is a spherical harmonic that is a polynomial. From Müller (1968) we have that the space of spherical harmonic polynomials of degree j is a finite-dimensional, normed linear space.

Let $\bar{\mathcal{K}}_j$ be the set of all convex, compact subsets of L_o whose support functions are spherical harmonic polynomials of degree j or less. Schneider (1971) shows that the union of $\bar{\mathcal{K}}_j$, j = 0, 1, ..., is dense in $\mathcal{K}$.

Take $\mathcal{K}_j = \bar{\mathcal{K}}_j \cap \mathcal{K}^1$.

This completes step 1.

Step 2

This step is divided into two parts. In part B we will use Choquet's theorem to obtain the desired representation. In Part A, we do the necessary preliminaries to show that Choquet's theorem can be applied. For a discussion of this theorem, see Phelps (1966).

A. We will show first that $\bar{F}_j$ is a compact convex subset of a locally convex space. Only the compactness part of this statement needs proof. We will appeal to Ascoli's theorem, which requires that $\bar{F}_j$ be an equicontinuous family, and that it be closed and bounded.

equicontinuity and boundedness

Take a basis for the r-dimensional space in which $\mathcal{K}_j$ is imbedded. Associate with each $\bar{f}_j$ the vector-valued coefficients, v_k, of its representation in this basis. That is, if $\mathcal{K}_j \longmapsto \alpha_1 e_1 + \ldots + \alpha_r e_r$ is the representation of K_j in the basis $(e_1, \ldots, e_r)$, then

$\bar{f}_j(K_j) = \alpha_1 v_1 + \ldots + \alpha_r v_r$, where $v_k \in L_0$, $k = 1, \ldots, r$.

Consider the dependence of v_k on $\bar{f}_j \in \bar{F}_j$. We claim that v_k must lie in a bounded subset of L_0, for if not, then $\bar{f}_j(K) \notin K$ for some $\bar{f}_j \in \bar{F}_j$. From this bound, the equicontinuity and the boundedness of $\bar{F}_j$ follow directly.

closedness

To show that $\bar{F}_j$ is closed, we consider a sequence in $\bar{F}_j$ $\langle \bar{f}_j^k \rangle$, $k = 1, \ldots,$ converging to $\bar{f}_j^o$, and we must prove that $\bar{f}^o_j \in \bar{F}_j$. It is obvious that $\bar{f}^o_j$ is linear on $\mathscr{K}_j$ and that $\bar{f}^o_j(K) \in K$ for all $K \in \mathscr{K}_j$. It remains to be proven that $\bar{f}^o_j$ is the restriction to $\mathscr{K}_j$ of some $\bar{f}^o \in \bar{F}$. We will define $\bar{f}^o$ explicitly as follows. The space of spherical harmonics of different degrees are mutually orthogonal, see Müller (1966). Therefore, for $K \in \mathscr{K}^1$, $K \notin \mathscr{K}_j$, we can find a unique $K_j(K) \in \mathscr{K}_j$ having as its support function the orthogonal projection of the support function of K onto the space of spherical harmonics of degree j or less. Setting $\bar{f}^o(K) = \bar{f}^o(K_j(K))$ we have shown that $\bar{f}^o_j$ is the restriction of $\bar{f}^o$ to $\mathscr{K}_j$ and that $\bar{f}^o \in F$.

This completes step 2A.

B. By Choquet's theorem, each $\bar{f}_j \in \bar{F}_j$ can be written as

$$\bar{f}_j(K) = \int \hat{f}_j(K)\, d\lambda(\hat{f}) \qquad \text{for all } K \in \mathscr{K}_j$$

where λ is a measure supported by the extreme points of $\bar{F}_j$.

We must show that the extreme points of $\bar{F}_j$ are precisely the functions

$$f_u : \bar{\mathscr{K}}_j \to L_o$$

defined by

$$f_u(K) = \arg\max_{x \in K} x \cdot u$$

for $u \in \hat{S}^{n-1}$.

It is easy to see that any f_u given above is extreme in $\bar{F}_j$. Observe that each $K \in \mathscr{K}_j$ is strictly convex (if the support function had a kink it could not be a polynomial). Therefore, $\arg\max_{x \in K} x \cdot u$ is singleton, and is an extreme point of K. If $f_u = \frac{1}{2} f^+ + \frac{1}{2} f^-$ for $f^+, f^- \in \bar{F}_j$, where $f^+ \neq f^-$, then, on any K where $f^+(K) \neq f^-(K)$, it would not be possible that both $f^+(K) \in K$ and $f^-(K) \in K$.

We now show that all extreme points of $\bar{F}_j$ are of the form f_u.

Let $\mathscr{H}_j$ be the support functions of sets in $\mathscr{K}_j$. As $\mathscr{H}_j$ is finite-dimensional, we can parameterize it by $\alpha = (\alpha_1, \ldots, \alpha_r) \in R^r$ in such a way that $\bar{f}_j$ is linear in α. Moreover, since $\mathscr{H}_j$ is compact and convex, the set $A \subseteq R^r$ of parameters α form a compact convex set.

We write $K(\alpha)$ to be the set in $\bar{\mathscr{K}}_j$ whose parameters are α.

Let $A' \subseteq A$ be the subset of parameter values on which $f(K) \in \text{bdy } K$.

We will show that

If $\alpha_1, \alpha_2 \in A'$ are such that u_1 and u_2 are the supporting vectors to $K(\alpha_1)$ and $K(\alpha_2)$ at $\bar{f}_j(K(\alpha_1))$ and $\bar{f}_j(K(\alpha_2))$, respectively, then $u_1 = u_2$.

Let $\alpha = (1+\varepsilon)\alpha_1 - \varepsilon\alpha_2 \in \mathbb{R}^r$. Define the function h to be the linear combination of the support functions of $K(\alpha_1)$ and $K(\alpha_2)$ with weights $(1+\varepsilon)$ and $-\varepsilon$. For ε sufficiently small, it follows from the fact that $K(\alpha_1)$ and $K(\alpha_2)$ have everywhere finite radii of curvature that h is the support function of a convex set $K(\alpha)$; that is, h is a convex function restricted to $\hat{S}^{n-1}$. (Note: This convex set is <u>not</u>

$(1+\varepsilon)K(\alpha_1) - \varepsilon K(\alpha_2)$, in the usual sense of Minkowski addition.) However, $K(\alpha) \in \bar{\mathcal{K}}_j$, because h is a spherical harmonic (though perhaps $K(\alpha) \notin \mathcal{K}_j$, as h(u) may exceed unity for some u). Recall that $\bar{f}_j$ is the restriction to $\mathcal{K}_j$ of some $f \in \bar{F}$. Let $\bar{\bar{f}}_j$ denote this function f restricted to $\bar{\mathcal{K}}_j$. Clearly $\bar{\bar{f}}_j$ is linear on $\bar{\mathcal{K}}_j$. Thus

$$\bar{\bar{f}}_j(K(\alpha)) = (1+\varepsilon)f(K(\alpha_1)) - \varepsilon f(K(\alpha_2)) .$$

On the other hand, we have from the definitions that

$$u_1 \cdot \bar{\bar{f}}_j(K(\alpha_2)) < \phi(K(\alpha_2), u_1)$$

$$u_1 \cdot \bar{\bar{f}}_j(K(\alpha_1)) = \phi(K(\alpha_1), u_1) ,$$

therefore

$$\begin{aligned} u_1 \cdot \bar{\bar{f}}_j(K(\alpha)) &= u_1((1+\varepsilon)\bar{\bar{f}}_j(K(\alpha_1)) - \varepsilon\bar{\bar{f}}_j(K(\alpha_2)) \\ &> (1+\varepsilon)\phi(K(\alpha_1),u_1) - \varepsilon\phi(K(\alpha_2),u_1) \\ &= \phi(K(\alpha),u_1) . \end{aligned}$$

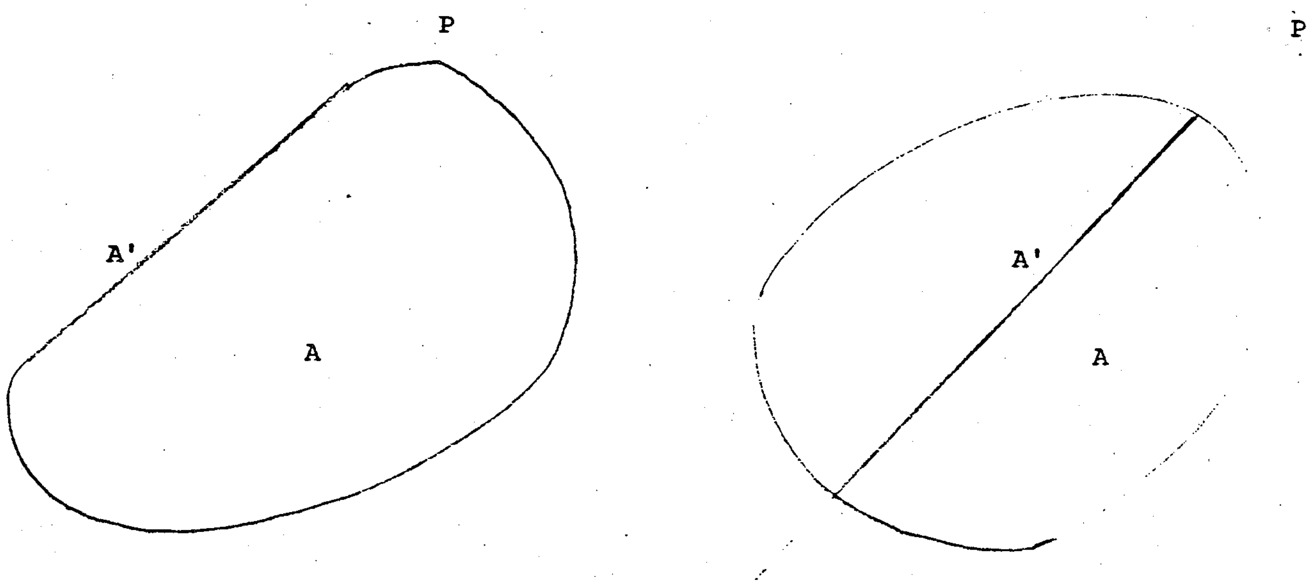
Thus

$$\bar{\bar{f}}_j(K(\alpha)) \notin K(\alpha) ,$$

contradicting the fact that $\bar{\bar{f}}_j$ is the restriction of a function in $\bar{F}$.

By the result just demonstated, we know that for all $\alpha \in A'$, $\bar{f}_j(K)$ maximizes $x \cdot u$ over $x \in K$ for some u independent of $\alpha \in A'$. We now note that $A' = A \cap P$ for some affine subspace P of R^r. This follows directly from the linearity of $\bar{f}_j$ and the fact that A is convex.

If the theorem were false, P would have to have dimension lower than r. Let it have dimension r'.



Define the real-valued function $e : A \to \mathbb{R}$ by

$$e(\alpha) = \sup\{\varepsilon \mid B_\varepsilon + \bar{f}_j(K(\alpha)) \subseteq K(\alpha)\}.$$

where B_ε is the closed ball of radius ε and center 0.

By construction $e(\alpha) = 0$ iff $\alpha \in A'$. By the linearity of $\bar{f}_j$ it follows that $e(\alpha)$ is affine in α.

To prove the result of step 2, it suffices to show that we can construct two functions, f^+ and f^-, on $\mathscr{K}_j$ such that $\bar{f}_j(K(\alpha)) = \frac{1}{2} f^+(K(\alpha)) + \frac{1}{2} f^-(K(\alpha))$ and such that f^+ and f^- are in $\bar{F}_j$.

Reparameterize $\mathscr{K}_j$ by a vector $\beta \in \mathbb{R}^r$ as follows. Let $\beta = 0$ correspond to a point $\alpha \in A'$. Now take a basis for $\mathbb{R}^r$ by selecting the first r' elements in P, and the remaining $r - r'$ in the orthogonal complement of P in such a way as to span $\mathbb{R}^r$. Call the basis $w = (w_1, \ldots, w_r)$.

Consider any set K such that its α parameter lies in the orthogonal complement of P. Expressed in the basis w, we have that w_k has non-zero weight for some $k > r'$, say $k = r$. Define $f^+(K)$ to be $\bar{f}_j(K) + \bar{\varepsilon}\beta_r(K)$ where $\beta_r(K)$ is the r^{th} component of the parameterization in the basis w, and $\bar{\varepsilon}$ is a vector in L. For $\bar{\varepsilon}$ sufficiently small,

$$\|f^+(K) - \bar{f}_j(K)\| \leqslant e(\alpha)$$

and hence $f^+(K) \in K$.

Linearity of f^+ is obvious, and the same applies to the construction of f^-, so $\bar{f}_j$ is not extreme unless $r' = r$, that is $P = \mathbb{R}^r$, in which case we have verified step 2.

Step 3

For all $K \in \bigcup_{j=1}^{\infty} \mathscr{K}_j$, $\arg\max_{x \in K} x \cdot u$ is continuous in μ.

Therefore, for every sequence, $\langle \mu_j \rangle$, in M_j, converging to μ,

$$\int \arg\max_{x \in K} x \cdot u \, d\mu_j \quad \text{converges to} \quad \int \arg\max_{x \in K} x \cdot u \, d\mu .$$

In particular, if the former is identically $\bar{f}_j(K)$ (i.e. $\mu_j \in M_j$) then so is the latter (i.e. $\mu \in M_j$).

Thus, M_j is a closed subspace of the space of all unit measures on $\hat{S}^{n-1}$, and is hence compact.

Step 4

Note that the correspondence

$$\psi : \mathscr{K}^1 \times \hat{S}^{n-1} \to R^n$$

defined by

$$\psi(K,u) = \arg\max_{x \in K} \; x \cdot u$$

is upper hemicontinuous, and for every $K \in \mathscr{K}^1$ is point-valued for almost every $u \in S^{n-1}$. Therefore,

$$\int \psi(K,u) \, d\mu$$

is discontinuous in K only when an atom of μ is a point of discontinuity of $\psi(K, \cdot)$.

By choosing $K \in \mathscr{K}^1$ to have a nondegenerate part of its boundary normal to an atom of μ, we can see that the continuity postulate (A.4) requires that μ be atomless.

Conversely, if μ is atomless, $\int \psi(K,u) \, d\mu$ will be continuous in K and singleton-valued throughout $\mathscr{K}^1$. Atomless measures that commute with permutations exist whenever $n \geqslant 3$.

This completes the proof of Proposition 1, from which Theorem 1 follows immediately.

Now we prove Theorem 2, restated below:

Theorem 2

The correspondence F is upper hemicontinuous, compact-valued, and, for each nonsingleton $K \in \mathscr{K}_0$, F(K) contains no extreme point of K.

Proof of Theorem 2

Since F is, by construction, the correspondence obtained through closing the graph Γ, and since Γ has a bounded range, it follows that F is upper hemi-continuous and compact valued.

Let $K \in \mathscr{K}_0$ be nonsingleton and let x be an extreme point of K such that $x \in F(K)$. Without loss of generality we can take $x = 0$, by virtue of the translation invariance of all solutions.

The supposition that $x \in F(K)$ implies the existence of $u \in \hat{S}^{n-1}$ such that $y \cdot \pi(u) \leqq x \cdot \pi(u) = 0$ for all permutations π. Note that $\sum_{\pi} \pi(u) = 0$ for any $u \in \hat{S}^{n-1}$, where the summation runs over all permutations. Hence $\sum_{\pi} y \cdot \pi(u) = 0$, and thus $y \cdot \pi(u) = 0$ for all permutations π.

We will now contradict this conclusion. Note that for $y \neq x$, $y \in K \subseteq \mathscr{K}_0$, we must have $y_j > y_k$ for some pair of components j, k. Also, as $u \in \hat{S}^{n-1}$, we can find some permutation π for which $\pi(u)_j > \pi(u)_k$. Let π' differ from π solely by the reversal of components j and k. Then $y \cdot \pi(u) > y \cdot \pi'(u)$, which establishes the desired contradiction.

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