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Source: *The Review of Economic Studies*, Vol. 43, No. 3 (Oct., 1976), pp. 537-542
Published by: Oxford University Press
Stable URL: https://www.jstor.org/stable/2297232

Notes and Comments

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1. EFFICIENCY CONCEPTS FOR ALTERNATIVE INSTITUTIONAL ARRANGEMENTS

In the last few years much work has been done on equilibrium theory in which the assumption of costless transactions has been dropped. One thread of this line of research was begun by the path-breaking paper of F. Hahn [3]. In this paper, he presented a concept of equilibrium for an economy with costly transactions which proceeds over time, and proved that it exists under certain conditions. Though the model is one with a finite horizon, and some perhaps undesirable assumptions are made, we shall not, address ourselves in the present brief note to these fundamental questions.[1] Instead, we shall direct our attention to the concept of efficiency, and in particular to the question whether or not the efficiency of the system can be improved by changing the institutional structure in which it is embedded.

The equilibrium of this model is defined to be a sequence of price vectors, one for every trading date, each describing the equilibrium buying and selling prices on all markets from that trading date through the horizon, with the property that aggregate demand does not exceed aggregate supply at any trading date for any spot or futures contract. Accepting this definition of equilibrium, we turn to the question of defining efficiency in this model. Hahn shows that the equilibrium realized by this system *may* not lie on the Pareto-frontier attainable from the same initial distribution, under the same technological constraints, but through a system of markets in which only a single budget constraint encompassing all periods, need be satisfied. It is argued that this is a case for the inefficiency of the system of a sequence of markets, and that, at least in the case in which introducing money balances is a costless institution, one should opt for this alternative.

This idea is further explored by D. Starrett in [7], where he presents an example of the Hahn system having the properties just described. Starrett then goes on to prove the general theorem that, barring coincidences, the equilibrium of the sequence-economy will always lie below the Pareto frontier of the economy with costless unit-of-account balances available to all participants.

In a recent paper, Green and Sheshinski [1] have proposed an alternative definition for efficiency of an institutional arrangement as compared to that of an alternative. Though their paper is addressed to the problem of equity and bond markets in situations involving uncertainty, the same definition of efficiency can be applied to the choice between monetary and non-monetary economies in the sense of Hahn [3], Kurz [6], and Starrett [7].

In this note we address ourselves to a particular example which shows that the substitution of the " sequential " by an " over-all " budget constraint may not lead to a Pareto improvement of the equilibria. This is, of course, not in contradiction with either the theory or the examples presented by the above authors. Nevertheless, we believe that there is some motivation for being interested in the alternative concept of efficiency which we have presented.

The idea of Green and Sheshinski can be formalized as follows:

Let A, B be two different institutional arrangements. Let $C(A)$ and $C(B)$ be the sets of competitive equilibrium allocations of A and B—assuming $C(A)$ and $C(B)$ are non-empty. Let " $\nsim_p$ " stand for Pareto non-comparability. Let x, y, z stand for allocations, x^i the ith individual's component of the allocation x, and $\succcurlyeq_i$ the ith individual's preference relation. We first define Pareto non-comparability for allocations and then for sets of allocations.

Definition:

(A) $x \nsim_p y$ iff $\exists i \;\&\; j \ni x^i \succcurlyeq_i y^i \;\&\; y^j \succcurlyeq_j x^j$.

Definitions:

(B1) $A \nsim_p^1 B$ iff $\exists(\bar{x}, \bar{y}) \in C(A) \times C(B) \ni \bar{x} \nsim_p \bar{y}$.

(B2) $A \nsim_p^2 B$ iff $\exists \bar{x} \in C(A) \ni \forall(\bar{x}, y) \in C(A) \times C(B)$, $\bar{x} \nsim_p y$.

(B3) $A \nsim_p^3 B$ iff $\exists \bar{y} \in C(B) \ni \forall(x, \bar{y}) \in C(A) \times C(B)$, $x \nsim_p \bar{y}$.

(B4) $A \nsim_p^4 B$ iff $\forall(x, y) \in C(A) \times C(B)$, $x \nsim_p y$.

Note the following:

(1) According to the above definition if $\forall i x^i \approx y^i$ then we say $x \nsim_p y$. This is probably questionable, but if it were changed the resulting comparisons would be essentially the same, but more cumbersome to state.

(2) In case $C(B)$ contains only a single element, $\nsim_p^1 \equiv \nsim_p^2$ and $\nsim_p^3 \equiv \nsim_p^4$.

(3) In case $C(A)$ contains only a single element, $\nsim_p^1 \equiv \nsim_p^3$ and $\nsim_p^2 \equiv \nsim_p^4$.

If both $C(A)$ and $C(B)$ are singletons, then all four definitions coincide.

In our example we will show that the two economies, $\mathscr{E}^d$ and $\mathscr{E}^s$ (using the notation developed in Hahn [3]) may be Pareto non-comparable in the strong sense, (B4):

$$C(\mathscr{E}^d) = \{y\} \text{ and } \forall x \in C(\mathscr{E}^s)\; x \nsim_p y.$$

2. EXAMPLE

The economy to be considered can be described as follows: Households are denoted h; $h = 1, 2, 3$.

Endowments: $w_h \in R^3_+ \; h = 1, 2, 3;$

$$w_1 = (A, 0, 0)$$

$$w_2 = (0, D, 0)$$

$$w_3 = (0, 0, A).$$

Commodities: C_1, d, C_2. $\phi_h \in R^3_+$: net consumption vector for h; Time periods are denoted by t; $t = 1, 2$.

C_1, d are period 1 goods, C_2 is a period 2 good.

Utility functions: $u_h = \min(\phi_{h_1}, \phi_{h_2})$ for all h.

Transaction costs: The purchase of one unit of C_2 at $t = 1$ requires one unit of d. Formally, $\mathscr{T}$, the set of technologically feasible transactions for all h is of the form:[2]

$$\mathscr{T} = \{(x^1_{h1}, x^1_{hd}, x^1_{h2}, y^1_{h1}, y^1_{hd}, y^1_{h2}, x^2_{h2}, y^2_{h2}) \colon x^1_{h2} \geqq x^1_{hd} - y^1_{hd} + w^1_{hd}\}$$

x^t_{hj}: purchase of good j at time t by household h; $x^t_{hj} \geqq 0$.

y^t_{hj}: sales of good j at time t by household h; $y^t_{hj} \geqq 0$.

$x^2_{h1} = y^2_{h1} = x^2_{hd} = y^2_{hd} = 0$ – no ex-post trading.

$p^1_1, p^1_d, p^1_2, p^2_2$: buying prices.

$q^1_1, q^1_d, q^1_2, q^2_2$: selling prices.

In the case of a *sequence economy*, the budget constraints of the households are,

$$\text{period 1: } (p^1_1, p^1_d, p^1_2)(x^1_{h1}, x^1_{hd}, x^1_{h2}) \leqq (q^1_1, q^1_d, q^1_2)(y^1_{h1}, y^1_{h2}, y^1_{h3}) \quad \ldots(1)$$

$$\text{period 2: } (p^2_2)(x^2_{h2}) \leqq (q^2_2)(y^2_{h2}) \quad \ldots(2)$$

for $h = 1, 2, 3$.

In the case of the *Debreu economy*,[3] the single budget constraint is, for each h

$$p^1_1x^1_{h1} + p^1_dx^1_{hd} + p^1_2x^1_{h2} + p^2_2x^2_{h2} \leqq q^1_1y^1_{h1} + q^1_dy^1_{hd} + q^1_2y^1_{h2} + q^2_2y^2_{h2}. \quad \ldots(3)$$

In deriving the equilibrium prices and allocations we will, for the sake of brevity, skip the formal proof whenever no ambiguity arises.

Equilibria for the Debreu Economy

At equilibrium, the buying and selling prices in a Debreu economy will satisfy the following conditions:

$$p^1_1 = q^1_1,\ p^1_d = q^1_d,\ p^2_2 = q^2_2$$

since spot markets are free of transaction costs.

$$p^1_2 = p^2_2,\ q^1_2 = q^2_2$$

since in a Debreu economy prices are, by definition, independent of the trading date.

$$p^1_2 \geqq q^1_2 + q^1_d$$

by the zero-profit condition for futures transactions, and with equality holding whenever such trades are made.

All p's and q's are non-negative by the desirability of all commodities. Combining these we find that $p^1_d = q^1_d = 0$, reflecting the fact that d is socially useless in this institutional setting.

Therefore, utility maximization leads to,

$$p^1_1 = p^2_2 = 1;$$

$$q^1_1 = q^2_2 = 1;$$

and the final allocation is

$$\left.\begin{array}{lll} \text{for } h = 1: & (A/2, 0, A/2), & u_1 = A/2 \\ \text{for } h = 2: & (0, D, 0), & u_2 = 0 \\ \text{for } h = 3: & (A/2, 0, A/2), & u_3 = A/2 \end{array}\right\} \quad (y)$$

Equilibria of the Sequence Economy

At trading date 2, there is only one economic good in the system, hence no meaningful trades can be made at this date, and we may as well assume,

$$x^2_{h2} = y^2_{h2} = 0 \quad \text{for all } h.$$

Since the first period market will be the only active one, we may suppress the superscripts. The possibilities for equilibrium are divided into cases below.

Case 1 ($p_d > 0$).

We may normalize and let $p_d = 1$; Then,

$$q_d = 1;$$

$$p_2 = q_2 + 1;$$

$$q_1 = q_1.$$

The households' problems then become:

$$h=1 \quad \max: \min(A-y_{11}, x_{12})$$
$$\text{s.t.} \quad p_2 x_{12} - p_1 y_{11} \leqq 0. \qquad \ldots(5)$$
$$h=2 \quad \max: \min(x_{21}, x_{22})$$
$$\text{s.t.} \quad p_1 x_{21} + p_2 x_{22} \leqq D. \qquad \ldots(6)$$
$$h=3 \quad \max: \min(x_{31}, A-y_{32})$$
$$\text{s.t.} \quad p_1 x_{31} - (p_2-1) y_{32} \leqq 0. \qquad \ldots(7)$$

We may assume (5)-(7) hold with equality by the desirability of goods 1 and 2 and derive the following:

From (5),

$$p_1 y_{11} - p_2(A - y_{11}) = 0$$

implies

$$y_{11} = p_2 A/(p_1+p_2); \; x_{12} = p_1 A/(p_1+p_2). \qquad \ldots(8)$$

From (6),

$$x_{21} = x_{22} = D/(p_1+p_2). \qquad \ldots(9)$$

From (7),

$$p_1(A-y_{32}) - (p_2-1)y_{32} = 0$$

and therefore

$$y_{32} = p_1 A/(p_1+p_2-1); \; x_{31} = p_2 A - A/(p_1+p_2-1). \qquad \ldots(10)$$

By definition: $$y_{11} = x_{21} + x_{31}. \qquad \ldots(11)$$

Equations (8)-(11) imply

$$\frac{p_2 A}{p_1+p_2} = \frac{D}{p_1+p_2} + \frac{p_2 A - A}{p_1+p_2-1}, \text{ hence, } \frac{p_1}{p_1+p_2-1} = \frac{D}{A} \text{ and thus,}$$

$$p_2 = p_1((A/D)-1)+1. \qquad \ldots(12)$$

Therefore, if we normalize prices by setting $p_d = 1$ all equilibria are the form $(p_1, 1, p_2)$ where p_2 can be expressed parametrically as a function of p_1 by $p_2 = p_1((A/D)-1)+1$.[4]

The final consumptions are:

$$\left.\begin{array}{ll} \text{for } h=1: & (p_1 A/(p_1+p_2), \quad \ldots, p_1 A/(p_1+p_2)) \\ \text{for } h=2: & (D/(p_1+p_2), \quad \ldots, D/(p_1+p_2)) \\ \text{for } h=3: & (A(p_2-1)/(p_1+p_2-1), \ldots, A(p_2-1)/(p_1+p_2-1)) \end{array}\right\} \quad (x^1)$$

For $D>0$, we have $u_2>0$ and $u_1+u_3<A$ and consequently the allocation is not Pareto comparable with (7) —$x^1 \nsim_p y$.

In the degenerate case of $D = 0$, there is an equilibrium of the form $(0, 1, p_2)$ which gives all goods to household 3, and this is also non-comparable to y.

Case 2 $(p_d = 0)$.

Subcase [A] $p_1 > 0$—hence we may normalize $p_1 = 1$.

For $D \geqq A$, we can show by an argument analogous to that of case 1, that all the equilibria are of the form:

$$(1, 0, p_2) \text{ for all } p_2 \geqq \max((A-D)/D, 1)$$

with final consumption:

$$\left.\begin{array}{ll} \text{for } h=1: & (A/(1+p_2), \quad 0, A/(1+p_2)) \\ \text{for } h=2: & (0, \quad x, 0) \\ \text{for } h=3: & (p_2A/(1+p_2), 0, p_2A/(1+p_2)) \end{array}\right\} \quad (x^2)$$

where x is a residual amount for good d left over after performing transactions between the other households. When $D<A/2$, $p_2 = 1$ is not compatible with equilibrium, and hence $x^2 \nsim_p y$. If $D \geqq A/2$, then there is one equilibrium identical to y—but still, our definition has $x^2 \nsim_p y$.

In cases where $D>A$, then (1, 0, 0) is also an equilibrium, with consumptions:

$$\left.\begin{array}{ll} \text{for } h=1: & (A, 0, A) \\ \text{for } h=2: & (0, x, 0) \\ \text{for } h=3: & (0, 0, 0) \end{array}\right\}. \quad (x^3)$$

Clearly: $x^3 \nsim_p y$.

Subcase [B] $p_1 = 0$. The unique equilibrium is (0, 0, 1) with final consumptions:

$$\left.\begin{array}{ll} \text{for } h=1: & (0, 0, 0) \\ \text{for } h=2: & (0, D, 0) \\ \text{for } h=3: & (A, 0, A) \end{array}\right\}. \quad (x^4)$$

It must be observed that x^4 is an equilibrium only because household 1's endowment of C_1 is equal to (in general, greater than or equal to) household 3's endowment of C_2. If that were not the case, x^4 would not be an equilibrium. Hence it is an artifact of our particular example, and really not of relevance—though its existence does not affect any of our conclusions.

Clearly $x^4 \nsim_p y$.

3. CONCLUSION

In this brief note we have tried to demonstrate that adopting an alternative concept of efficiency for choices among economic institutions may lead us to inconclusive results when choosing between institutions, one of which embodies costless financial systems unavailable in the other. This negative conclusion is, in some ways, parallel to that of Green and Sheshinski in that in this paper the addition of a costless institution led to Pareto noncomparability of the equilibria, whereas in the latter paper the addition of a costly institution resulted in the addition of a pareto inferior point being added to the set of equilibria.

Of course our result is not all that surprising because one must expect that if transactions are costly in terms of real resources, then when these transactions become unnecessary, the owners of these resources will experience adverse income effects and may be made worse off. Drawing attention to this simple fact and to the complexities it may cause in choosing among institutions was the primary goal of this exercise.

First version received July 1975*: final version accepted September* 1975 (*Eds.*).

Financial support by National Science Foundation through grant SOC 71-03803 is gratefully acknowledged.

NOTES

1. Questions concerning non-convexities are explored in Heller and Starr [5]; a model with an infinite horizon is studied in Hayashi [4].

2. In general, superscripts refer to the date of transaction and subscripts refer to households and commodities.

3. The prices in a Debreu economy are independent of the date of transaction, as we discuss below. This economy has been shown to be consumption equivalent to one in which there is a sequence of budget constraints but costless money balances can be used to transfer wealth between periods, subject to a constraint on terminal money balance.

4. To ensure positivity of prices we must assume $A \geqq D$. If this does not hold, $p_d = 1$ is not compatible with equilibrium.

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