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Source: *Econometrica*, Jan., 1974, Vol. 42, No. 1 (Jan., 1974), pp. 21-34

Published by: The Econometric Society

Stable URL: https://www.jstor.org/stable/1913682

THE STABILITY OF EDGEWORTH'S RECONTRACTING PROCESS

By Jerry R. Green[1]

The core is the set of all unblocked allocations. Implicit in this definition is the idea that if an allocation is proposed which could be blocked, some coalition will form and issue a counterproposal which it can enforce. A process of successive counterproposals based on this idea is shown to converge in a finite period of time (amost surely) to the core.

1. INTRODUCTION

THE OBJECT OF this paper will be to show that the core is a stable solution. In order to study this problem we must, of course, have a theory of disequilibrium behavior and a corresponding adjustment mechanism. Every solution concept rests on some set of behavioral postulates for the participants in the system and "equilibrium" represents a feasible situation in which all participants are acting according to these postulates. We also require that the equilibrium notion be a stationary point of the adjustment mechanism. For example, the competitive equilibrium is based on the hypothesis that individuals will maximize their preferences subject to a budget constraint. Further, if we suppose that prices change when excess demand is non-zero and remain constant when excess demand is zero, then a price system leading to zero excess demand, an "equilibrium," is stationary with respect to this adjustment hypothesis.

The core is defined to be the set of all unblocked allocations. That is, it is the set of all allocations such that no subset of the participants can improve the position of all its members by withdrawing from the system and using only its own resources. The word "allocation" in the above sentences has been deliberately vague. In this paper we consider allocations to be assignments of utility to the various individuals. These could arise from an underlying economic system though we shall not restrict the nature of the underlying process through which the stated utilities are generated. It could be pure exchange and we shall be justifying our assumptions with reference to this case. But productive economies (with no externalities) or even non-economic (e.g., political) underlying processes could be considered.

Implicit in the definition of the core is the hypothesis that if a proposed allocation is not in the core, then one of the coalitions that can block it will, in fact, do so. We assume that they will introduce a counterproposal which is better for them and which they can enforce by withdrawal. This disequilibrium process will be called recontracting; we shall be more explicit about this below.

One point that should be noted is the following: The recontracting process as defined above is based on the same behavioral postulate, blocking by coalitions, that is used to define the solution concept, the core. This seems to be a desirable property. It is, however, not shared by most studies of disequilibrium price dynamics because these involve price changes brought about by a market manager or other artificiality. Prices do not vary as a consequence of the maximizing behavior of individuals. Recent studies by Fisher [4], Diamond [1], and Rothschild [11] have attempted to rectify this shortcoming of the traditional conceptualizations of the price adjustment process.

2. THE RECONTRACTING PROCESS

The fundamental data of the system we are considering are the utility possibility sets of the coalitions in the economy. We are assuming that the welfare of individuals can be represented by a real-valued index such that higher levels of this index are associated with more preferred situations. We shall suppose that there are n individuals in the economy and will denote the set of all these participants by N. Coalitions are usually denoted by S or T and are subsets of N. We use $\subseteq$ to mean "is a subset of" and $\subset$ to mean "is a subset of but is not equal to."

For each $S \subseteq N$, $V(S)$ will be the set of all utility combinations attainable by the members of S on their own. Allocations are written as vectors in R^n, and we denote by R^S the subspace of R^n indexed by the members of S; thus $V(S) \subset R^S$. We use the symbol $x|_S$ to mean the projection of the vector x into R^S; that is, $x|_S$ is the vector of utility levels that allocation x assigns to the members of S. Thus we can write the definition of blocking as: A coalition $S \subseteq N$ is said to *block* $x \in R^n$ if there exists $y \in V(S)$ and $y \geqslant x|_S$.[2] The *core*, denoted $\mathscr{C}$, is the set of all $x \in V(N)$ such that no blocking coalitions for x exist.

Our dynamic process will generate a set of proposals $\{x_t\}$ beginning from an arbitrary x_0 such that for each t there exists a coalition S_t such that S_t blocks x_{t-1} and further that $x_t|_S \in V(S)$ and $x_t|_S \geqslant x_{t-1}|_S$. If for some x_t there are no blocking coalitions (x_t is in the core), then the process stops and x_t is the final allocation.

Several questions must be settled before the process outlined above is completely defined.

(i) If several blocking coalitions are possible for x_t, which one actually forms and proposes the blocking allocation?

(ii) In general, if x is blocked by S, there will be many allocations in $V(S)$ superior to $x|_S$. Which of these will actually be chosen by S to form the blocking allocation?

(iii) Suppose that $x_t|_S$ is selected by S to block x_{t-1}; what allocation is given to the people not in S? (We denote them by S^c hereafter.) This must be determined before the specification of x_t is complete.

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[2] We adopt the following definitions for vector inequalities: $x > y$ implies $x_i > y_i$ for all i; $x \geqslant y$ implies $x_i \geqslant y_i$ for all i and $x_i > y_i$ for at least one i; and $x \geq y$ implies $x_i \geqslant y_i$ for all i.

Before stating the hypotheses we shall make with respect to these questions, we shall state some assumptions on the sets $V(S)$ and introduce some further notation.

Define

$$\overline{V}(S) = \{z \mid z \in V(S), z' \geqslant z \text{ implies } z' \notin V(S)\}.$$

$$B_S(x) = \{z \mid z \in \overline{V}(S) \text{ and } z \geqslant x|_S\}.$$

The set of all Pareto efficient allocations for coalition S is $\overline{V}(S)$ and $B_S(x)$ is the subset of those that could be used to block x. We shall assume that all the sets $V(S)$ are $|S| - 1$ rectifiable. That is, there exists a Lipschitzian function mapping some bounded subset of $R^{|S|-1}$ onto $V(S)$.[3]

3. ASSUMPTIONS ON THE UTILITY POSSIBILITY SETS

ASSUMPTION 1: *For each $i \in S$ and $S \subseteq N$ such that $S \neq \{i\}$ there is $x^{i,S} \in \overline{V}(S)$ such that $x^{i,S}|_{\{i\}} < \overline{V}(\{i\})$.*

This says that there are efficient allocations for every coalition such that individuals are given less than what they could obtain by themselves. If the sets $V(S)$ are obtained from an exchange economy with individual's endowments $\omega_i \in R^k_+$ and monotone utility functions $u_i(x_i)$ over commodity bundles in the consumption set assumed to be R^k_+, then this would mean that $u_i(\omega_i) > u_i(0)$ for all i.

ASSUMPTION 2: *For each $S \subseteq N$ such that $|S| > 1$, there exists $x^S \in \overline{V}(S)$ such that $x^S_{(i)} > \overline{V}(\{i\})$ for all $i \in S$.*

This means that all coalitions can improve the positions of all of their members. If the $V(S)$ are derived from an exchange economy this rules out the situation in which the marginal rates of substitution for any two individuals at their endowment points are identical. Thus there could always be some positive gains to trade for everyone.

If Assumption 2 holds, then for each S there exists $\varepsilon^S > 0$ such that $y \in N_{\varepsilon^S}(x^S)$ implies $y|_{\{i\}} > \overline{V}(\{i\})$ for all $i \in S$. We shall let $\bar{\varepsilon} = \min_{S \subseteq N} \varepsilon^S$ and retain this notation for use in the proof.

ASSUMPTION 3: *If $z \in V(S) \cap \{u \mid u \in R^S, u_i \geqslant V(\{i\}) \text{ for all } i \in S\}$ and $z + \bar{\varepsilon}\delta_i \in V(S)$ where δ_i is an $|S|$-vector of zeros except for a 1 in the ith place, and $\varepsilon > 0$, then there exists a constant $|S|$-vector $\kappa > 0$ dependent only on S and $\bar{\varepsilon}$ such that $z + \kappa \in V(S)$.*

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[3] I am indebted to a referee for pointing out that rectifiability is necessary if one is to be able to define Lebesgue measure on a curve in Euclidean space of higher dimension.

This assumption would be fulfilled in a trading economy with strictly monotone preferences if the utility functions were appropriately chosen. We take it as an abstract hypothesis here.

We also take the hypotheses on closure, boundedness, and free disposal used by Scarf in [12]. These are:

(S1) $V(S)$ is closed for all $S \subseteq N$.

(S2) $u \in V(S)$ and $y \leqslant u$ implies $y \in V(S)$.

(S3) $V(S) \cap \{u | u \in R^S, u_i \geqq \overline{V}\{i\}$ for all $i \in S\}$ is non-empty and bounded.

Assumption 4 is called the strong-superadditivity assumption. The force of this assumption is to insure that the core has the same dimensionality as the set of Pareto optima. In an exchange economy with individual endowments $\omega_i \in R_+^k$ we can write the endowment vector for the economy as $(\omega_1, \omega_2, \ldots, \omega_n) = \omega \in R_+^{nk}$. Then it is a theorem that (see [6]) for almost every $\omega \in R_+^{nk}$ (in the sense of Lebesgue measure on R^{nk}) the strong superadditivity assumption is valid for the associated economy. This assumption is stated as follows:

ASSUMPTION 4: *There exists $x^* \in \mathscr{C}$ such that $x^*|_S \notin V(S)$ for any $S \subset N$.*

This completes the assumptions on the sets $V(S)$. We now turn to assumptions on the recontracting process itself and in particular to the three questions stated in the last section.

4. ASSUMPTIONS ON THE RECONTRACTING PROCESS

Let x_t be the proposal at time t. Let $\mathscr{T}(x) = \{S | B_S(x) \neq \varnothing\}$ be the set of all possible blocking coalitions for x. We suppose that over each collection of possible blocking coalitions $\mathscr{T} \neq \varnothing$, there is a strictly positive probability distribution that determines which coalition will, in fact, block the current proposal.

ASSUMPTION 5: *For every collection $\mathscr{T}$ of coalitions that form a set of possible blocking coalitions for some proposal, the probability that coalition $T \in \mathscr{T}$ forms is strictly positive and depends only on the collection $\mathscr{T}$ and the coalition in question. In particular, it is not explicitly dependent on the proposal (of course it depends on x through the sets $B_S(x)$). We shall let these be denoted $p(\mathscr{T}, T)$ defined for $T \in \mathscr{T}$ and all possible collections $\mathscr{T}$ of blocking coalitions. Thus $\Sigma_{T \in \mathscr{T}}\, p(\mathscr{T}, T) = 1$.*

One possible justification for this follows: Blocking requires informational exchange within a coalition and informational exchange takes time. For each potential blocking coalition, there is a non-decreasing, non-negative function that gives the probability that this coalition has completed the informational exchange required for blocking by the stated time. From these functions, we can derive the probability that a given coalition has formed before any other. We suppose that the first coalition to complete the requisite informational exchange process will become the blocking coalition. Clearly, if these cumulative informational exchange functions are independent of the proposal, the probability of a

given coalition becoming the actual blocking coalition will depend only on the other coalitions engaged in informational exchange—that is, on the other members of $\mathcal{T}(x)$. We do not posit that things necessarily happen as sketched above, but it offers some claim to plausibility for our assumption.

In answer to the second question, we suppose that a blocking coalition S will select a point in $\overline{V}(S)$. That is, sufficient information will be available for the coalition to select a Pareto efficient point. The coalition must, of course, select something superior to the current proposal. Thus the point chosen will lie in $B_S(x)$. We shall not specify the nature of the informational exchange process used by the coalition to choose the point in $B_S(x)$ but shall note that there are several such processes known that will result in probability distributions over the point selected in $B_S(x)$. These include studies by Reiter [**10**] and Hurwicz, Radner, and Reiter [**8**].

We assume that if E is a Lebesgue measurable subset of $B_S(x)$ (assumed to be non-empty), then the probability that coalition S will select a point in E to block x, given that S is the blocking coalition, is given by $v_1^S(E|x)$. If the $V(S)$ satisfy assumptions 1–4, then $\overline{V}(S)$ will be a manifold of dimension $|S| - 1$. Then $v_1^S(\cdot|x)$ will be a probability measure on this manifold for all x such that $B_S(x) \neq \varnothing$. Let $\lambda^S(\cdot)$ be Lebesgue measure on $\overline{V}(S)$. This is possible in any one of several equivalent ways, because of the rectifiability assumption (see Federer [**3**, pp. 169–74 and 261–62; in particular, Section 3.2.26, p. 261]). We make the following assumption on $v_1^S(\cdot|x)$:

ASSUMPTION 6: *The $v_1^S(\cdot|x)$ satisfy*

(i) $v_1^S(E|x) = 0$ *if* $\lambda^S(E) = 0$,
(ii) $v_1^S(B_S(x)|x) = 1$, *and*
(iii) $v_1^S(E|x) > \varepsilon \lambda^S(E \cap B_S(x))/\lambda^S(B_S(x))$

for any measurable $E \subseteq \overline{V}(S)$, where $\varepsilon > 0$ is a constant independent of x and E.

Part (i) is the absolute continuity of v_1^S with respect to Lebesgue measure. Part (ii) says that the process moves to a point in $B_S(x)$ with probability 1. Part (iii) is less transparent: It means that if a subset of B_S makes up a certain proportion of B_S (in the sense of its measure relative to the measure of B_S), then the probability of arriving at this set cannot be smaller than ε times this proportion. A sufficient condition for (iii) is that the density functions of the measures $v_1^S(\cdot|x)$ are bounded away from zero (almost everywhere) uniformly in x. (Density functions exist by virtue of (S3), 6(i), and the Radon-Nikodym theorem.)

With respect to the third question on p. 4, namely what allocation is given to the complement of a blocking coalition, we propose that they be allowed to respond by selecting a point in the set of Pareto efficient points for themselves. In general, this will not be superior to the allocation they obtained in the previous proposal. Further, the stochastic mechanism through which these are selected, $v_2^{S^c}(\cdot)$, satisfies the following assumption.

ASSUMPTION 7:

(i) $v_2^{S^c}(F) = 0$ if $\lambda^{S^c}(F) = 0$,

(ii) $v_2^{S^c}(\overline{V}(S^c)) = 1$, and

(iii) $v_2^{S^c}(F) \geqslant \varepsilon \lambda^{S^c}(F \cap \overline{V}(S^c))/\lambda^{S^c}(\overline{V}(S^c))$

for all measurable $F \subseteq \overline{V}(S^c)$ where $\varepsilon > 0$ is independent of F.

Note that the point selected is implicitly assumed to be independent of the previous proposal and the blocking proposal selected by S.

Let

$$X(S, x) = \{z | z|_S \in B_S(x), z|_{S^c} \in \overline{V}(S^c)\}.$$

The transition probability of the process given that S is the blocking coalition is defined by the product measure

$$(v_1^S \times v_2^{S^c})(\cdot)$$

on $X(w, x)$. The probability of reaching a measurable rectangle $E \times F$ in $X(S, x)$ if the previous proposal was x is simply

$$v_1^S(E|x) \times v_2^{S^c}(F),$$

because of our assumption that v_1^S and $v_2^{S^c}$ are independent.

Thus the barter processes we shall consider are defined by the distributions $v_1^S(\cdot|x)$ and $v_2^{S^c}(\cdot)$ for all $x \in V(N)$ and $S \subseteq N$. We now state the main theorem and the proof. Discussion of the results and further comments are reserved for Section 6.

5. MAIN THEOREM AND THE PROOF

THEOREM: *Let x_0 be an arbitrary proposal. Then the probability that x_t is not in the core approaches zero with t.*

Discussion of the Method of Proof

The proof is accomplished by dividing the possible characteristics of the initial proposal into six cases. Together these exhaust the possibilities, but they are not mutually exclusive. If a proposal x_0 should fall under more than one case, any of the proofs associated with these cases would work. Case 1 is when the only blocking coalition for x_0 is N; the proof is easy. Case 2 is when $x_0 \in \mathscr{C}$; this is included only for completeness. Cases 3 and 4 exhaust the possibilities when $n \geqslant 4$. Case 3 treats the instance in which a possible blocking coalition of size less than $n - 1$ exists for x_0. Case 4 is the instance in which the only possibilities for blocking x_0 are through N or a coalition of $n - 1$ individuals. Cases 5 and 6 are $n = 2$ and $n = 3$ respectively. It is clear that this is an exhaustive classification of the possible economies and initial states. In every instance we follow the procedure of showing that there is a finite number, say τ_i, and a probability, say $\delta_i > 0$, such that the probability that x_{τ_i} is in the core given that x_0 was in case

i is at least δ_i. If we fail to reach the core in period τ_i, then we must be in one of the cases 1, 3, 4, 5, or 6, and we can begin the analysis again. Clearly if $\tau = \max_{i=1,6} \tau_i$ and $\delta = \min_{i=1,6} \delta_i$, then the probability of not being in the core goes to zero as $t \to \infty$ at least as rapidly as $(1 - \delta)^{t/\tau}$.

CASE 1 : $\mathscr{T}(x_0) = \{N\}$. In this case the coalition of the whole will now select a Pareto optimum x_1 such that $x_1 \geqslant x_0$. Clearly $x_1 \in \mathscr{C}$, since if not, then $x_1|_S \in$ int $V(S)$ for some $S \neq N$. But $x_1 \geqslant x_0$ implies $x_1|_S \geqslant x_0|_S$, a contradiction, for then $\mathscr{T}(x_0)$ would have contained S by the disposability of utility assumption (S2).

CASE 2 : $\mathscr{T}(x_0) = \varnothing$. Therefore $x_0 \in \mathscr{C}$; it is therefore stationary, and the theorem is trivially true.

CASE 3 : $\mathscr{T}(x_0)$ contains $S \neq N$ and $|S^c| > 1$. Further $n \geqslant 4$. (We shall treat the cases $n = 2$ and $n = 3$ separately as cases 5 and 6 respectively.) We shall need the following lemma.

LEMMA 1: *If the conditions of Case 3 are satisfied, and $i \in S^c$ is arbitrary, then there exists $\delta > 0$ such that the probability that $x_2 \in X(\{i\}, x_1)$ is at least δ.*

PROOF OF THE LEMMA : Since $S \in \mathscr{T}(x_0)$, we know that $p(x_1 \in (S, x_0)) = p(\mathscr{T}(x_0), S) > 0$ by hypothesis. Thus, $x_1|_{S^c} \in \overline{V}(S^c)$ with probability at least $p(\mathscr{T}(x_0), S)$. By Assumption 1 there exists $x^{i,S^c} \in \overline{V}(S^c)$ such that $x^{i,S^c}|_{\{i\}} < \overline{V}(\{i\})$. For $\varepsilon > 0$ sufficiently small we shall have $N_\varepsilon(x^{i,S^c}) \cap \overline{V}(S^c) \subseteq \{z | z \in \overline{V}(S^c), z_i < \overline{V}(\{i\})\} \equiv Z_{i,S^c} \subset \overline{V}(S^c)$. Since $\varepsilon > 0$, the above intersection is open in $\overline{V}(S^c)$ and hence Z_{i,S^c} is non-null (Lebesgue measure $\lambda^{S^c}(\cdot)$ on $\overline{V}(S^c)$). Let $\lambda^{S^c}(Z_{i,S^c}) = \alpha > 0$; then the probability that $x_1|_{S^c}$ is in Z_{i,S^c} is at least $\varepsilon \alpha / \lambda^{S^c}(\overline{V}(S^c)) = p_{1,i} > 0$. Hence the probability that $\{i\}$ is among the possible blocking coalitions for x_1 is at least $p_{1,i} \cdot p(\mathscr{T}(x_0), S)$. Let $p_{2,i} = \min_{\mathscr{T} \supset \{i\}} p(\mathscr{T}, \{i\}) > 0$. Thus, the probability that $\{i\}$ actually blocks x_1, given that $S \not\ni \{i\}$ blocked x_0, is at least $p_{1,i} \cdot p_{2,i}$. Therefore, the probability that $i \in S^c$ will block x_1, given that $S \in \mathscr{T}(x_0), S \neq N$, is at least $p_{1,i} \cdot p_{2,i} \cdot p(\mathscr{T}(x_0), S) = \delta(i) > 0$. Let $\delta = \min_i \delta(i)$. This completes the proof of Lemma 1.

We shall complete the proof of the theorem in Case 3 below (Lemma 3) by showing that it is possible, with probability bounded away from zero, to arrive at the core directly from a proposal in which a single individual is the blocking coalition. We first consider Case 4 because Lemma 3 will be used to prove the main theorem in both cases.

CASE 4: $S \in \mathscr{T}(x_0)$ implies $S = N$ or $|S| = n - 1$. Further, $n \geqslant 4$. We shall proceed by showing that the probability of x_2 being in Case 3 is at least ξ, for some $\xi > 0$ independent of x_0. Then, the lemma above will apply and the probability of x_3 being blocked by a single individual will be positive and bounded away from zero by $\xi\delta$.

LEMMA 2: *Let $\mathcal{T}(x_0)$ consist entirely of conditions of size $n-1$, and perhaps N as well; then there exists $\xi > 0$ such that the probability that x_2 is blocked by a coalition of size smaller than $n-1$ is at least ξ.*

The proof of Lemma 2 depends on the following fact: Let L be an $n-1$ dimensional surface contained in the n-dimensional cube K with faces parallel to the coordinate hyperplanes such that there exists an n-cube of side κ in the lowest corner of K which is strictly contained on the lower side of L. (This is the situation asserted in Assumption 3.) Then a lower bound on the surface area of L is one half the surface area of the hypercube with side κ. This can be seen as follows: Take each of the n, $n-1$ dimensional surfaces with side κ forming the top of the cube and project it along the coordinate axis to which it is perpendicular, generating a cylinder in n-space. For each dimension, i, let the intersection of this cylinder and L be L_i. Since the cube with side κ is strictly beneath L, the L_i do not intersect. But the area of each of the L_i is clearly greater than that of the $n-1$ dimensional surface from which it was generated.[4] Thus the surface area of L is greater than the sum of the areas of these surfaces.

PROOF OF LEMMA 2: Let $\mathcal{Y}$ be the family of all possible collections of blocking coalitions of the form stated in the hypothesis of this lemma. Let $\mathcal{Y}' \subset \mathcal{Y}$ be the subfamily of all those that contain N. Define $\alpha = \max_{\mathcal{T} \in \mathcal{Y}'} p(\mathcal{T}, N)$. Then for $\mathcal{T} \in \mathcal{Y}$, the probability that a coalition of size $n-1$ will block x_0 is at least $1 - \alpha$. Let $x_1|_{S_1}$ be the allocation proposed by S_1, where $|S_1| = n-1$ and $S_1^c = \{i\}$, who receives $\overline{V}(\{i\})$. Consider $\mathcal{T}(x_1)$. If $\mathcal{T}(x_1)$ falls into any of the first three cases we are done. Thus suppose $\mathcal{T}(x_1) \in \mathcal{Y}$, the only remaining possibility since $n \geqslant 4$. We know that $S_1 \notin \mathcal{T}(x_1)$ since $x_1|_{S_1} \in \overline{V}(S_1)$. Thus there exists $S_2 \in \mathcal{T}(x_1)$ such that $|S_2| = n-1$ and $|S_1 \triangle S_2| = 2$ ($\triangle$ is the symmetric difference operation: $A \triangle B = ((A \setminus B) \cup (B \setminus A))$). By the above analysis, the probability of such an S_2 forming is at least $1 - \alpha$. Since $i \in S_2$, $x_2|_i \geqslant \overline{V}(\{i\})$. Let $\{i'\} = S_2^c$ and consider $\{i, i'\}$. By Assumption 2, $\overline{V}(\{i, i'\})$ contains an $\bar{x}$ such that $\bar{x} > (\overline{V}(\{i\}), \overline{V}(\{i'\}))$. Let $\min_{j=i,i'} (\bar{x}_j - \overline{V}(\{j\})) = \bar{\varepsilon} > 0$. Let

$$M(S_2, i) = \{z|_{S_2} \mid z|_{S_2} \in \overline{V}(S_2), \overline{V}(\{i\}) + \bar{\varepsilon} \geqslant z_i \geqslant \overline{V}(\{i\})\}.$$

We shall show that, with probability bounded away from zero, S_2 will select a point $M(S_2, i)$ and thus $\{i, i'\}$ will be a possible blocking coalition. This will prove the lemma since $n \geqslant 4$ and $|\{i, i'\}| = 2$. Consider

$$B_{S_2}(x_1) = \{z|_{S_2} \mid z|_{S_{2_j}} \geqslant x_1|_j, \text{ for all } j \in S_2 \text{ and } z|_{S_2} \in \overline{V}(\{S_2\})\}.$$

Let the largest possible increment in utility for individual i, given that S_2 is the blocking coalition and x_1 was formed by S_1, be given by

$$\chi(x_1) = \sup_{z_2|_{S_2} \in B_{S_2}(x_1)} (z_2|_i - \overline{V}(\{i\})).$$

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[4] This can be shown rigorously by using the Caratheodory construction of Gross measure (see [3, pp. 169 ff.]).

SUBCASE A: $\chi(x_1) \leqslant \bar{\varepsilon}$. In this case $B_{S_2}(x_1) \subseteq M(S_2, i)$; therefore

$$v_1^{S_2}(M(S_2, i)|x_1) = 1,$$

since S_2 must choose a point in $B_{S_2}(x_1)$ by Assumption 6(ii).

SUBCASE B: $\chi(x_1) > \bar{\varepsilon}$.

We shall use Assumption 6(iii) to bound the probability of reaching $M(S_2, i)$ away from zero, and thus the probability that $\{i, i'\}$ can block x_2 will be bounded away from zero. By this assumption

$$v_1^{S_2}(M(S_2, i)|x_1) > \varepsilon \frac{\lambda^{S_2}(M(S_2, i) \cap B_{S_2}(x_1))}{\lambda^{S_2}(B_{S_2}(x_1))};$$

since $\lambda^{S_2}(B_{S_2}(x_1)) \leqslant \lambda^{S_2}(\overline{V}(S_2))$, it will suffice to find a positive lower bound for $\lambda^{S_2}(M(S_2, i) \cap B_{S_2}(x_1))$ that is independent of x_1. Define

$$A(S_2, i, x_1, y) = \lambda^{S_2 \setminus \{i\}}((B_2(x_1) \cap \{z|_{S_2} | z_i = y\}).$$

The area of the cross section of $B_{S_2}(x_1)$ taken at $z_i = y$ is A. Note that since S_2 has $n - 1$ members and $n \geqslant 4$, $\lambda^{S_2 \setminus \{i\}}(\cdot)$ is Lebesgue measure of at least one dimension. Since $\chi(x_1) > \bar{\varepsilon}$, we have $A(S_2, i, x_1, y) > 0$ for each $y \in [\overline{V}(\{i\}) + \bar{\varepsilon}, \overline{V}(\{i\})]$. Let $\bar{\bar{\varepsilon}} = \bar{\varepsilon}/3$. For $y \in [\overline{V}(\{i\}) + \bar{\bar{\varepsilon}}, \overline{V}(\{i\})]$, we have that there exists $z \in B_{S_2}(x_1)$ such that $z_i > y + \bar{\bar{\varepsilon}}$ by the definition of Subcase B. By Assumption 3, there exists an $|S_2|$-vector $\kappa > 0$ such that $x_1|_{S_2} + \kappa$ is in $V(S_2)$. Hence $B_{S_2}(x_1)$ bounds an $|S_2|$-cube with faces parallel to the coordinate hyperplanes generated by x_1 and $x_1 + \kappa$. Thus for each $y \in [\overline{V}(\{i\}) + \bar{\bar{\varepsilon}}, \overline{V}(\{i\})]$, $B_{S_2}(x_1) \cap \{z|_{S_2} | z_i = y\}$ bounds an $|S_2|$-1-cube of side κ. By the fact above (p. 28), the $\lambda^{S_2 \setminus \{i\}}$ measure of this set is bounded below by one-half the area of this cube. Further, by the above, κ is independent of x_1 in Subcase B and of $y \in [\overline{V}(\{i\}) + \bar{\bar{\varepsilon}}, \overline{V}(\{i\})]$. Call this lower bound on $A(S_2, i, x_1, y)$, $\underline{A}(S_2, i)$. Upon integrating over $y \in [\overline{V}(\{i\}) + \bar{\bar{\varepsilon}}, \overline{V}(\{i\})]$ we observe that

$$\lambda^{S_2}(M(S_2, i) \cap B_{S_2}(x_1)) \geqslant \bar{\bar{\varepsilon}} \underline{A}(S_2, i).$$

Now utilizing Assumption 6 and the expression above,

$$v_1^{S_2}(M(S_2, i)|x_1) > \frac{\varepsilon \lambda^{S_2}(M(S_2, i) \cap B_{S_2}(x_1))}{\lambda^{S_2}(B_{S_2}(x_1))} > \frac{\varepsilon \bar{\bar{\varepsilon}} \underline{A}(S_2, i)}{\lambda^{S_2}(\overline{V}(S_2))} > 0,$$

which is a positive constant independent of x_1. Call it η. Thus, with probability at least η, $x_2|_{\{i, i'\}} < \bar{x} \in \overline{V}(\{i, i'\})$ by definition of M. Hence, $\{i, i'\} \in \mathscr{T}(x_2)$. Since $n \geqslant 4$, $n - 1 > |\{i, i'\}^c| > 1$, and we are in Case 3 with probability at least

$$(1 - \alpha)^2 \cdot \hat{p} \cdot \eta = \xi > 0,$$

where

$$\hat{p} = \min_{\{\mathscr{T} | S \in \mathscr{T}, |S| = 2\}} p(\mathscr{T}, S).$$

Since ξ has been shown to be independent of x_0, the lemma is proven.

We shall now use Lemmas 1 and 2 to conclude the theorem in Cases 3 and 4. In either instance, we have shown that within at most four stages of barter there is at least a probability of $\delta\xi$ that the blocking coalition consists of a single individual. This situation is sufficient to conclude the main theorem in these cases through use of the following lemma. The purpose of this lemma is to show that if x_t is reached because a single individual blocked x_{t-1}, then N is a possible blocking coalition for x_t, and further that N can propose a core allocation.

LEMMA 3: *Let* $x_t \in Z^i = \{z | z_i = \overline{V}(\{i\}), z|_{\{i\}^c} \in \overline{V}(\{i\}^c)\}$. *Then there exists* $D_i \subset Z^i$ *open relative to* Z^i *such that for an open (relative to* $\mathscr{C}$) *set* $D' \subset \mathscr{C}$ *we have that* $d \in D_i, d' \in D'$ *implies* $d < d'$.

PROOF OF LEMMA: By Assumption 4, let $x^* \in \mathscr{C}, x^*|_S \notin V(S)$ for any $S \subset N$. Thus there exists $Q \subset \overline{V}(\{i\}^c)$, an open set in the relative topology, such that $x^*|_{\{i\}^c} > q$ for all $q \in Q$. Further, $x^*|_i > \overline{V}(\{i\})$. Let $D_i = (\overline{V}(\{i\}), Q) \subset Z^i$. D_i is clearly open relative to Z^i since Q is open in $\overline{V}(\{i\}^c)$. Let $\beta > 0$ be such that $z \in N_\beta(x^*)|_{\{i\}^c}$ implies $z > q \in Q$. Let $D' = N_\beta(x^*) \cap \mathscr{C}$. D_i and D' as constructed fulfill the requirements of the lemma.

What we have shown in Lemmas 1 and 2 is that within four periods of barter, there is a probability of at least $\varepsilon\xi$ of being in Z^i, for some i. Further, for $x_t \in D_i \subset Z^i, \mathscr{T}(x_t)$ includes N. Let the probability of being in D_i, given that $\{i\}$ has blocked the previous proposal, be $\pi_i = v_2^{\{i\}^c}(Q) > 0$. Thus the probability of being in some D_i within five periods of barter is at least $\pi\delta\xi$, where $\pi = \min \pi_i$. Further, for each $d_i \in D_i$, the probability of moving to $\mathscr{C}$ is no less than the product of the probabilities that in fact N will block d_i and that some $d' \in D'$ will be selected by N.

Conclusion of the Proof of the Theorem in Cases 3 and 4

The first of these probabilities is at least $\bar{p} = \min_{\{\mathscr{T} | N \in \mathscr{T}\}} p(\mathscr{T}, N) > 0$, by Lemma 3. The latter is $v_1^N(D'|d_i)$, and

$$v_1^N(D'|d_i) > \frac{\varepsilon\lambda^N(D' \cap B_N(d_i))}{\lambda^N(B^N(d_i))} = \frac{\varepsilon\lambda^N(D')}{\lambda^N(B_N(d_i))}$$

since $d_i \in D_i, d' \in D'$ implies $d_i < d'$ and $D' \subset \mathscr{C}$,

$$\geqslant \frac{\varepsilon\lambda^N(D')}{\lambda^N(\overline{V}(N))} = \rho_i > 0,$$

and this is independent of d_i. Let $\rho = \min \rho_i$. Thus, once a single player coalition blocks, the probability of reaching the core is at least $\bar{p}\rho$. Hence, within five periods, there is a probability of reaching the core of at least $\bar{p}\rho\pi\delta\xi > 0$ from an arbitrary initial position. Call this γ. Thus, the $5t$-step transition probability of not reaching the core is

$$P^{(5t)}(x_{5t} \notin \mathscr{C}|x_0) > (1 - \gamma)^t$$

which clearly converges to zero with t.

CASE 5: $n = 2$.

Proof of the Theorem in Case 5

Let $\overline{V}(\{1\}) \times \overline{V}(\{2\}) = \omega$, and $x = (x_1, x_2)$ be a proposal for the market; then if $x \geqslant \omega$ the only possible blocking coalition is N and we move directly to the core as in Case 1. If $x_i < \overline{V}(\{i\})$ for some i, then $\{i\}$ might block, say with probability p, yielding the allocation ω. From this we go to the core directly. If N blocks when $\{i\}$ could have blocked, we are either in the core or in a position at which only $\{i\}$ could block. If the latter, we arrive at ω and then at the core.

CASE 6: $n = 3$.

Proof of the Theorem in Case 6

We need treat only the case in which $\mathcal{T}(x_0)$ contains only coalitions of two players and perhaps the coalition of the whole, since other situations are adequately handled by Lemma 3. Thus, suppose that $\mathcal{T}(x_0)$ contains no coalitions of single individuals. We shall let

$$\min_{\{\mathcal{T} \mid \{i,j\} \in \mathcal{T}\}} p(\mathcal{T}, \{i, j\}) = p_{i,j},$$

to shorten the notation. Further denote $\overline{V}(\{i\}) = \omega_i$. By Assumption 4 and the fact that the $V(S)$ are closed, there exists $\bar{\varepsilon} > 0$ such that $y \in N_{\bar{\varepsilon}}(x^*)$ implies $y|_S \notin V(S)$ for all $S \subset N$. Thus for any pair $\{i, j\}$, we can find $\bar{x}_{i,j} \in \overline{V}(\{i, j\})$ such that $\bar{x}_{i,j} < x^*|_{\{i,j\}}$. Hence, there exists $\tilde{\varepsilon}_{ij}$ such that $z \in N_{\tilde{\varepsilon}_{ij}}(\bar{x}_{i,j})$ implies $z < y|_{\{i,j\}}$ for all $y \in N_{\bar{\varepsilon}}(x^*)$. Let $\tilde{\varepsilon} = \min \tilde{\varepsilon}_{ij}$.

Suppose that, without loss of generality, $\{1, 2\} \in \mathcal{T}(x_0)$. Then $x_1 \in X(\{1, 2\}, x_0)$ with probability at least $p_{1,2}$. Consider $\mathcal{T}(x_1)$. Since neither $\{1\}$ nor $\{2\}$ could block x_0 and $x_1|_{\{1,2\}} > x_0|_{\{1,2\}}$, we have $\{1\} \notin \mathcal{T}(x_1)$ and $\{2\} \notin \mathcal{T}(x_1)$. Further, since $x_1|_3 = \omega_3$, $\{3\} \notin \mathcal{T}(x_1)$. Since $x_1|_{\{1,2\}} \in \overline{V}(\{1, 2\})$, $\{1, 2\} \notin \mathcal{T}(x_1)$. If $\mathcal{T}(x_1) = \{N\}$ we are in Case 1 and can conclude the theorem. Thus for concreteness, and again without loss of generality, we suppose $\{2, 3\} \in \mathcal{T}(x_1)$. Hence, with probability at least $p_{1,2} \cdot p_{2,3}$, $x_2 \in X(\{2, 3\}, x_1)$.

Consider

$$\zeta = \inf_{z \in N_{\tilde{\varepsilon}}(\bar{x}_{1,3})} z|_3.$$

We can choose $\tilde{\varepsilon} > 0$ smaller than $\min \tilde{\varepsilon}_{ij}$, if necessary, so that $\zeta > \omega_3$ and this will preserve the property that $z \in N_{\bar{\varepsilon}}(\bar{x}_{i,j})$ implies $z < y|_{\{i,j\}}$ for all $y \in N_{\bar{\varepsilon}}(x^*)$. Let $D_{23} = \{z \mid z \in X(\{2, 3\}, x_1), x_1 \in X(\{1, 2\}, x_0) \zeta > z|_3 > \omega_3\}$. Let $\eta = \inf_{z \in D_{23}} z|_2$.

SUBCASE A: $x_1|_2 > \eta$.

We have that $D_{23} \supset X(\{2, 3\}, x_1)$, since by Scarf's disposability of utility assumption $\overline{V}(\{2, 3\})$ is falling everywhere. Thus, the probability of being in D_{23} is one in this case, given that $\{2, 3\}$ is the blocking coalition.

SUBCASE B: $x_1|_2 \leqslant \eta$.

In this case

$$P(x_2 \in D_{23}|x_2 \in X(\{2, 3\}, x_1)) > \frac{\varepsilon\lambda^{\{2,3\}}(D_{23} \cap B_{\{2,3\}}(x_1))}{\lambda^{\{2,3\}}(B_{\{2,3\}}(x_1))}.$$

The conditions of subcase B imply that if $x_2 \in \overline{V}(\{2, 3\})$ and $x_2|_2 > \eta$, then $x_2 \in B_{\{2,3\}}(x_1)$. Thus the above expression is greater than or equal to

$$\frac{\varepsilon\lambda^{\{2,3\}}(D_{23})}{\lambda^{\{2,3\}}(\overline{V}(\{2, 3\}))} = p_{D_{23}} > 0.$$

Hence, in either subcase the probability that x_2 is in D_{23} is at least $p_{1,2}p_{2,3}p_{D_{23}}$, and this is bounded away from zero independently of the initial position. For $x_2 \in D_{23}$, $N_{\tilde{\varepsilon}}(\bar{x}_{1,3}) \subset B_{\{1,3\}}(x_2)$ by definition of ζ. Hence,

$$p(x_3 \in N_{\tilde{\varepsilon}}(\bar{x}_{1,3})|x_2 \in D_{23}) > \frac{\varepsilon\lambda^{\{1,3\}}(N_{\tilde{\varepsilon}}(\bar{x}_{1,3}))}{\lambda^{\{1,3\}}(\overline{V}(\{1, 3\}))},$$

and this is independent of the particular choice of $x_2 \in D_{23}$. Thus, $p(x_3 \in N_{\tilde{\varepsilon}}(\bar{x}_{1,3}))$ is bounded away from zero independently of the initial position. But by definition of $\tilde{\varepsilon}$, $N \in \mathcal{T}(x_3)$ and $N_{\tilde{\varepsilon}}(x^*) \subset B_N(x_3)$ for all $x_3 \in N_{\tilde{\varepsilon}}(\bar{x}_{1,3})$. Thus, the probability of reaching the core from such an x_3 is at least

$$\min_{\{\mathcal{T}|N\in\mathcal{T}\}} p(\mathcal{T}, N) \cdot \frac{\varepsilon\lambda^N(N_{\tilde{\varepsilon}}(x^*))}{\lambda^N(\overline{V}(N))},$$

which is again bounded away from zero independently of the choice of $x_3 \in N_{\tilde{\varepsilon}}(\bar{x}_{1,3})$. This is sufficient to conclude the theorem.

6. FURTHER DISCUSSION OF THE PROCESS AND RESULTS

We have conceptualized the process by which a final solution is reached as a sequence of proposals. Each proposal is superceded by a counterproposal that represents a feasible allocation that could be used to block it. In the end, no further blocking is possible and presumably the final proposal is actually implemented by the participants in the system. No real activity is going on while the recontracting process is still under way. The consequence of this is that the sets $V(S)$ are fixed throughout the course of barter. This is strongly reminiscent of the Walrasian tâtonnement in which excess demands are reported by the use of tickets and prices adjust until excess demand is indicated to be zero—only then does trading take place.

Traditional stability results for the tâtonnement process state that as $t \to \infty$ prices will become arbitrarily close to equilibrium prices. This means that under the tâtonnement system, the participants *never* actually perform any trades since they must wait forever for prices to adjust to equilibrium. The result we have

obtained is of a fundamentally different nature even though we keep the tâtonne-ment property. We have shown that the probability that the current proposal is not in the core approaches zero with t. Thus the realization of the outcome of barter (e.g., trade in commodities) will take place in a finite amount of time with probability one. This makes our tâtonnement-like assumption somewhat easier to swallow. Indeed, it seems that this is what Edgeworth had in mind:

> ... Rather there is conceived to be a certain normality about the proceedings. They need not be supposed to take up a long period; rather the contrary, since the disposition and circumstances of the parties are assumed to remain throughout constant. But it is supposed that agreements are renewed or varied many times. A "final settlement" is not reached until the market has hit upon a set of agreements which cannot be varied with advantage to all the recontracting parties . . . [2, pp. 313–314].

However, this is not the formulation of Edgeworth's process discussed in the literature [7, 9, 13]. Indeed, these writers envision barter as a non-tâtonnement adjustment. Because of this they obtain only a convergence to the set of Pareto optima, since the core depends on endowments and these are constantly changing. The point reached as an equilibrium may not even lie in the core of the original economy, and thus our theorem is of a stronger variety in this sense.

These writers envision the path of barter as determinate; it is the solution to a set of difference or differential equations. However, it was Edgeworth's opinion, and we have followed him on this point as well, that the path of barter is indeter-minate. Again:

> At what point on the tract of contract-curve between P and Q [see Edgeworth's diagram on p. 316] the process of bartering will come to a stop cannot be predicted. The position of equilibrium may be described as indeterminate. The essential condition of this indeter-minateness is the absence of competition [2, p. 317].

In the last sentence "competition" refers to price competition as in Marshall. Indeed Edgeworth believed that the fundamental distinction between barter and price competition was that the outcome of the latter was independent of initial conditions and that of the former was not.

This position is supported further by the results of Reiter [10] and Hurwicz, Radner, and Reiter [8]. Since barter surely involves informational exchange, if the informational exchange process has a stochastic element, then this will be reflected in the resulting path of barter.

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