Temporary General Equilibrium in a Sequential Trading Model with Spot and Futures Transactions

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TEMPORARY GENERAL EQUILIBRIUM IN A SEQUENTIAL TRADING MODEL WITH SPOT AND FUTURES TRANSACTIONS

By Jerry R. Green[1]

The existence of an equilibrium is proven for a two-period model in which there are spot transactions and futures transactions in the first period and spot markets in the second period. Prices at that date are viewed with subjective uncertainty by all traders. This introduces the possibility of speculation. Conditions for the existence of a competitive equilibrium include restriction on the nature of price expectations.

1. INTRODUCTION

MODELS OF GENERAL equilibrium have primarily dealt with situations in which prices are announced and decisions taken at an initial point in time. This paper considers the case in which actions are made sequentially and plans may be revised as time progresses.

Economic activity, in the traditional models, consists of carrying out the plans and contracts already made. Further, because these are optimal, if markets were instituted at later dates, no trading would take place: Markets would be unnecessary. In this sense, the institution of "once-and-for-all" trading is self-realizing. If, however, one introduces marketing costs, the appearance of new agents, new commodities, or unanticipated changes in tastes, this self-realization property may fail. Trade would take place, if it were possible. If agents knew about the existence of markets in the future, their plans and actions at the initial date would be changed. This leads to the study of sequential trading, which is a self-realizing institution in the above sense even in cases in which "once-and-for-all" trading is not.

The object of the present paper is to discover the class of cases in which this institutional structure is internally consistent. We shall not treat the more complex possibilities mentioned above. By examining a model in which all markets exist and in which there are no "imperfections," we hope to establish a framework in which they could be studied.

A model of this type has been studied by Grandmont [6]. He assumes the existence of all spot markets at each date. Wealth is transferred between periods by holding money which affords no direct utility. We shall study a system in which trades in all commodities, both present and future, are possible at the initial date, and all future commodities are again tradeable in the future. However, we do not

1103

consider financial assets of the Grandmont type. Borrowing and lending may occur, nevertheless, by trading commodities deliverable in the future against present ones.

Hahn [8] has treated a sequence of markets economy in the case of certainty. Radner [10] has treated uncertainty in a model somewhat similar to ours. He proves that expectations exist that lead to equilibria at all dates and are consistent with these equilibria, whereas we treat expectations as data of the system. A further difference is that Radner's equilibrium expectations are point forecasts. We shall explicitly rule out this type of belief and shall show that its presence is generally inconsistent with the existence of an equilibrium for our model.

Section 2 describes the model and gives a theorem on the determinateness of optimal individual actions. Section 3 is concerned with further properties of demand correspondences. Section 4 contains the equilibrium theorem. Section 5 gives two examples of the non-existence of equilibrium in cases in which our assumptions fail. A concluding discussion and some open questions are given in Section 6.

The results of this paper are embodied in theorems, lemmas, corollaries, examples, and remarks. They are numbered consecutively within each section. Assumptions and other noteworthy expressions and equations are also numbered consecutively, but their numbers appear in parentheses. Definitions are given in the body of the text. They are not numbered, but the defined term is italicized.

2. THE MODEL AND THE DETERMINATENESS OF INDIVIDUAL BEHAVIOR

In this section we study the behavior of the representative individual in our system. Assumptions made in this section are to be understood, in the sequel, to apply to all individuals.

The relevant span of economic activity covers two periods. Period 1, the present, is the date at which the decisions we study will be taken. Period 2 occurs in the future. There are l_1 commodities in the economy in period 1 and l_2 in period 2. We let $l_1 + l_2 = l$.

In period 1 there are markets for the l_1 currently deliverable commodities and markets for contracts for future delivery of the l_2 commodities of period 2. We assume that storage of commodities is impossible. Thus, even though some of the period 1 commodities may be identical to some of the period 2 commodities, physically, they must be regarded as separate economic entities.

The *endowment* of the individual is $\omega = (\omega^1, \omega^2) \in R_+^l$ where $\omega^1 \in R_+^{l_1}, \omega^2 \in R_+^{l_2}$, and both are known with certainty. We assume

$$(2.1) \qquad \omega \gg 0.$$

A *consumption stream*, $x = (x^1, x^2) \in R_+^l$, represents a feasible realization for the individual during the two periods. That is, we assume:

(2.2)   The consumption sets are the non-negative orthant of the commodity space each period.

Market prices at date 1 are written $p = (p^1, p^2) \in R^l_+ \setminus \{0\}$, where p^1 is the price vector for current commodities and p^2 is the price vector for future contracts. Since the set of available net trades for each individual is completely determined by the ratios between components of p, we normalize p so that

$$p \in \Delta^l = \{p \in R^l_+ | \Sigma\, p_i = 1\}.$$

An *action* of the individual at date 1 is written $z = (x^1, b)$, where $x^1 \in R^{l_1}_+$ and $b \in R^{l_2}$. We denote by x^1 the first part of the consumption stream that the individual will obtain; b represents the vector of futures contracts traded. If $b_j > 0$, the individual has contracted at time 1 to receive commodity j at time 2, and conversely. At time 2, the market meets again for the l_2 commodities of this period. The endowment of the individual at this time, given that he took the action (x^1, b) at time 1, is $\omega^2 + b$. In general, $\omega^2 + b$ will not be non-negative. It is allowable for an individual to sell contracts for a commodity in excess of his endowment, but he must be prepared to purchase enough of this commodity at time 2 to fulfill his obligations. This is made precise below.

The individual has expectations about the prices he will face at time 2. A price vector at time 2 is written $q \in R^{l_2}_+ \setminus \{0\}$. Since, given an action taken in period 1, the set of trading opportunities is invariant to a change in all prices by the same proportion, we can take

$$q \in \Delta^{l_2} = \{q \in R^{l_2}_+ | \Sigma q_j = 1\}.$$

For every $p \in \Delta^l$, the individual's *subjective beliefs* or *expectations* about $q \in \Delta^{l_2}$ can be summarized by

$$\psi : \Delta^l \to M(\Delta^{l_2}, \mathscr{B}_{\Delta^{l_2}})$$

where $M(\Delta^{l_2}, \mathscr{B}_{\Delta^{l_2}})$ is the set of all probability measures on Δ^{l_2} with its Borel σ-field. Let "int" denote "interior," understood to be taken relative to the manifold $\{q \in R^{l_2} \Sigma q_i = 1\}$. We assume

(2.3)   (i) for all $p \in \Delta^l$, int co supp $\psi(\rho) \neq \varnothing$;

(ii) for all $p \in \Delta^l$, $\psi(p)(\text{int } \Delta^{l_2}) = 1$.

This assumption rules out the possibility of point expectations, that is, the situation in which the individual is uncertain about prices, yet "expects" a particular price with certainty, except in the vacuous case $l_2 = 1$. It also implies that no zero price is ever given any positive weight in anyone's expectations. The rationale for this is that they are forecasting future *equilibria* which they know cannot have zero prices since they are insatiable in all commodities themselves.

Strictly speaking, we shall not actually need this assumption in what follows. However, without it, the statement of Theorem 2.1 becomes more complex. (We would have to say that demand is determinate at prices p if and only if $p \gg 0$ and $\bar{p}(p) \in \text{int}_L$ co supp $\psi(p)$, where int_L means the interior of the indicated set relative to the smallest linear manifold L containing it; see below for notation. Similar technical changes would have to be made later in the presentation. On the whole,

the increased generality of dropping (2.3) does not seem to be worth it. The spirit of the generalized conditions, that there must be "sufficient" variability of prices for demand to be determinate, remains the same; see Remark 2.8.

We assume that the preferences of the individual can be represented by a von Neumann-Morgenstern utility function, u, on the space of all consumption streams, R_+^l. Further we assume that

$$u : R_+^l \to R^1$$

satisfies

(2.4a) u is continuous,

(2.4b) u is concave,

(2.4c) u is strictly monotone,

(2.4d) u is bounded

(see Grandmont [5]).

Suppose an individual has taken an action $z = (x^1, b)$ in period 1 and now faces a price system $q \in \Delta^{l_2}$ in period 2. He will choose $x^2 \in R_+^{l_2}$ to solve

$$\max u(x^1, x^2)$$

subject to

$$q \cdot x^2 \leqq q \cdot (\omega^2 + b).$$

In order for a solution to exist to this problem, it is necessary that $q \cdot (\omega^2 + b) \geqq 0$ and that $q \gg 0$. If a solution exists, we shall denote the value of the objective function at the optimum by $\phi(x^1, b, q)$. Given (x^1, b), the function $(x^1, b, \cdot)$ is well-defined, continuous and bounded on the set $\{q \in \Delta^{l_2} | q \gg 0, q \cdot (b + \omega^2) \geqq 0\}$.

The individual then maximizes

$$\int_{\Delta^{l_2}} \phi(x^1, b, q) \psi(p)(dq)$$

subject to $p^1 \cdot x^1 + p^2 \cdot b \leqq p^1 \cdot \omega^1$.

The set of feasible $(x^1, b) \in R^l$ must be such that the objective function is defined at these points. That is, in addition to the budget constraint, we must have

(2.5) $q \cdot (b + \omega^2) \geqq 0$ for all $q \in \operatorname{supp} \psi(p)$,

for if this were violated at some q, then it would be violated in a neighborhood of q that is assigned positive weight by $\psi(p)$. Hence, the above integral would not be well-defined. This is formally analogous to the statement that the optimal consumption bundle must lie in the consumption set in standard consumer theory. It is not an assumption on behavior as such. We denote, for all $x^1 \geqq 0$ and b satisfying (2.5),

$$v(x^1, b, p) = \int_{\Delta^{l_2}} \phi(x^1, b, q) \psi(p)(dq).$$

We call v the *expected utility index*. Let $p \in \Delta^l$, $p = (p^1, p^2)$. If $p^2 \neq 0$, write

$$\tilde{p}(p) = \frac{p^2}{\sum_j p_j^2} \in \Delta^{l_2}.$$

THEOREM 2.1: *A solution to the maximization problem of a trader described as above exists if and only if*

(2.6)   (i) $p \gg 0$,

          (ii) $\tilde{p}(p) \in int\ co\ supp\ \psi(p)$.

Theorem 2.1 is proven in separate parts by Lemmas 2.2–2.6. Lemmas 2.2 and 2.3 prove the sufficiency of (2.6) and Lemmas 2.4–2.6 prove its necessity.

LEMMA 2.2: *Under conditions (2.6) the set of feasible actions for the consumer is compact.*

PROOF: Let $p = (p^1, p^2)$ satisfy (2.6); let $B = \{b \in R^{l_2} | q \cdot (b + \omega^2) \geqq 0$ for all $q \in supp\ \psi(p)\}$; and let $H(x^1) = \{b \in R^{l_2} | p^2 \cdot b \leqq p^1 \cdot \omega^1 - p^1 \cdot x^1\}$. Thus the set of all feasible actions with prices p is

$$A = \{(x^1, b) | x^1 \geqq 0, b \in B \cap H(x^1)\}.$$

Let $x^{1k} \to \bar{x}^1$, $b^k \to \bar{b}$, such that $(x^{1k}, b^k) \in A$ for all k. Clearly, $\bar{x}^1 \geqq 0$ and $\bar{b} \in B$. Also,

$$p^2 \cdot b^k \leqq p^1 \omega^1 - p^1 x^{1k} \quad \text{for all } k.$$

Taking the limit, we have that $\bar{b} \in H(\bar{x}^1)$. Hence A is closed.

Let $H(0) \equiv H$; $H(x^1) \subseteq H$ for all $x^1 \geqq 0$.

First we shall show that $B \cap H$ is bounded. Assume not. Then there exists $\langle b^k \rangle$, $k = 1, \ldots$, such that $b^k \in B \cap H$ and $\|b^k\| \equiv \Sigma_j |b_j^k| \to \infty$. Thus there exists j such that $|b_j^k| \to \infty$. Let $J_0^+ = \{j | \{b_j^k\}$ cannot be bounded above$\}$. $J_0^+ \neq \varnothing$, since otherwise we would have $qb^k \to -\infty$ for any $q \gg 0$, and this would contradict $b^k \in B$, for all k. Let $j_1 \in J_0^+$ and let $\langle k_1 \rangle$ index a subsequence $\langle b_{j_1}^{k_1} \rangle$ that is monotone increasing and unbounded. Let $J_1^+ = \{j | b_j^{k_1}$ cannot be bounded above$\}$. If $J_1^+ \neq \{j_1\}$, then let $j_2 \neq j_1, j_2 \in J_1^+$, and let $\langle k_2 \rangle$ be a subsequence of $\langle k_1 \rangle$ such that $\langle b_{j_2}^{k_2} \rangle$ is monotone increasing and unbounded. Continue in this way until obtaining a set $J_m^+ = \{j_1, \ldots, j_m\}$ and a sequence $\langle k_m \rangle$ such that $\langle b_j^{k_m} \rangle$ is monotone increasing and unbounded for all $j \in J_m^+$ and bounded above for all $j \notin J_m^+$.

Let $J_0^- = \{j | b_k^{k_m}$ cannot be bounded below$\}$, $J_0^- \neq \varnothing$ because otherwise $p \cdot b^{k_m} \to +\infty$, contradicting $b^{k_m} \in H$ for all k_m. Let $j_{m+1} \in J_0^-$ and $\langle k_{m+1} \rangle$ be a subsequence of $\langle k_m \rangle$ such that $\langle b_{j_{m+1}}^{k_{m+1}} \rangle$ is monotone decreasing and unbounded. As above, construct $\langle k_{m+2} \rangle, \ldots, \langle k_{m+m'} \rangle$ and $J_1^-, \ldots, J_{m'}^-$ such that $\{b_j^{k_{m+m'}}\}$ is montone decreasing and unbounded for all $j \in J_{m'}^-$ and bounded below for all $j \notin J_{m'}^-$.

Denote the subsequence $\langle k_{m+m'} \rangle$ by $\langle r \rangle$. Thus if $j \in J_m^+$, $b_j^r \to +\infty$, if $j \in J_{m'}^-$, $b_j^r \to -\infty$, and if $j \notin J_m^+ \cup J_m^-$, b_j^r is bounded. Let $B' = \{b \in R^{l_2} | q \cdot b \geqq -q \cdot \omega_2$ for all $q \in$ co supp $\psi(p)\}$. Clearly $B' \subseteq B$. Take $b \in B$ and $q \in$ co supp $\psi(p)$. Then $q = \Sigma \alpha_j q^j$ for some $\{\alpha_j\} \subset [0, 1]$ with $\Sigma \alpha_j = 1$ and $q^j \in$ supp $\psi(p)$. Thus $q^j \cdot b \geqq -q^j \cdot \omega^2$ for all j. Combining these inequalities, $q \cdot b \geqq -q \cdot \omega^2$ or $b \in B'$. Hence, $B = B'$.

Since $\tilde{p}(p) \in$ int co supp $\psi(p)$, by assumption, there exists $q \in$ co supp $\psi(p)$ such that $q \gg 0$, and $q_j < \tilde{p}_j(p)$ for all $j \in J_m^+$, $q_j = \tilde{p}_j(p)$ for all $j \in J_m^+ \cup J_{m'}^-$, and $q_j > \tilde{p}_j(p)$ for all $j \in J_{m'}^-$.

Hence $\langle (\tilde{p}(p) - q) \cdot b^r \rangle$ is monotone and diverges to $+\infty$. If $p^2 \cdot b^r \leqq p^1 \cdot \omega^1 - p^1 \cdot x^1 \leqq p^1 \cdot \omega^1$ for all r, then $q \cdot b^r \to -\infty$. Thus $b^r \notin B'$ for r sufficiently large, and as $B' = B$ we have a contradiction. If $q \cdot b^r \geqq -q \cdot \omega^2$ for all r, then $\tilde{p}(p) \cdot b^r \to +\infty$, or $p^2 \cdot b^r \to +\infty$, contradicting $b^r \in H$ for all r. Thus $B \cap H$ is bounded.

Let $\{I | I = p^2 \cdot b$ and $b \in B \cap H\}$ be bounded below by $\underline{I}$. If $(x^1, b) \in A$, x^1 must satisfy $p^1 \cdot x^1 \leqq p^1 \cdot \omega^1 - \underline{I}$ and $x^1 \geqq 0$. Hence, A is contained in the product of two bounded sets. Thus A, the set of all feasible actions with prices p, is compact.

$$Q.E.D.$$

LEMMA 2.3 : $v(x^1, b, p)$ *is a continuous function of* (x^1, b), *on the set* A *of feasible actions at prices* p.

PROOF : See Grandmont [5].

LEMMA 2.4 : *If* $p_j^1 = 0$ *for some* j, *then no solution to the individual's maximization problem exists.*

PROOF : Let $(x^1, b) \in A$ be such a solution, and let $\tilde{x}_k^1 = x_k^1$ for $k \neq j$ and $\tilde{x}_j^1 > x_j^1$. By strict monotonicity of u, $\phi(\tilde{x}^1, b, q) > \phi(x^1, b, q)$ for all $q \in$ supp $\psi(p), q \gg 0$. Clearly $(\tilde{x}^1, b) \in A$. Hence, (x^1, b) cannot be optimal.

LEMMA 2.5 : *If* $p^2 \neq 0$ *and* $\tilde{p}(p) \notin$ int co supp $\psi(p)$, *then no solution to the individual's maximization problem exists.*

PROOF : Let $\Psi = \{y \in R_+^{l_2} | y = \alpha q$ for some $q \in$ co supp $\psi(p)$ and $\alpha \geqq 0\}$. Then Ψ is clearly closed and convex. Further, int co supp $\psi(p) \neq \varnothing$ implies int $\Psi \neq \varnothing$, and the hypothesis of this lemma implies $\tilde{p}(p) \notin$ int Ψ. Thus there exists $z \in R^{l_2}$, $z \neq 0$, such that $\tilde{p}(p) \cdot z = 0$ and $y \cdot z \geqq 0$ for all $y \in \Psi$. In particular $q \cdot z \geqq 0$ for all $q \in$ co supp $\psi(p)$.

Suppose that for all $q \in$ supp $\psi(p), q \cdot z = 0$. Clearly $q \cdot z = 0$ for all $q \in$ co supp $\psi(p)$. Take $\bar{q} \in$ int co supp $\psi; \bar{q} \gg 0$. If z is a vector with all components the same, then $\bar{q} \cdot z \neq 0$ follows from $z \neq 0$. Since this would contradict our assumption, let $z_j > z_{j'}$ for some j, j'. We can then find $\tilde{q} \in$ int co supp $\psi(p)$ near $\bar{q}$ such that $\tilde{q}_j > \bar{q}_j$, $\tilde{q}_{j'} < \bar{q}_{j'}$, and $\tilde{q}_k = \bar{q}_k$ for all other k. Thus either $\bar{q} \cdot z \neq 0$ or $\tilde{q} \cdot z \neq 0$, contradicting $q \cdot z = 0$ for all $q \in$ co supp $\psi(p)$.

Thus there exists $q' \in$ supp $\psi(p)$ such that $q' \cdot z \neq 0$. Let $\bar{N}(q')$ be a closed ball with center q' such that $q \in \bar{N}(q')$ implies $q \cdot z \neq 0$. Suppose $\psi(p)(\bar{N}(q')) = 0$; this would

contradict $q' \in \operatorname{supp} \psi(p)$. Let $V = \{q \in \Delta^{l_2} | q \cdot z > 0\}$, and let $\overline{V} = \{q \in V | q \gg 0\}$. Since $\psi(p) (\operatorname{int} \Delta^{l_2}) = 1$, and V contains $\overline{N}(p)$, we have $\psi(p)(\overline{V}) = \psi(p)(V) > 0$.

Suppose now that $(x^1, b) \in A$ is an optimal action, contrary to the conclusion of this lemma. Consider $(x^1, b + z)$. By definition of $\tilde{p}(p)$, $p^2 \cdot (b + z) \leqq p^1 \cdot \omega^1 - p^1 x^1$. By the choice of z, $q \cdot (b + z) \geqq -q \cdot \omega^2$ for all $q \in \operatorname{supp} \psi(p)$. Thus $(x^1, (b + z) \in A$.

It follows from the strict monotonicity of u that $\phi(x^1, b, q) \leqq \phi(x^1, b + z, q)$ for all $q \in \operatorname{supp} \psi(p)$, $q \gg 0$, and that strict inequality holds for $q \in V$. This contradicts the assumption that (x^1, b) is an optimal action. *Q.E.D.*

LEMMA 2.6: *If $p^2 = 0$, then no solution to the individual's maximization problem exists.*

PROOF: Let $z \gg 0$. If $(x^1, b) \in A$ is an optimal solution, consider $(x^1, b + z)$. Clearly $(x^2, b + z) \in A$ since $p^2 = 0$. Also $\phi(x^1, b + z, q) > \phi(x^1, b, q)$ for all $q \in \operatorname{supp} \psi(p)$, $q \gg 0$. Hence (x^1, b) was not optimal in A. *Q.E.D.*

REMARK 2.7: The case in which $p_j^2 = 0$ for some j but not $p^2 = 0$ is covered under the conditions of Lemma 2.5. We therefore did not consider this negation of (2.6(i)) separately.

REMARK 2.8: Assumption (2.3) rules out point expectations. If we were to assume that the individual held such expectations (i.e., $\operatorname{supp} \psi(p)$ is a single point), it is clear that demand would be determinate only for $p \in \Delta$ such that $\{\tilde{p}(p)\} = \operatorname{supp} \psi(p)$. It is clear that equilibrium in this case could exist only if, for some $p \in \Delta$, every individual held the same beliefs—a most unlikely instance.

We find this conclusion heartening since point expectations mean that the individual is "sure," and this seems contrary to the very spirit of the uncertainty question. In the case considered, in which the uncertainty is genuine, equilibrium will be shown to exist for a much wider class of environments.

3. DEMAND CORRESPONDENCES

We shall now study the dependence of the set of optimal actions on p. Our analysis is for the representative individual and all assumptions made are assumed to apply to all individuals in the economy.

Let $\mathscr{C}(\Delta^{l_2})$ be the class of compact subsets of Δ^{l_2}. Define

$$\sigma: \Delta^l \to \mathscr{C}(\Delta^{l_2})$$

by

$$\sigma(p) = \operatorname{supp} \psi(p).$$

The following two assumptions are made on ψ and σ:

(3.1) ψ is continuous in the weak topology.

(3.2) σ is an upper hemi-continuous correspondence.

REMARK 3.1: Assumption (3.1) implies that σ is lower hemi-continuous as follows: Take $\bar{q} \in$ supp $\psi(\bar{p})$. Then if G is a neighborhood of $\bar{q}$, $\psi(\bar{p})(G) > 0$, since otherwise $\Delta^{l_2} \backslash G$ would contain supp $\psi(\bar{p})$. Let $p^k \to \bar{p}$. If σ were not lower hemi-continuous, then there would exist an open neighborhood of $\bar{q}$, $\bar{G}$, such that

$$\bar{G} \cap \text{supp } \psi(p^k) = \varnothing$$

for infinitely many k. Hence, $\psi(p^k)(\bar{G}) = 0$ for infinitely many k. But weak continuity of ψ is equivalent to $\lim_k \inf \psi(p^k)(G) \geqq \psi(\bar{p})(G)$ for all open sets G (see Parthasarathy [9]). Thus $\psi(\bar{p})(\bar{G}) = 0$. This is a contradiction.

Hence, the combination of (3.1) and (3.2) implies

(3.3) σ is a continuous correspondence.

Denote

$$B(p) = \{b \in R^{l_2} | q \cdot (b + \omega^2) \geqq 0 \quad \text{for all } q \in \text{supp } \psi(p)\},$$

$$H(x^1, p) = \{b \in R^{l_2} | p^2 \cdot b \leqq p^1 \cdot \omega^1 - p^1 \cdot x^1\}, \text{ and}$$

$$A(p) = \{(x^1, b) \in R^l | x^1 \geqq 0, b \in B(p) \cap H(x^1, p)\}.$$

These sets correspond to B, $H(x^1)$, and A of the last section, except that we now consider p to be a variable.

Denote

$$S = \{p \in \Delta^l | p \gg 0, \tilde{p}(p) \in \text{int co supp } \psi(p)\}.$$

The result of the last section is that the individual's maximization problem has a solution if and only if $p \in S$.

We now assume

(3.4) S is convex.

REMARK 3.2: This assumption is, of course, a condition on $\psi(\cdot)$. If supp $\psi(\cdot)$ is a constant, it is trivially satisfied. The only role of this assumption is in the proof of Lemma 4.7. Although it may be weakened considerably without affecting 4.7, it may not be done away with altogether; see Remark 4.8 and Example 5.1.

LEMMA 3.3: (i) *The correspondence* $A(\cdot)$ *is convex-valued, lower hemi-continuous on* Δ^l, *and has a closed graph*; (ii) $A(\cdot)$ *is compact-valued and upper hemi-continuous on* S.

PROOF: That $A(p)$ is non-empty and convex for all p in Δ^l is obvious. The fact that $A(\cdot)$ has a closed graph follows directly from the lower hemi-continuity of σ; see assumption (3.1) and Remark 3.1. It remains to be shown that:

(i) $A(\cdot)$ is lower hemi-continuous on Δ^l. Let $p \in \Delta^l$ and $z \in A(\bar{p})$. Let $\{p^k\} \in \Delta^l$ be a sequence tending to $\bar{p}$. We wish to find a sequence $\{z^k\} \in A(p^k)$ which tends to z. For any $b \in R^{l_2}$, let $f(b)$ be the minimum of $q \cdot (b + \omega^2)$ when $q \in \Delta^{l_2}$. Then f is a

continuous function of b and $f(0) > 0$ since $\omega^2 \gg 0$. Therefore, it is possible to find $\bar{b} \ll 0$ such that $f(\bar{b}) > 0$. Let $\bar{z} = (0, \bar{b})$; choose $0 \leqslant \lambda < 1$, and define $\tilde{z} = (\tilde{x}^1, \tilde{b}) = \lambda z + (1 - \lambda)\bar{z}$. We claim that $\tilde{z} \in A(\bar{p})$ and $\tilde{z} \in A(p^k)$ for k large enough. First, $\bar{p}^2 \cdot \bar{b} < \bar{p}^1 \cdot \omega^1$, since either $\bar{p}^1 = 0$, in which case $\bar{p}^2 \cdot \bar{b} < 0$, or $\bar{p}^1 \neq 0$, in which case $\bar{p}^2 \cdot \bar{b} \leqslant 0 < p^1 \cdot \omega^1$. It follows that

$$\bar{p}^1 \cdot \tilde{x}^1 + \bar{p}^2 \cdot \tilde{b} < \bar{p}^1 \cdot \omega^1.$$

On the other hand, for every $q \in \text{supp } \psi(\bar{p})$,

$$q \cdot (\tilde{b} + \omega^2) \geqslant (1 - \lambda)q \cdot (\bar{b} + \omega^2) \geqslant (1 - \lambda)f(\bar{b}) > 0.$$

Therefore, $\tilde{z} \in A(\bar{p})$. For k large enough, one has

$$p^{1k} \cdot \tilde{x}^1 + p^{2k} \cdot \tilde{b} < p^{1k} \cdot \omega^1.$$

Given $\tilde{b}$, and an arbitrary $p \in \Delta^l$, let $g(p)$ be the minimum of $q \cdot (\tilde{b} + \omega^2)$ when $q \in \text{supp } \psi(p)$. Since $\text{supp } \psi(\cdot)$ is a continuous correspondence, $g(\cdot)$ is continuous on Δ^l. We have $g(\bar{p}) \geqslant (1 - \lambda)f(\bar{b}) > 0$. Therefore, for k large enough,

$$q \cdot (\tilde{b} + \omega^2) \geqslant g(p^k) > 0 \quad \text{for all } q \in \text{supp } \psi(p^k).$$

This shows that $\tilde{z} \in A(p^k)$ for k large enough.

As this is true for any $\lambda < 1$, one can construct a sequence $z^k \in A(p^k)$ which tends to z, as follows. Let λ^n be a sequence $0 \leqslant \lambda^n < 1$ which tends to 1. For any $n \geqslant 1$, let $\bar{z}^n = \lambda^n z + (1 - \lambda^n)\bar{z}$. Consider $k_1 = \min\{\bar{k}|\bar{z}^1 \in A(p^k), \text{ all } k \geqslant \bar{k}\}$, and let $z^k = (0, 0)$ for $k \leqslant k_1$. Then consider $k_2 = \min\{\bar{k}|\bar{k} \geqslant k_1 + 1, \bar{z}^2 \in A(p^k) \text{ for all } k \geqslant \bar{k}\}$, and let $z^k = \bar{z}^1$ for $k_1 \leqslant k < k_2$. Proceed by induction: $k_n = \min\{\bar{k}|\bar{k} \geqslant k_{n-1} + 1, \bar{z}^n \in A(p^k) \text{ for all } k \geqslant \bar{k}\}$, and $z^k = \bar{z}^{n-1}$ for $k_{n-1} \leqslant k < k_n$. The sequence z^k satisfies all requirements. This shows (i) of the lemma.

(ii) $A(\cdot)$ is compact-valued and upper hemi-continuous on S. Let $p^k \in S$ converging to $\bar{p} \in S$. Consider $z^k = (x^k, b^k)$ in $A(p^k)$. It suffices to show that the sequence $\{z^k\}$ is bounded. As $p^{1k} \cdot x^k + p^{2k} \cdot b^k \leqslant p^{1k} \cdot \omega^1$, it is sufficient to show that the sequence $\{b\}^k$ is bounded. Assume that this is not true. One could find a subsequence (retain the same notation) such that $\|b^k\|$ diverges to $+\infty$. One can then proceed as in the proof of Lemma 2.2 to find J^+, J^-, and a subsequence (same notation) such that $b_j^k \to +\infty$ for all $j \in J^+$, $b_j^k \to -\infty$ for all $j \in J^-$, and b_j^k is bounded for $j \notin J^+ \cup J^-$. Certainly, $J^+ \neq \varnothing$, for $\tilde{p}(p^k) \cdot b^k \geqslant -\tilde{p}(p^k) \cdot \omega^2$ for all k, and $\tilde{p}(p^k)$ tends to $\tilde{p}(\bar{p}) \gg 0$. On the other hand, $J^- \neq \varnothing$, since $\tilde{p}(p^k) \cdot b^k \leqslant (p^{1k}/\Sigma_j p_j^{2k}) \cdot \omega^1$ for all k. Since $\tilde{p}(\bar{p}) \in \text{int co supp } \psi(\bar{p})$, there exists $\bar{q}$ in co supp $\psi(\bar{p})$ such that $\bar{q}_j < \tilde{p}_j(\bar{p})$ for all $j \in J^+$, $\bar{q}_j > \tilde{p}_j(\bar{p})$ for all $j \in J^-$, and $\bar{q}_j = \tilde{p}_j(\bar{p})$ for all $j \notin J^+ \cup J^-$.

Since co supp $\psi(\cdot)$ is lower hemi-continuous at $\bar{p}$, there exists a sequence $\{q^k\} \in$ co supp $\psi(p^k)$ such that $q^k \to \bar{q}$. Then $(\tilde{p}(p^k) - q^k) \cdot b^k$ diverges to $+\infty$. However, this is impossible, since, for all k:

$$(\tilde{p}(p^k) - q^k) \cdot b^k \leqslant \left(p^{1k}\Big/\sum_j p_j^{2k}\right) \cdot \omega^1 + q^k \cdot \omega^2. \qquad Q.E.D.$$

LEMMA 3.4: (i) If (x^{1k}, b^k, p^k) be such that $(x^{1k}, b^k) \in A(p^k)$ and $p^k \in \Delta^l$ for all k, then (x^{1k}, b^k, p^k) converging to (x^1, b, p) implies $\lim v(x^{1k}, b^k, p^k) = v(x^1, b, p)$. (ii) $v(x^1, b, p)$ is concave in (x^1, b) for all $p \in S$. (iii) $v(x^1, b, p)$ is strictly monotone in (x^1, b) for all $p \in S$.

PROOF: These properties can be shown using the methods of Grandmont [6, Section 3.7, Propositions 1, 2, 3] or Sondermann [11, Lemmas 7.1, 7.2]. We shall not reproduce the proof here. Weak continuity of ψ is necessary for (i), (ii), and (iii) follow from the corresponding properties of u.

The *demand correspondence* is defined for each $p \in S$ as

$$\xi(p) = \{(x^1, b) \in A(p) | v(x^1, b, p) \geqq v(x^{1'}, b', p) \quad \text{for all } (x^{1'}, b') \in A(p)\}.$$

To shorten the notation slightly we shall write the generic element of $A(p)$ as $z = (x^1, b) \in R^l$.

THEOREM 3.5: Let $p^k \in S$, $p^k \to \bar{p}$, and $\bar{z}^k \in \xi(p^k)$, $\bar{z}^k \to \bar{z} = (\bar{x}^1, \bar{b})$. Then $\bar{z} \in A(\bar{p})$ and $v(\bar{x}^1, \bar{b}, \bar{p}) \geqq v(x^1, b, \bar{p})$ for all $(x^1, b) \in A(\bar{p})$.

PROOF: Since $A(\cdot)$ has a closed graph, $\bar{z} \in A(\bar{p})$. Let $z \in A(\bar{p})$. Since $A(\cdot)$ is lower hemi-continuous (see Lemma 3.3), there is a sequence $z^k \in A(p^k)$, $z^k \to z$. Now for all k, $v(\bar{z}^k, p^k) \geqq v(z^k, p^k)$. By (i) of Lemma 3.4, $v(\bar{z}, \bar{p}) \geqq v(z, \bar{p})$.   Q.E.D.

THEOREM 3.6: (i) $\xi(\cdot)$ is convex and compact-valued and upper hemi-continuous on S. (ii) $p \cdot \xi(p) = p^1 \cdot \omega^1$ for all $p \in S$.

PROOF: The first two parts of (i) follow from the fact that $A(\cdot)$ is convex and compact-valued on S and that $v(\cdot, \cdot, \cdot)$ is continuous and concave in its first two arguments. Since ξ has a closed graph, it is sufficient to show that for any sequences $p^k \in S$ $p^k \to \bar{p} \in S$, $z^k \in \xi(p^k)$, then $\{z^k\}$ is bounded. This follows from the upper hemi-continuity and compact valuedness of $A(\cdot)$ on S.

Part (ii) follows from strict monotonicity of $v(\cdot, \cdot, \cdot)$ in its first two arguments. A proof can be constructed paralleling that of Grandmont [6, Section 3.8, Proposition 1].

LEMMA 3.7: Let A be convex and $N_\varepsilon(a) \subset N_{\varepsilon/2}(A)$ for some a. Then $N_{\varepsilon/2}(a) \subset A$.

PROOF: If not, then there exists x such that $x \in N_{\varepsilon/2}(a)$ and $x \notin A$. Since A is convex, we can find $z \neq 0$ such that $z \cdot x = 0$ and $z \cdot a' < 0$ for all $a' \in A$. Further z can be chosen such that $\Sigma |z_i| = 1$. Since $x \in N_{\varepsilon/2}(a)$, $N_{\varepsilon/2}(x) \subset N_\varepsilon(a)$ and, therefore, by the hypothesis of the lemma, $N_{\varepsilon/2}(x) \subset N_{\varepsilon/2}(A)$. Let $y \in N_{\varepsilon/2}(x)$ such that $z \cdot y = \varepsilon/2$. But for all $\tilde{a} \in N_{\varepsilon/2}(A)$, $z \cdot \tilde{a} < \varepsilon/2$. Hence $y \notin N_{\varepsilon/2}(A)$, contradicting $N_{\varepsilon/2}(x) \subset N_{\varepsilon/2}(A)$.   Q.E.D.

Denote $\bar{\sigma}(p) \equiv \text{co } \sigma(p) \equiv \text{co supp } \psi(p)$.

LEMMA 3.8: S is open in Δ^l.

PROOF: Since $S = \{p \in \Delta^l | p \gg 0\} \cap \{p \in \Delta^l | \tilde{p}(p) \in \text{int } \bar{\sigma}(p)\}$, it suffices to show that the latter set is open. Let $p^* \in \Delta^l$ be such that $\tilde{p}(p^*) \in \text{int } \bar{\sigma}(p)$. Then there exists $\varepsilon > 0$ such that $N_\varepsilon(\tilde{p}(p^*)) \subset \text{int } \bar{\sigma}(p^*) \subset \bar{\sigma}(p^*)$. Since $\tilde{p}(\cdot)$ is continuous, there exists $\delta' > 0$ such that $|p - p^*| < \delta'$ implies $\tilde{p}(p) \in N_{\varepsilon/2}(\tilde{p}(p^*))$. By the triangle inequality we have that $N_{\varepsilon/2}(\tilde{p}(p)) \subset N_\varepsilon(\tilde{p}(p^*))$ and hence

$$N_{\varepsilon/2}(\tilde{p}(p)) \subset \bar{\sigma}(p^*).$$

Since $\sigma(\cdot)$ is lower hemi-continuous, so is $\bar{\sigma}(\cdot)$. Hence, there exists $\delta'' > 0$ such that

$$\bar{\sigma}(p^*) \subset N_{\varepsilon/2}(\bar{\sigma}(p))$$

for $|p - p^*| < \delta''$. Combining the two equations above, $N_{\varepsilon/2}(\tilde{p}(p)) \subset N_{\varepsilon/4}(\bar{\sigma}(p))$ for p such that $|p - p^*| < \min(\delta', \delta'') \equiv \delta$. Since $\bar{\sigma}(p)$ is convex we apply the last lemma, obtaining $N_{\varepsilon/4}(\tilde{p}(p)) \subset \bar{\sigma}(p)$ for $|p - p^*| < \delta$. Thus there is a neighborhood of p^* in $\{p \in \Delta^l | \tilde{p}(p) \in \text{int } \bar{\sigma}(p)\}$.   Q.E.D.

4. EQUILIBRIUM

The previous sections have studied the theory of individual behavior when the market is known to meet again in the future. We now study the question of existence of an equilibrium for an economy with a finite number, I, of individuals, all of whom behave as above. We denote the index set of all individuals $\mathscr{I} = \{1, \ldots, I\}$ and the generic element of $\mathscr{I}$ by i. Any entity used in the previous sections will now be written with a presuperscript i when it refers to the ith individual. For example, ^iS is the set of prices on which i's demand correspondence, $^i\xi$, is non-empty valued.

Let

$$P = \bigcap_{\mathscr{I}} {}^iS,$$

where P is the set of prices on which everyone's excess demand correspondence is well-defined. We write the *aggregate excess demand correspondence*

$$\zeta: P \to R^l,$$

defined as

$$\zeta(p) = \sum_{\mathscr{I}} {}^i\xi(p) - \left(\sum_{\mathscr{I}} {}^i\omega^1, 0\right).$$

Note that the aggregate excess demand for a currently deliverable commodity j is $\sum {}^i\xi_j(p) - \sum {}^i\omega_j^1$ which corresponds to the usual notion of demand minus supply; and the aggregate excess demand for a futures contract, j, is $\sum {}^i\xi_j(p)$, which is the sum of offers to buy such contracts minus offers to sell them.

An *equilibrium* is an $I + 1$ tuple in $\Delta^l \times R^{Il}$, $(p^*, {}^1z^*, \ldots, {}^Iz^*)$ such that $^iz^* \in {}^i\xi(p^*)$ for all i and

$$\sum_{\mathscr{I}} {}^iz^* = \left(\sum_{\mathscr{I}} {}^i\omega^1, 0\right).$$

In order to have some hope of finding an equilibrium, it is necessary to assume

(4.1) P is non-empty.

That is, there must be some price at which all excess demand correspondences have a non-empty value. Thus (4.1) is a condition on the compatibility of the individual's expectations.

REMARK 4.1: Since ^iS is open and convex for each i, by Lemma 3.8 and assumption (3.4), it follows that P is open and convex, since it is the intersection of a finite number of open, convex sets.

One can construct examples, and we shall present one such in Section 5, such that equilibrium does not exist even under all the assumptions presented thus far. It is therefore necessary to postulate the following:

(4.2)   There exists $C \subseteq \varDelta^{l_2}$ such that (i) C has non-empty interior in $\varDelta^{l_2}$; (ii) for
             all $p \in \varDelta^l$ and all $i \in \mathscr{I}$, $C \subseteq$ int co supp $^i\psi(p)$.

It is clear that for each $p \in \varDelta^l$ one can find an open set in int co supp $^i\psi(p)$ for all i. Thus the force of assumption (4.2) is to assert that one can choose the same such set for all $p \in \varDelta^l$. We shall say that expectations are *common on C* if (4.2) holds. Intuitively, it means that some futures prices are always given positive weight, irrespective of the current price vector. A further discussion of (4.2) will follow an example of Section 5, in which it fails to hold, all of our other assumptions hold, and there is no equilibrium.

REMARK 4.2: Assumption (4.2) is, of course, a joint condition on all of the $^i\psi(\cdot)$. If each of the supp $^i\psi(\cdot)$ were constant, it would be trivially satisfied. It is interesting to note that (3.4) is also implied by the same condition. One can easily observe that (3.4) and (4.2) are independent, yet both are implied by the same assumption on supp $^i\psi(\cdot)$. This assumption is far too strong, however. We therefore assume (3.4) and (4.2) separately. One can easily verify that there is a wide class of cases in which supp $^i\psi(\cdot)$ are not constant and yet in which (3.4) and (4.2) are both satisfied.

Assumption (4.2) plays two distinct roles in the proof. It is first used to conclude unboundedness of the aggregate excess demand correspondence, as prices go to the boundary of P, from unboundedness of the individual demand correspondences; see Theorem 4.4, Remark 4.5, and Example 4.6.

The second use is to apply a compactness argument to obtain convergence of a subsequence of fixed points in the existence theorem 4.13. Example 5.2 shows how the proof breaks down, and in fact equilibrium fails to exist, in the absence of this assumption.

We shall write the closure of P in $\varDelta^l$ as $\bar{P}$.

LEMMA 4.3: *Let* $\{p^k\} \in P$, $p^k \to \bar{p}$, *and* $\bar{p} \in \bar{P} \setminus P$. *Let* $^iz^k \in {}^i\xi(p^k)$ *for all i. Then for some i,* $\|^iz^k\| \to \infty$.

PROOF: If the lemma were false, $\{\|^iz^k\|\}_k$ would lie in a compact set for every i. Hence, a subsequence would converge to $^i\bar{z} = ({}^i\bar{x}, {}^{1i}\bar{b})$ for every i. Now for some i, $\bar{p} \notin {}^iS$. By (3.5), $^iv({}^i\bar{z}, \bar{p}) \geqq {}^iv(z, \bar{p})$ for all $z \in {}^iA(\bar{p})$. This would contradict Theorem 2.1.   *Q.E.D.*

THEOREM 4.4: *Let* $\{p^k\} \subset P$, $p^k \to \bar{p}$, *and* $\bar{p} \in \bar{P} \setminus P$. *Then* $\|\zeta(p^k)\| \to \infty$.

REMARK 4.5: In the theory of general equilibrium with a single market date, this theorem follows immediately from the above lemma since the commodity space is bounded below. In our case, the theorem is proven using the assumption of common expectations (4.2). The following example shows why one may not be able to conclude the theorem from the above lemma in the absence of such an assumption.

EXAMPLE 4.6: Let $l_2 = 2$, and let $I = 2$. Suppose the supports of their expectations are given by

$$\text{supp } {}^1\psi(p) = \{q \in \varDelta^2 | q_1 \geqq \tfrac{1}{2}\},$$

$$\text{supp } {}^2\psi(p) = \{q \in \varDelta^2 | q_1 \leqq \tfrac{5}{4}\tilde{p}_1(p) - \tfrac{1}{8}\},$$

for all p. Consider a sequence $p^k \to \bar{p}$ such that $\tilde{p}_1(p^k) > \tfrac{1}{2}$ for all k and $\tilde{p}_1(\bar{p}) = \tfrac{1}{2}$. This means that, in the limit, individual 1 is sure (almost sure) that the relative price of future deliverable commodity 1 will be higher at the next market date than at the current market date. He will thus buy futures for this commodity and finance their purchase with sales of the other future. That is $^1b_1^k \to +\infty$ and $^1b_2^k \to -\infty$. But individual 2 believes the opposite. He thinks, in the limit, that long positions in commodity 2 are sure to make unbounded arbitrage profit. Hence $^2b_1^k \to -\infty$ and $^2b_2^k \to +\infty$. Thus although $\|^i\xi(p^k)\| \to \infty$ for all i, their speculations might "cancel out," so that we cannot conclude $\|\zeta(p^k)\| \to \infty$. But this example fails to satisfy the assumption that expectations are common on a set with non-empty interior. The only set in $\bigcap_{i=1,2} \text{supp } {}^i\psi(p^k)$ for all k is $\{\tfrac{1}{2}\}$.

We now proceed to a proof of Theorem 4.4.

PROOF OF THEOREM 4.4: Assume that the theorem is false. Then one can find a subsequence (keep the same notation) $z^k = (x^k, b^k) \in \zeta(p^k)$ which converges to $\bar{z}$. We shall derive a contradiction by showing that, for this subsequence, $\|b^k\| \to \infty$. Choose $^iz^k = ({}^ix^{1k}, {}^ib^k) \in \xi(p^k)$ for every i, such that $\Sigma_i({}^iz^k - ({}^i\omega^1, 0)) = z^k$. For each i, the sequence $\{^ix^{1k}\}_k$ is bounded. This implies, in view of the last lemma, that $\|^ib^k\| \to \infty$ for some i.

Let C be such that int $C \neq \varnothing$ and $C \subseteq \text{co supp } {}^i\psi(p^k)$ for all i and k. The existence of such a C follows from assumption (4.2). Let $Y = \{b \in R^{l_2} | q \cdot (b + \omega^2) \geqq 0$ for

all $q \in C$}. Since $C \subseteq$ co supp ${}^i\psi(p^k)$, we have $Y \supseteq {}^iB(p^k)$ for all i and k. Let $\tilde{q} \in$ int C and suppose $\tilde{q} \cdot (\tilde{b} + \omega^2) = 0$ for some $\tilde{b} \neq -\omega^2$, $\tilde{b} \in Y$. Then we can find a q' in C such that $q' \cdot (\tilde{b} + \omega^2) < 0$ since $\tilde{q} \in$ int C. Hence, for all $b \in Y$, $b \neq -\omega^2$, we have that $\tilde{q} \cdot (b + \omega^2) > 0$. Further $\tilde{q} \in$ int C implies $\tilde{q} \gg 0$. Hence, $\tilde{q} \cdot ({}^ib^k + \omega^2) > 0$ for all ${}^ib^k \in {}^iB(p^k)$ such that ${}^ib^k \neq -\omega^2$. Since $\|{}^ib^k\| \to \infty$ for some i by the last lemma, and $\tilde{q} \gg 0$, $\tilde{q} \cdot ({}^ib^k + {}^i\omega^2) \to \infty$ for this i. Hence, $\tilde{q} \cdot (\Sigma_i ({}^ib^k + {}^i\omega^2)) \to \infty$. Therefore for some j, $\Sigma_i {}^ib_j^k \to \infty$.   Q.E.D.

LEMMA 4.7: *There exists a non-decreasing sequence of sets $\langle P^n \rangle$, $n = 1, \ldots$, such that $P^n \subset P$, $P = \bigcup_{n=1}^{\infty} P^n$ and each of the P^n is compact, convex, and has non-empty interior.*

PROOF: Since P is open, $P = \bigcup_{k=1}^{\infty} K^k$ where K^k is closed for all k. Let $P^n = $ co $\bigcup_{k=1}^{n} K^k$. Since P is convex and $\bigcup_{k=1}^{n} K^k \subseteq P$, $P^n \subseteq P$, for all n. Since co $\bigcup_{k=1}^{n} K^k \supseteq \bigcup_{k=1}^{n} K^k$, $P \subseteq \bigcup_{n=1}^{\infty} P^n$. Clearly P^1 can be chosen to have non-empty interior. Therefore, $\langle P^n \rangle$ has all desired properties.   Q.E.D.

REMARK 4.8: Convexity of each P^n is far more than we shall need in the existence proof. It suffices that each P^n has the fixed point property. For this purpose it would suffice that P be homeomorphic to the interior of the simplex. However, we know of no economically meaningful assumption other than the convexity of each ${}^iS$ that will insure this. In Example 5.1 we show that without convexity of ${}^iS$ one may generate sets P that are not homeomorphic to the interior of the simplex and such that there is no approximating sequence of sets each having the fixed point property. This approximating sequence is necessary for an existence proof along the lines of 4.11, or indeed any other proof involving fixed point theorems of a more general nature than Kakutani's.

LEMMA 4.9: *Let P^n be one of the sets whose existence is asserted in the last lemma. Let $\zeta^n(\cdot)$ be $\zeta(\cdot)$ restricted to the domain P^n. Then the range of $\zeta^n(\cdot)$ is bounded.*

PROOF: This follows directly from Theorem 3.6 (i) and the proof is therefore omitted.

COROLLARY 4.10: *ζ^n is upper hemi-continuous on P^n since the range of ζ^n is contained in a compact set.*

THEOREM 4.11: *The economy has an equilibrium.*

PROOF: Let us write $z = (x^1 - \omega, b) \in \zeta(p)$ where $(x^1, b) \in \Sigma^i \xi(p)$ and $\omega = \Sigma^i \omega$. By Lemma 4.9, we can choose $\bar{Z}^n$ to be a compact, convex set containing the range of ζ^n for each n. By this choice of $(P^n, \bar{Z}^n)$, Theorem 3.6, and Corollary 4.10, the conditions necessary for an application of Debreu's result [1] are satisfied. Thus there exists for each n, (p^n, z^n) such that $z^n \in \zeta^n(p^n)$, $p^n \cdot z^n = 0$, and $p \cdot z^n \leqslant 0$ for all $p \in P^n$. Since $p^n \in \Delta^l$, there exists a convergent subsequence (retain the index n)

such that $p^n \to p^*$. Suppose $p^* \in \bar{P} \backslash P$. By Theorem 4.4, $\|z^n\| \to \infty$. We shall derive a contradiction by showing that $\langle z^n \rangle$ is bounded.

First it will be shown that b^n is bounded. Let C be the set whose existence is asserted in assumption (4.2). Let $\bar{p}^2 \in \operatorname{int} C$, which is non-empty by this assumption, and let $\bar{p} = (0, \bar{p}^2) \in \Delta^l$.

Hence, $\bar{p} \in {}^i\tilde{S} = \{p \in \Delta^l | \tilde{p}(p) \in \operatorname{int} \operatorname{co} \operatorname{supp} {}^i\psi(p)\}$ for all i. Since ${}^i\tilde{S}$ is open for all i (see the proof of Lemma 3.8), there exists $\bar{\bar{p}} \gg 0$ such that $\tilde{p}(\bar{\bar{p}}) = \bar{p}^2$ and $\bar{\bar{p}} \in {}^iS$ for all i. Then, since $\bar{\bar{p}} \in P^n$ for sufficiently large n, we have (i) $q \cdot (b^n + \omega^2) \geqq 0$ for all $q \in C$; and (ii) $\bar{p}^2 \cdot b^n \leqq \bar{\bar{p}}^1 \cdot \omega^1 + \bar{p}^2 \cdot \omega^2 \leqq \Sigma_j \omega_j$. Combining (i) and (ii) and using $\bar{\bar{p}} \gg 0$, we have shown that $\langle b^n \rangle$ is bounded.

To show that $\langle x^{1n} \rangle$ is bounded, choose $p \gg 0$ in P^1; then $p^1 \cdot x^{1n} \leqq p^1 \cdot \omega^1 - p^2 \cdot b^n$ for all n, $\langle b^n \rangle$ bounded, and $x^{1n} \geqq 0$.

Therefore, $\langle z^n \rangle$ is bounded, and this contradiction establishes $p^* \in P$.

By Remark 4.1 (a consequence of Lemma 3.8), P is open. Since ζ has closed graph and z^n remain bounded, there exists a further subsequence converging to $z^* \in \zeta(p^*)$. It remains to be shown only that $z^* = 0$.

Since $p^n \cdot z^n = 0$ for all n, $p^* \cdot z^* = 0$. Suppose $z_j^* > 0$ for some j. Then if $z^* > 0$, $p^* \cdot z^* > 0$ because $p^* \in P$ implies $p^* \gg 0$. Similarly $z^* < 0$, is not possible. If $z_j^* > 0$ for $j \in J^+ \neq \varnothing$, and $z_j^* < 0$ for $j \in J^- \neq \varnothing$, then since P is open, we can find $p \in P$ such that $p \cdot z^* > 0$. But $p \in P^n$ for some n; hence, $p \cdot z^n > 0$ for large n, contradicting $p \cdot z^n \leqq 0$ for all $p \in P^n$. Thus $z^* = 0$.   Q.E.D.

5. TWO EXAMPLES

EXAMPLE 5.1: This example is constructed to show that, without assumption (3.4), the demand correspondence may be non-empty-valued only on a set, S, which cannot be approximated in the sense of Lemma 4.7 by sets with the fixed-point property.
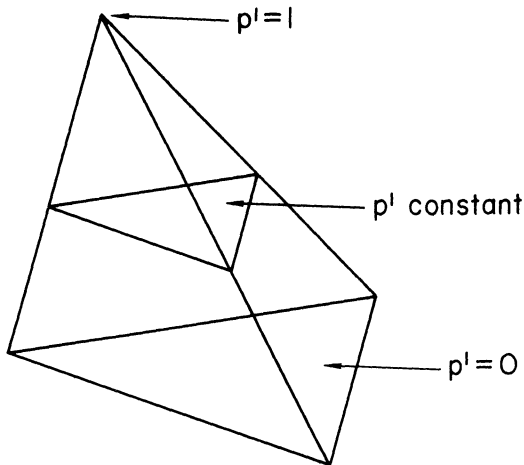

FIGURE 1.

Let $l_1 = 1$ and $l_2 = 3$. Then a current price vector may be thought of as a point in the unit tetrahedron (with barycentric coordinates). Let it be oriented, for the purposes of our discussion, such that the vertical axis corresponds to the currently deliverable good.

We may think of the base of the tetrahedron, $\{p \in \Delta^4 | p^1 = 0\}$, as also representing Δ^3, the space of prices in period 2.

The example begins by constructing a correspondence between points of Δ^3 and (barycentric) spheres in Δ^3 as follows: Consider Δ^3 and two symmetric subsets, α and β, as shown in Figure 2. The point in the center is (1/3, 1/3, 1/3). Let $\varepsilon > 0$ be fixed.
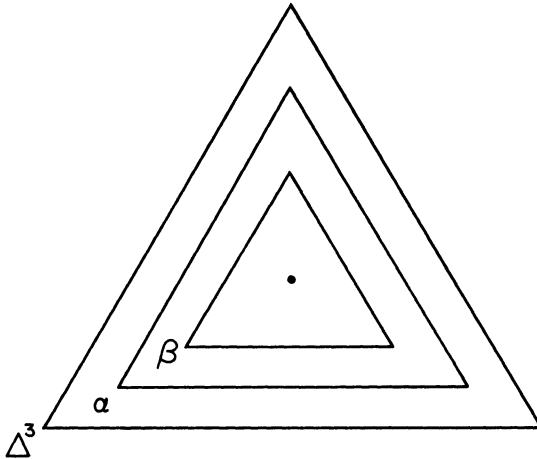

FIGURE 2.

Consider any point, q, between α and the boundary of Δ^3. Associate to this point an ε-sphere such that q and it are geometrically similar to q and Δ^3. Clearly all q not on the boundary of Δ^3 are in the interior of their associated neighborhoods.

For q between α and β construct an image sphere as follows: Let the line from the center point through q cut α and β at a and b, respectively. Let the neighborhood associated with a have "top corner" (say, where q_2 is maximized), c. Let d be the point between a and c defined by

$$\frac{\overline{ad}}{\overline{dc}} = \frac{\overline{aq}}{\overline{qb}},$$

where a line over two letters means the length of the line segment between them.

Let the ε-sphere associated to q be such that q and it are geometrically similar to d and the sphere associated with a. Since d is, by construction, in the interior of this sphere, q will be in the interior of its sphere for all q outside β. It is easy to see that for all q on the boundary of β, q is the "top corner" of its associated sphere, and hence not in its interior. For all q in the interior of β, let this situation persist.

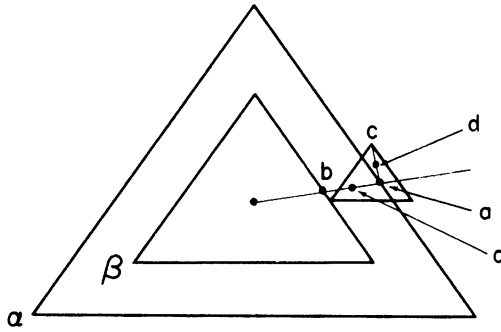Denote the correspondence thus defined from Δ^3 to itself by Q. It is clear that Q is continuous.

FIGURE 3.

Now consider a point in the tetrahedron, p. Consider the projection of this point in the vertical direction (recall the orientation of Δ^4) into Δ^3. Denote this mapping by $\Pi : \Delta^4 \to \Delta^3$. Note that this is *not* the projection $\tilde{p}$.

The correspondence from Δ^4 to spheres in Δ^3, which is to be thought of as co supp $\psi(p)$, is defined by $Q(\Pi(p))$. $Q(\Pi(\cdot))$ satisfies all of our assumptions on co supp $\psi(p)$ except (3.4).

We will now show that this mapping gives rise to a set S that is pathological in the sense that it cannot be approximated by sets with the fixed point property. For $p^1 > 0$ but small, the part of the cross-section of the tetrahedron at p^1 fixed that is in S is an annular region close to that between β and the boundary of Δ^3. But the line from $(1, 0, 0, 0)$ to $(0, 1/3, 1/3, 1/3)$ has no points in S, since on this line $\Pi(p) = \tilde{p}(p) = (1/3, 1/3, 1/3)$ and $\Pi(p) \notin \text{int } Q(\Pi(p))$; it is the "top corner" of $Q(\Pi(p))$. Hence, S has a torus-like shape, at least near its base.

As yet there are no known properties of the correspondence described above that can account for the pathological behavior of S. Thus we have chosen to adopt (3.4) which is, however, stronger than one actually needs. Nevertheless (3.4) is implied by "supp $\psi(p)$ is constant" and also by "$\tilde{p}(p) \in \text{int co supp } \psi(p)$ for all p." However, neither of these is necessary for (3.4); and no assumption sufficient for the approximation of P but weaker than (3.4) with any intuitive economic meaning is known.

EXAMPLE 5.2: This example is to demonstrate that there may be no equilibria in the absence of assumption (4.2). Suppose there are three commodities, two in the future and one at present. To ease the computations (though they will still be too long to present fully herein), let us choose a different normalization of p and q. If $p = (p_1^1, p_1^2, p_2^2)$ and $q = (q_1, q_2)$, let $p_1^1 \equiv 1$ and $q_1 \equiv 1$.

Since $p_1^1 = 0$ is incompatible with determinate demand correspondences, this normalization involves only the assumption that $q_1 = 0$ is never given any positive probability.

Suppose there is only one individual and his utility function is

$$u(x) = \sum_j \log x_j$$

(log is taken to the base e).

In period 2, having already consumed x_1 and having bought contracts b_1 and b_2, the problem faced will be

$$\max \log x_1^2 + \log x_2^2,$$

subject to $x_1^2 + q_2 x_2^2 \lesseqgtr b_1 + \omega_1^2 + q_2(b_2 + \omega_2^2)$.

Let the right hand side of the constraint be $W \geqslant 0$. Then

$$x_1^2 = \frac{W}{2},$$

$$x_2^2 = \frac{W}{2q_2},$$

is the solution at time 2.

At time 1 the problem is $\max \int u(x)\psi(p)(dq)$, subject to $x_1^1 + p_1^2 b_1 + p_2^2 b_2 \leqslant \omega_1^1$, where (x_1^2, x_2^2) in the maximand take the values given above as a function of q_2.

Assume that $\omega_1^1 = \omega_1^2 = \omega_2^2 = 1$ for simplicity. The first order conditions for the above maximization are:

$$0 = \int \frac{-p_1^2}{1 - p_1^2 b_1 - p_2^2 b_2} + \frac{2}{1 + b_1 + q_2(b_2 + 1)}\psi(p)(dq_2),$$

$$0 = \int \frac{-p_2^2}{1 - p_1^2 b_1 - p_2^2 b_2} + \frac{2q_2}{1 + b_1 + q_2(b_2 + 1)}\psi(p)(dq_2),$$

and

$$x_1^2 + p_1^2 b_1 + p_2^2 b_2 = 1.$$

Let us temporarily assume that $\psi(p)$ is the uniform distribution on an interval of q_2 values, say $[\theta_1, \theta_2]$. With this assumption the first-order conditions above can be solved (with considerable algebra) yielding equilibrium values of p_1^2 and p_2^2 as functions of θ_1 and θ_2. These equilibrium functions are denoted $\bar{p}_1^2$ and $\bar{p}_2^2$:

$$\bar{p}_1^2(\theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1} \log \frac{1 + \theta_2}{1 + \theta_1},$$

$$\bar{p}_2^2(\theta_1, \theta_2) = 2 - \frac{2}{\theta_2 - \theta_1} \log \frac{1 + \theta_2}{1 + \theta_1}.$$

Now suppose that $\psi(p)$ is always a uniform distribution on an interval but that the interval is a function of p. Suppose further that the end-points of the interval are functions only of p_2^2. That is, the interval is

$$[\theta_1(p_2^2), \theta_2(p_2^2)].$$

A temporary equilibrium price system (p_1^{2*}, p_2^{2*}) will be such that

$$p_1^{2*} = \bar{p}_1^2(\theta_1(p_2^{2*}), \theta_2(p_2^{2*})),$$

$$p_2^{2*} = \bar{p}_2^2(\theta_1(p_2^{2*}), \theta_2(p_2^{2*})).$$

We will show that functions $\theta_1(\cdot)$ and $\theta_2(\cdot)$ exist such that these equations (i.e., the second one) can never be satisfied.

Let $\theta_2 - \theta_1 \equiv 1$ and consider a shift in the interval by an amount $d\theta$. It can be shown that

(*) $\qquad \dfrac{d\bar{p}_2^2}{d\theta} = 2\dfrac{1}{(1 + \theta_1)(2 + \theta_1)}.$

Now consider functions $\theta_1(p_2^2), \theta_2(p_2^2)$ such that $\theta_2 \equiv \theta_1 + 1$, $\theta_1 = 1$ for $p_2^2 \in [0, 1 - \log 2]$, and $\theta_1(p_2^2)$ satisfies the differential equation

$$\frac{d\theta_1}{dp_2^2} = \tfrac{1}{2}(2 + 3\theta_1 + \theta_1^2)$$

for $p_2^2 \in [1 - \log 2, \infty)$ with boundary condition $\theta_1(1 - \log 2) = 1$. It is clear that $[\theta_1, \theta_2]$ is a continuous correspondence as a function of p_2^2.

Using equation (*) above,

$$\frac{d\bar{p}_2^2}{dp_2^2} = \frac{2(d\theta/dp_2^2)}{1 + 3\theta_1 + \theta_1^2} = 1$$

for $p_2^2 \in [1 - \log 2, \infty)$. But $\bar{p}_2^2(\theta_1(p_2^2), \theta_2(p_2^2)) = 2 - \log 3/2 > 1 - \log 2$ for $p_2^2 \in [0, 1 - \log 2]$. Thus if p is such that $p_2^2 \leqslant 1 - \log 2$, p cannot be an equilibrium. But if $p_2^2 > 1 - \log 2$, $\bar{p}_2^2 - p_2^2 \equiv 1 + \log 4/3$; hence $\bar{p}_2^2(\theta_1(p_2^2), \theta_2(p_2^2)) \neq p_2^2$ for all $p_2^2 \geqslant 0$, and there can be no equilibrium.

For every value of p_2^2, there is a p_1^2 such that

$$\frac{p_2^2}{p_1^1} \in [\theta_1(p_2^2), \theta_2(p_2^2)].$$

However,

$$\bigcap_{p_2^2 \geqslant 0} [\theta_1(p_2^2), \theta_2(p_2^2)] = \varnothing,$$

so that the example does not satisfy (4.2).

The problem in this example is that the demands cannot be bounded below. If one approximates the set S with sets, P^n, having the fixed-point property (this can be done, but we omit the demonstration), the sequence of $p^n \in P^n$ obtained using the method of Debreu [1] (see Theorem 4.11) will converge to a point in the boundary of S. The associated actions z^n become unbounded. There is no way to contradict this without assumption (4.2). Thus existence cannot be proven using our methods, and it is, in fact, false.

6. CONCLUSION

This section embodies some interpretations of the model, possibilities for extension and generalization, and general remarks that did not seem to fit in elsewhere. These are best introduced by stating our goal of a longer term nature, which is the study of equilibrium theory over time. As a first step in this direction,

we choose to begin with the exchange model and perfect competition. That is, all markets available and all individuals behaving as price takers. This is, of course, in the same spirit as the way in which equilibrium theory began in the static case; see Debreu [2]. Reasons for interest in the sequential trading case were given in Section 1.

The principal question is why should one study a case in which all markets exist, no a priori bounds are placed on the volume of trading or the asset positions of the traders, and no financial assets exist (that is, all assets are claims on real goods). With regard to the existence of markets, it is our position that the non-existence of a market is an economic phenomenon, just as is its existence. If a market fails to exist, it is because it is unprofitable to operate, infeasible, or not mutually beneficial. Therefore, rather than place a priori restrictions on the model, we have chosen to consider the "perfect" case, in the belief that generalizations including marketing technologies (as in Foley [4] and Hahn [8] for the certainty case) will produce, as a result of jointly maximizing behavior, a pattern of available markets approximating reality more closely.

Similar remarks apply to financial assets. We hope to show that the introduction of financial assets into the "perfect" exchange model will lead to outcomes that are socially superior in some way. Thus financial assets are a byproduct of socially maximizing behavior. In this way we hope to deduce why certain financial assets are available while others are not, rather than taking the set of such assets as a datum. It may also be possible to deduce normative conclusions about the optimality of a particular structure of active markets and financial assets.

The question of bounds on trading, particularly with regard to claims on future commodities, deserves special attention. As is clear from reading the proof above, difficulties encountered in showing the existence of an equilibrium in the sequential trading model are due principally to the fact that trades are not bounded below. If there were a priori bounds on trading futures contracts, the proof would be much simpler and technical assumptions, in particular (4.2), would be unnecessary. Surely (4.2) is undesirable in the sense that our analysis would be more general without it. Nevertheless, having shown that (4.2) is needed for equilibrium theory to be internally consistent in the "perfect" case is a conclusion of potential descriptive interest. Our institutions in the real world may be thought to have arisen to make the world more orderly, that is to insure that equilibrium exists when otherwise it might not. Thus the phenomenon of collateral loans insures that one does not take an extreme short position in a futures market. Even though such a position may be feasible for an individual (that is in his $A(p)$), his $\psi(p)$ may be quite different from that of the agent (bank, broker) who loans him the necessary funds. This agent requires collateral to guarantee the loan presumably because there is some region of *his* $\psi(p)$ in which his client might otherwise not be solvent, even though in the mind of the client this is unthinkable. Thus, while collateral is an aspect of maximizing behavior, it also serves to restrict the amount of loans possible, that is, to bound the individuals feasible set from below. Thus it may be that the "imperfections" in our world have arisen to protect us from the potential non-existence of equilibrium in a "perfect" world without some of our assumptions.

All of the above mentioned institutional possibilities are open questions that require further study.

Theoretical questions as to the extension of this model involve the consideration of production, and the purchase and sale of commodities by firms over time. This includes investment decisions with uncertain future prices, questions of the choice of durability of inputs, and a host of others. Another important generalization will be to extend this analysis to n-periods instead of only two.

An interpretation of our model in which commodities represent Fisherian "real income" can be applied to interest rate theory. In this way we have extended the results of de Montbrial [3] to the case of uncertainty without point expectations. Several special cases of this theory, emphasizing the differences between comparative static results obtained with this model and in the traditional theories of the term structure of interest rates, are given in Green [7].

A final crucial open question is whether, in a sequential trading model of this type, a temporary equilibrium at one date gives rise to an environment for which a temporary equilibrium exists at the next date.

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