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Working Paper 19-016

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Abstract

Various forms of substitutability are essential for establishing the existence of equilibria and other useful properties in diverse settings such as matching, auctions, and exchange economies with indivisible goods. We extend earlier models' definitions of substitutability to settings in which each agent can be both a buyer in some transactions and a seller in others, and show that all these definitions are equivalent. We then introduce a new class of substitutable preferences that allows us to model intermediaries with production capacity. We also prove that substitutability is preserved under economically important transformations such as trade endowments, mergers, and limited liability.

JEL Classification: C62, C78, D44, D47

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1 Introduction

Various forms of substitutability are essential for establishing the existence of equilibria and other useful properties in diverse settings such as matching, auctions, exchange economies with indivisible goods, and trading networks (Kelso and Crawford, 1982; Roth, 1984; Bikhchandani and Mamer, 1997; Gul and Stacchetti, 1999, 2000; Milgrom, 2000; Ausubel and Milgrom, 2006; Hatfield and Milgrom, 2005; Sun and Yang, 2006, 2009; Ostrovsky, 2008; Hatfield et al., 2013; Fleiner et al., 2017). Substitutability arises in a number of important applications, including matching with distributional constraints (Abdulkadiroğlu and Sönmez, 2003; Hafalir et al., 2013; Ehlers et al., 2014; Echenique and Yenmez, 2015), supply chains (Ostrovsky, 2008), markets with horizontal subcontracting (Hatfield et al., 2013), "swap" deals in exchange markets (Milgrom, 2009), and combinatorial auctions for bank securities (Klemperer, 2010; Baldwin and Klemperer, 2018).

The diversity of settings in which substitutability plays a role has led to a variety of different definitions of substitutability, and a number of restrictions on preferences that appear in some definitions but not in others.[1] In this paper, we show how the different definitions of substitutability are related to each other, while dispensing with some of the restrictions in the preceding literature.

We consider agents with quasilinear utility who can simultaneously be buyers in some transactions and sellers in others; this allows us to embed the focal substitutability concepts from the matching, auctions, and exchange economy literatures.[2] As our setting is more general than those used in most of the previous work (which considered two-sided markets), we generalize the previous substitutability concepts where necessary. Our main result then shows that all of the generalized substitutability concepts are equivalent;[3] we call preferences satisfying these conditions *fully substitutable*.[4]

Preferences are fully substitutable if goods or services bought and sold act as substitutes

[1]For instance, some definitions assume "free disposal"/"monotonicity," under which an agent is always weakly better off with a larger set of goods than with a smaller one, while other definitions do not; some definitions assume that all bundles of goods are feasible for each agent, while others do not.

[2]While all of the results in our paper consider the preferences of a single agent, and thus do not depend on the details of the agent's setting, for concreteness, notational simplicity, and continuity with prior literature, we state and prove these results in the general trading network setting of Hatfield et al. (2013).

[3]Note that the concept of substitutability has also been extended to settings with externalities (see, e.g., Pycia and Yenmez (2015) and Rostek and Yoder (2017, 2018)); we do not address such settings in this paper. Likewise, we do not address various strengthenings or weakenings of the substitutability condition (see, e.g., Klaus and Walzl (2009); Hatfield and Kojima (2010); Hatfield and Kominers (2017)), or settings in which utility is not transferable.

[4]We use the modifier "fully" to highlight the possibility that under such preferences, an agent can be both a buyer in some transactions and a seller in others, whereas under the "gross substitutes" preferences of Kelso and Crawford (1982), an agent can be only a buyer or only a seller.

for each other. That is, whenever an agent is offered a new opportunity as a buyer, he is neither induced to take any previously-rejected opportunities as a buyer, nor to reject any previously-taken opportunities as a seller; similarly, whenever an agent is offered a new opportunity as a seller, he is neither induced to take any previously-rejected opportunities as a seller, nor to reject any previously-taken opportunities as a buyer.

We introduce a rich new class of fully substitutable preferences that models the preferences of intermediaries with production capacity; to show that these preferences are fully substitutable, we rely on several properties of fully substitutable preferences that we establish. We also introduce a novel proof technique that uses "dummy layers," which simplifies modeling the preferences of a firm with several alternative production technologies.

We also prove that full substitutability is preserved under several economically important transformations: trade endowments and obligations, mergers, and limited liability. We show that full substitutability can be recast in terms of submodularity of the indirect utility function, the single improvement property, and a condition from discrete convex analysis called $\mathrm{M}^\natural$-concavity. Finally, we prove that (again, when the utilities of the agents are quasilinear) full substitutability implies two key monotonicity conditions—the Laws of Aggregate Supply and Demand—as well as a slightly stronger condition called monotone–substitutability.

All of our results explicitly incorporate economically important features that were not fully addressed in the earlier literature, such as indifferences, non-monotonicities, and unbounded utility functions. In particular, unbounded utility functions allow us to model firms with technological constraints under which some production plans are infeasible (and will therefore never be undertaken under any vector of prices).

The full substitutability conditions we unify here are exactly the conditions needed for the existence of competitive equilibria in the trading networks setting we examined in earlier work (Hatfield et al., 2013). Moreover, under full substitutability, the requirements of competitive equilibrium are essentially equivalent to those of matching-theoretic stability and chain stability (Hatfield et al., 2013, 2017). Furthermore, as the Hatfield et al. (2013) trading network framework generalizes both exchange economy (such as Gul and Stacchetti (1999)) and combinatorial auction settings (such as Ausubel and Milgrom (2002)), our work here gives a fully unified interpretation of substitutability for those applications as well.

1.1 History and Related Literature

For two-sided settings, Kelso and Crawford (1982) introduced the (demand-theoretic) gross substitutability condition, under which substitutability is expressed in terms of changes in an agent's demand as prices change. In exchange economies with indivisible goods, gross substitutability guarantees the existence of core allocations and competitive equilibria (Kelso and Crawford, 1982; Gul and Stacchetti, 1999, 2000). Roth (1984) introduced a related (choice-theoretic) substitutability concept, which is expressed in terms of how an agent's choice responds to changes in the set of available options. In two-sided matching settings, choice-theoretic substitutability guarantees the existence of stable outcomes (Roth, 1984; Hatfield and Milgrom, 2005; Hatfield and Kominers, 2013).

In a matching setting, Hatfield and Milgrom (2005, Theorem 2) showed that the choice- and demand-theoretic substitutability conditions are essentially equivalent. Ausubel and Milgrom (2002) offered an alternative definition of gross substitutability for an auction setting with continuous prices, in which demand is not guaranteed to be single-valued, and showed that gross substitutability is equivalent to submodularity of the indirect utility function. Gul and Stacchetti (1999) showed in an exchange economy setting that (gross) substitutability is equivalent to submodularity of the indirect utility function, as well as a single improvement property and a "no complementarities" condition. The various substitutability conditions for two-sided settings were subsequently extended and generalized, giving rise to two (mostly) independent literatures.

Ostrovsky (2008) generalized the choice-theoretic substitutability conditions to the context of supply chain networks by introducing a pair of related assumptions, same-side substitutability and cross-side complementarity, which impose two constraints: First, when an agent's opportunity set on one side of the market expands, that agent does not choose any options previously rejected from that side of the market. Second, when an agent's opportunity set on one side of the market expands, that agent does not reject any options previously chosen from the other side of the market. Ostrovsky (2008) and Hatfield and Kominers (2012) showed that under same-side substitutability and cross-side complementarity, a stable outcome always exists if the contractual set has a supply chain structure (see also Fleiner et al. (2017)).

Meanwhile, Sun and Yang (2006) generalized the demand-theoretic substitutability conditions to indivisible objects allocation settings with a certain structured form of complementarity. More specifically, the Sun and Yang (2006) gross substitutability and complementarity condition, akin to the combination of same-side substitutability and cross-side complementarity, requires that objects can be divided into two groups such that objects in the same group are substitutes and objects in different groups are complements; this condition guarantees the existence of competitive equilibria.

Hatfield, Kominers, and Jagadeesan (2017) showed that same-side substitutability and cross-side complementarity are together equivalent to the assumption of weak quasisubmodularity of the indirect utility function—an adaptation of submodularity to the setting without transfers.[5] Sun and Yang (2009) showed that the gross substitutability and complementarity condition is equivalent to submodularity of the indirect utility function.

In the present work, we introduce a generalization of the same-side substitutability and cross-side complementarity conditions of Ostrovsky (2008) to settings with indifferences. We also extend the gross substitutes and complements condition of Sun and Yang (2006) to settings with fully general intermediary preferences. Then, using a notation originally introduced by Hatfield and Kominers (2012), we give a reinterpretation of both the choice- and demand-theoretic substitutability conditions, in terms of indicator functions that track the underlying goods in the economy; in the process, we show how to "fold" the general economy we consider into a Kelso and Crawford (1982) economy in which the underlying goods are (gross) substitutes.[6] We show moreover that all of the previously established connections between gross substitutability and more technical conditions such as submodularity and the single improvement property extend to full substitutability as well (for suitable generalizations of the technical conditions).

Subsequent to our work, Baldwin and Klemperer (2018) obtained additional insights on the underlying mathematical structure of fully substitutable preferences using the techniques of tropical geometry. Baldwin and Klemperer (2018) studied the set of price vectors for which the demand correspondence is multi-valued, and associated them with convex-geometric objects called tropical hypersurfaces. Then, using the normal vectors that determine agents' tropical hypersurfaces, they distinguish among preferences that are strongly substitutable, are gross substitutable, or have complementarities. Full substitutability corresponds to the "strong substitutes demand" condition of Baldwin and Klemperer (2018).[7]

The discrete mathematics literature has explored several other concepts that are equivalent to substitutability in certain settings. We connect to that literature in Section 6.4, where we establish the equivalence of full substitutability and $M^\natural$-concavity in our setting.[8]

1.2 Organization of the Paper

Our paper is organized as follows. In Section 2, we present our framework. In Section 3, we present three definitions of full substitutability and show that they are all equivalent. In

[5] This is a correction of a result of Hatfield and Kominers (2012).

[6] This transformation, which we also applied in our previous work (Hatfield et al., 2013) generalizes an idea introduced by Sun and Yang (2006).

[7] Baldwin and Klemperer (2018) also consider several other economically important classes of preferences for which the existence of competitive equilibria is guaranteed.

[8] Candogan et al. (2016) use this equivalence result to recast the problem of finding a competitive equilibrium in a network economy as a discrete concave optimization problem, which in turn allows them to construct computationally efficient algorithms for finding an equilibrium. Paes Leme (2017) provides a detailed survey that covers the discrete-mathematical substitutability concepts and their algorithmic properties.

Section 4, we present and explore several economically important classes of fully substitutable preferences. In Section 5, we discuss transformations that preserve full substitutability. In Section 6, we provide several alternative characterizations of full substitutability. In Section 7, we show that full substitutability implies monotone–substitutability and, thus, the Laws of Aggregate Supply and Demand. Section 8 concludes the main body of the paper. In Appendix A, we present six additional equivalent definitions of full substitutability, which explicitly deal with indifferences in preferences. All proofs are presented in Appendix B.

2 Model

The results we present consider the preferences of an individual agent, and thus do not depend on the environment in which that agent is located. However, for notational convenience and for continuity with the related literature, we present these results in the trading network setting of Hatfield et al. (2013).[9]

There is an economy with a finite set I of *agents* and a finite set Ω of *trades*. Each trade $\omega \in \Omega$ is associated with a *buyer* $b(\omega) \in I$ and a *seller* $s(\omega) \in I$, with $b(\omega) \neq s(\omega)$. We allow Ω to contain multiple trades associated to the same pair of agents, and allow for the possibility of trades $\omega \in \Omega$ and $\psi \in \Omega$ such that the seller of ω is the buyer of ψ, i.e., $s(\omega) = b(\psi)$, and the seller of ψ is the buyer of ω, i.e., $s(\psi) = b(\omega)$.

A *contract* x is a pair $(\omega, p_\omega) \in \Omega \times \mathbb{R}$ that specifies a trade and an associated price. For a contract $x = (\omega, p_\omega)$, we denote by $b(x) \equiv b(\omega)$ and $s(x) \equiv s(\omega)$ the buyer and the seller associated with the trade ω of x. The set of possible contracts is $X \equiv \Omega \times \mathbb{R}$. A set of contracts $Y \subseteq X$ is *feasible* if it does not contain two or more contracts for the same trade: formally, Y is feasible if $(\omega, p_\omega), (\omega, \hat{p}_\omega) \in Y$ implies that $p_\omega = \hat{p}_\omega$. We call a feasible set of contracts an *outcome*. An outcome specifies a set of trades along with associated prices, but does not specify prices for trades that are not in that set. An *arrangement* is a pair $[\Psi; p]$, with $\Psi \subseteq \Omega$ and $p \in \mathbb{R}^\Omega$. Note that an arrangement specifies prices for *all* the trades in the economy. For any arrangement $[\Psi; p]$, we denote by $\kappa([\Psi; p]) \equiv \cup_{\psi \in \Psi}\{(\psi, p_\psi)\} \subseteq X$ the outcome induced by $[\Psi; p]$.

For a set of contracts $Y \subseteq X$ and agent $i \in I$, we let $Y_{\to i} \equiv \{y \in Y : i = b(y)\}$ denote the set of *upstream* contracts in Y for i, i.e., the subset of contracts in Y for which i is the buyer. Similarly, we let $Y_{i \to} \equiv \{y \in Y : i = s(y)\}$ denote the set of *downstream* contracts in Y for i, i.e., the subset of contracts in Y for which i is the seller; we let $Y_i \equiv Y_{\to i} \cup Y_{i \to}$. We use

[9] In particular, presenting the results in the framework of Hatfield et al. (2013) allows us to apply the results of Hatfield et al. (2013) in the proof of Theorem 4 in Section 5.2 (on "mergers" of agents with fully substitutable preferences). In turn, Theorem 4 allows us to prove the full substitutability of preferences in the "intermediary with production capacity" preference class that we introduce in Section 4.2.

analogous notation with regard to sets of trades $\Psi \subseteq \Omega$; that is, $\Psi_{\to i} \equiv \{\psi \in \Psi : i = b(\psi)\}$, $\Psi_{i\to} \equiv \{\psi \in \Psi : i = s(\psi)\}$, and $\Psi_i \equiv \Psi_{\to i} \cup \Psi_{i\to}$. For a set of contracts $Y \subseteq X$, we let $\tau(Y) \equiv \{\omega \in \Omega : (\omega, p_\omega) \in Y \text{ for some } p_\omega \in \mathbb{R}\} \subseteq \Omega$ denote the set of trades associated with contracts in Y.

2.1 Preferences

Each agent i has a *valuation* (or *preferences*) $u_i : \wp(\Omega_i) \to \mathbb{R} \cup \{-\infty\}$ over the sets of trades in which he is involved, with $u_i(\varnothing) \in \mathbb{R}$.[10] The utility functions we allow here are quite general: We do not impose any monotonicity/free disposal conditions on agents' preferences, and thus trades can represent both "goods" and "bads". Moreover, we allow the utility of agent i to equal $-\infty$, formalizing the idea that i, due to technological constraints, may only be able to produce or sell certain outputs contingent upon procuring appropriate inputs; e.g., if $\psi, \omega \in \Omega$ with $b(\psi) = s(\omega) = i$ and agent i cannot sell ω unless he has procured ψ, then $u_i(\{\omega\}) = -\infty$.[11] The assumption that $u_i(\varnothing)$ is finite for each $i \in I$ implies that no agent is obligated to engage in market transactions at highly unfavorable prices; he can always choose a (finite) outside option.

The valuation u_i over bundles of trades gives rise to a quasilinear utility function U_i over bundles of trades and associated transfers. Specifically, for any feasible set of contracts $Y \subseteq X$, we define

$$U_i(Y) \equiv u_i(\tau(Y)) + \sum_{(\omega,p_\omega)\in Y_{i\to}} p_\omega - \sum_{(\omega,p_\omega)\in Y_{\to i}} p_\omega,$$

and, slightly abusing notation, for any arrangement $[\Psi; p]$, we define

$$U_i([\Psi; p]) \equiv u_i(\Psi) + \sum_{\psi\in\Psi_{i\to}} p_\psi - \sum_{\psi\in\Psi_{\to i}} p_\psi.$$

Note that by construction, $U_i([\Psi; p]) = U_i(\kappa([\Psi; p]))$.

The *choice correspondence* of agent i from the set of contracts $Y \subseteq X$ is defined by

$$C_i(Y) \equiv \underset{Z\subseteq Y_i;\ Z \text{ is feasible}}{\arg\max} \{U_i(Z)\}.$$

Note that the choice correspondence can be multi-valued because of indifferences; that is, it is possible that $|C_i(Y)| > 1$. Moreover, each element of the choice correspondence is a *set* of contracts, and so for $Y^* \in C_i(Y)$ we have $Y^* \in \wp(Y)$; in particular, we may have

[10] We assume that trades in $\Omega \smallsetminus \Omega_i$ do not affect i, and abuse notation slightly by writing $u_i(\Psi) \equiv u_i(\Psi_i)$ for $\Psi \subseteq \Omega$. We use $\wp(\cdot)$ to denote the power set.

[11] In the classical exchange economy literature (Bikhchandani and Mamer, 1997; Gul and Stacchetti, 1999), the valuation of an agent i is defined over bundles of objects Ω as $u_i : \wp(\Omega_i) \to \mathbb{R}$, and is normalized such that $u_i(\varnothing) = 0$. While these assumptions are completely innocuous and natural in the context of exchange economies, they immediately rule out the kinds of technological constraints discussed above.

$|Y^*| > 1$ (even when $|C_i(Y)| = 1$).[12] Also, the choice correspondence may be empty-valued (e.g., if Y is the set of all contracts with prices strictly between 0 and 1 for a particular trade, i.e., $Y = \{(\omega, p_\omega) : p_\omega \in (0,1)\}$, and $u^i(\{\omega\}) = 1$). When the set Y is finite, the choice correspondence is also guaranteed to contain at least one element.

The *demand correspondence* of agent i, given a price vector $p \in \mathbb{R}^\Omega$, is defined by

$$D_i(p) \equiv \arg\max_{\Psi \subseteq \Omega_i} \{U_i([\Psi; p])\}.$$

Like the choice correspondence, the demand correspondence can be multi-valued; that is, $\Psi \in D_i(p)$ is an optimal set of trades for agent i given prices p. (Unlike the choice correspondence, the demand correspondence always contains at least one element.)

3 Substitutability Concepts

We now introduce three substitutability concepts that generalize the existing definitions from matching, auctions, and exchange economies. In the matching literature, it is standard to formulate substitutability in terms of choice functions and to consider expansions of the set of available contracts on one side. In the literature on economies with indivisible goods, it is standard to formulate substitutability in terms of demand functions and to consider disadvantageous price changes, i.e., increases in input prices or decreases in output prices. Finally, in auction theory it is standard to formulate substitutability in terms of demand functions and to consider an increase (or decrease) of the entire price vector.

In this section, for the of ease the exposition and to allow for a more direct comparison of the different substitutability concepts, we follow the approach of Ausubel and Milgrom (2002) and restrict attention to opportunity sets and vectors of prices for which choices and demands are single-valued. In Appendix A, we introduce additional definitions that explicitly deal with indifferences and multi-valued correspondences, and prove that those definitions are equivalent to each other and to the definitions given in this section.

3.1 Choice-Language Full Substitutability

First, we define full substitutability in the language of choices, adapting and merging the Ostrovsky (2008) same-side substitutability and cross-side complementarity conditions. Setting conditions on how each agent's optimal choice changes as that agent's opportunity set expands originated in the matching literature, where it is natural to consider an expansion

[12] Note that this notation is distinct from the standard notation used in settings without transferable utility, where (as preferences are usually assumed to be strict) $C_i(\cdot)$ is defined as a function instead of a correspondence, and so an element of $C_i(Y)$ is a contract in the unique optimal choice from Y.

in the set of available trades and thus an expansion in the set of available contracts (see Ostrovsky (2008), Westkamp (2010), Hatfield and Kominers (2012), Hatfield et al. (2013), and Hatfield, Kominers, and Westkamp (2017)). In choice language, we say that a choice correspondence C_i is fully substitutable if, when attention is restricted to sets of contracts for which C_i is single-valued, whenever the set of options available to i on one side expands, i rejects a larger set of contracts on that side (same-side substitutability) and selects a larger set of contracts on the other side (cross-side complementarity).

Definition 1. The preferences of agent i are *choice-language fully substitutable (CFS)* if:

1. for all sets of contracts $Y, Z \subseteq X$ such that $|C_i(Z)| = |C_i(Y)| = 1$, $Y_{i\rightarrow} = Z_{i\rightarrow}$, and $Y_{\rightarrow i} \subseteq Z_{\rightarrow i}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $Y_{\rightarrow i} \smallsetminus Y^*_{\rightarrow i} \subseteq Z_{\rightarrow i} \smallsetminus Z^*_{\rightarrow i}$ and $Y^*_{i\rightarrow} \subseteq Z^*_{i\rightarrow}$;
2. for all sets of contracts $Y, Z \subseteq X$ such that $|C_i(Z)| = |C_i(Y)| = 1$, $Y_{\rightarrow i} = Z_{\rightarrow i}$, and $Y_{i\rightarrow} \subseteq Z_{i\rightarrow}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $Y_{i\rightarrow} \smallsetminus Y^*_{i\rightarrow} \subseteq Z_{i\rightarrow} \smallsetminus Z^*_{i\rightarrow}$ and $Y^*_{\rightarrow i} \subseteq Z^*_{\rightarrow i}$.

Note that here we restrict attention to situations in which the choice correspondence is single-valued. As we describe after Theorem 1, CFS is equivalent to two seemingly more restrictive definitions that explicitly deal with the multi-valued parts of the choice correspondence (see Section 3.4 and Appendix A).

3.2 Demand-Language Full Substitutability

Our second definition uses the language of prices and demands, adapting the gross substitutes and complements condition (GSC) of Sun and Yang (2006).[13] We say that a demand correspondence D_i is fully substitutable if, when attention is restricted to prices for which demands are single-valued, a decrease in the price of some inputs for agent i leads to a decrease in his demand for other inputs and to an increase in his supply of outputs, and an increase in the price of some outputs leads to a decrease in his supply of other outputs and an increase in his demand for inputs.

Definition 2. The preferences of agent i are *demand-language fully substitutable (DFS)* if:

1. for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\rightarrow}$, and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\rightarrow i}$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have $\{\omega \in \Psi'_{\rightarrow i} : p_\omega = p'_\omega\} \subseteq \Psi_{\rightarrow i}$ and $\Psi_{i\rightarrow} \subseteq \Psi'_{i\rightarrow}$;

[13] The definition of full substitutability that corresponds directly to GSC is demand-language contraction full substitutability (Definition A.4); that said, there is a subtlety in interpreting the relationship between the Sun and Yang (2006) model and ours (see Appendix A.2).

2. for all price vectors $p, p' \in \mathbb{R}^{\Omega}$ such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$, and $p_\omega \le p'_\omega$ for all $\omega \in \Omega_{i\to}$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ and $\Psi_{\to i} \subseteq \Psi'_{\to i}$.

3.3 Indicator-Language Full Substitutability

Our third definition is essentially a reformulation of Definition 2, using a convenient vector notation due to Hatfield and Kominers (2012). For each agent i, for any set of trades $\Psi \subseteq \Omega_i$, define the *(generalized) indicator function* $e_i(\Psi) \in \{-1, 0, 1\}^{\Omega_i}$ to be the vector with component $e_{i,\omega}(\Psi) = 1$ for each upstream trade $\omega \in \Psi_{\to i}$, $e_{i,\omega}(\Psi) = -1$ for each downstream trade $\omega \in \Psi_{i\to}$, and $e_{i,\omega}(\Psi) = 0$ for each trade $\omega \notin \Psi$. The interpretation of $e_i(\Psi)$ is that an agent buys a strictly positive amount of a good if he is the buyer in a trade in Ψ, and "buys" a strictly negative amount if he is the seller of such a trade.

Definition 3. The preferences of agent i are *indicator-language fully substitutable (IFS)* if for all price vectors $p, p' \in \mathbb{R}^{\Omega}$ such that $|D_i(p)| = |D_i(p')| = 1$ and $p \le p'$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have $e_{i,\omega}(\Psi) \le e_{i,\omega}(\Psi')$ for each $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.

Definition 3 clarifies the reason for the term "full substitutability"—an agent is more willing to "demand" a trade (i.e., keep an object that he could potentially sell, or buy an object that he does not initially own) if prices of other trades increase.

3.4 Equivalence of the Definitions

The main result of this section is that the three definitions of full substitutability presented are all equivalent. Subsequently, we use the term *full substitutability* to refer to all our substitutability concepts.

Theorem 1. *Choice-language full substitutability (CFS), demand-language full substitutability (DFS), and indicator-language full substitutability (IFS) are all equivalent.*

The intuition behind Theorem 1 is that, under all three substitutability concepts, when i receives new options as a buyer, i becomes weakly less interested in other purchase opportunities and weakly more interested in other sale opportunities. In choice-language substitutability, this is explicit, as Part 1 of CFS considers expanding the set of available contracts as a buyer. Meanwhile, in demand-language substitutability, this is implicit, as Part 1 of DFS considers reducing the prices of trades available to i as a buyer. Similarly, indicator-language substitutability requires that, when prices for trades for which i is a buyer fall, i becomes weakly less interested in any purchase opportunity for which the price does not fall and weakly more interested in any sale opportunity for which the price does not fall.

Recall that in order to ease the exposition and to allow for an easier and more transparent comparison of different substitutability notions, throughout this section we restricted attention to opportunity sets and vectors of prices for which choices and demands are single-valued. In Appendix A, we introduce additional definitions of full substitutability that explicitly deal with the multi-valued nature of the choice and demand correspondences. We then prove that the definitions introduced in Appendix A, as well as CFS, DFS, and IFS, are all equivalent (Theorem A.1). Theorem 1 thus follows immediately from Theorem A.1. Moreover, since Definitions 1–3 are equivalent to the definitions that explicitly consider the multi-valued nature of the choice and demand correspondences, it is sufficient to work with the conditions in Definitions 1–3; in particular, requiring that all agents' preferences satisfy any of Definitions 1–3 is sufficient to guarantee the existence of stable outcomes and competitive equilibria (Hatfield et al., 2013).

4 Classes of Fully Substitutable Preferences

Full substitutability is a natural condition, but it does rule out certain classes of preferences. For instance, full substitutability rules out situations in which an agent would only be willing to sell multiple units of a good at a particular price, but not one unit (at that same price). Similarly, situations in which one agent's multiple inputs are complements are also precluded. More generally, complementarities in production are ruled out because inputs are required to be substitutes for each other. Similarly, full substitutability rules out economies of scale because it requires that outputs cannot complement each other, as they would in the case when producing more outputs makes it possible to overcome initial fixed costs and/or adopt more efficient production technology.[14] Full substitutability also places more subtle restrictions: in Section 4.2, we provide an example which shows that the full substitutability assumption can also rule out the case of preferences representing an agent who is capacity constrained and requires different types of inputs to produce different types of outputs.[15]

At the same time, full substitutability also allows for many rich types of preferences. By construction, fully substitutable preferences include, as a special case, "one-sided" preferences that satisfy the gross substitutability condition of Kelso and Crawford (1982). Gross

[14]For an extended discussion, see Ostrovsky (2008, Section I.A) and Hatfield et al. (2013, Section II.B).

[15]To overcome the modeling difficulties associated with allowing for complementarities, authors have made various assumptions on functional forms of production, or worked in either large market limit environments (Azevedo and Hatfield, 2015; Che et al., 2017; Jagadeesan, 2017) or settings in which goods are perfectly divisible (Hatfield and Kominers, 2015).

substitutability has been extensively studied in the literatures on matching, competitive equilibrium, and discrete concave optimization,[16] and a variety of examples and classes of preferences satisfying the gross substitutability condition have been presented.[17]

Beyond one-sided preferences, it is easy to see that the full substitutability condition allows for environments with homogeneous goods in which agents have increasing marginal costs of production and diminishing marginal utilities of consumption. It also allows for richer classes of "two-sided" preferences that involve complementarities between the contracts an agent can execute as a buyer and those that he can execute as a seller; in Sections 4.1 and 4.2, we introduce, formally model, and discuss, two such classes of preferences.

4.1 Preferences of Intermediaries

We start with *intermediary* preferences, introduced by Hatfield et al. (2013) in the context of used car dealers, but applicable more generally.[18]

The preferences of an *intermediary* i are represented as follows: Consider an intermediary i who has access to a number of heterogeneous inputs $Y_{\to i}$ (e.g., used cars, raw diamonds, or temporary workers) and to a set of requests $Y_{i\to}$ (e.g., for used cars, for engagement rings, or for temp services). Each element $(\varphi, p_\varphi) \in Y_{\to i}$ specifies the characteristics of the particular input and the price at which this input is available to intermediary i. Each element $(\psi, p_\psi) \in Y_{i\to}$ specifies the characteristics required by the contract's customer and the price that customer is willing to pay. Some inputs φ and requests ψ are compatible with each other, while others are not.[19] For every compatible input–request pair (φ, ψ), there is also a cost $c_{\varphi,\psi}$ of preparing the input φ for resale to satisfy the compatible request ψ.[20] Intermediary i's objective is to match some of the inputs in $Y_{\to i}$ to some of the requests in $Y_{i\to}$ in a way that maximizes his profit, $\sum_{(\varphi,\psi)\in\mu}(p_\psi - p_\varphi - c_{\varphi,\psi})$, where μ denotes the set of compatible input–request pairs that the intermediary selects.

[16]See, e.g., Kelso and Crawford (1982), Hatfield and Milgrom (2005), Milgrom (2009), Milgrom and Strulovici (2009), Ostrovsky and Paes Leme (2015), and Paes Leme (2017).

[17]For instance, Klemperer (2010) makes use of grossly substitutable preferences in the design of the "Product-Mix" auction which has been and continues to be used by the Bank of England to allocate funds to banks via securitized loans. Similarly, preferences in electricity markets can often be expressed substituably via assignment messages (Milgrom, 2009).

[18]A closely related class of preferences was introduced by Sun and Yang (2006, Section 4) in the context of two-sided markets in which agents on one side (firms) have preferences over agents and objects on the other side (workers and machines) that are determined by the productivity of each worker on each machine (see discussion at the end of Section 4.2 below for more detail).

[19]For instance, a given raw diamond can only be turned into a polished diamond of certain grades. Similarly, a particular temp worker is only qualified to perform certain types of jobs.

[20]For example, $c_{\varphi,\psi}$ may be the cost of repairing a car, turning a diamond into an engagement ring, or training a worker to perform a specific set of tasks. Note that we could formally allow all pairs of inputs and requests to be compatible, and encode incompatibilities by setting $c_{\varphi,\psi} = \infty$ for some pairs (φ, ψ).

Formally, following Hatfield et al. (2013), define a *matching*, μ, as a set of pairs of trades (φ, ψ) such that φ is an element of $\Omega_{\rightarrow i}$ (i.e., an input available to intermediary i), ψ is an element in $\Omega_{i\rightarrow}$ (i.e., a request received by i), φ and ψ are compatible, and each trade in Ω_i belongs to at most one pair in μ. Slightly abusing notation, let the cost of matching μ, $c(\mu)$, be equal to the sum of the costs of pairs involved in μ (i.e., $c(\mu) = \sum_{(\varphi,\psi)\in\mu} c_{\varphi,\psi}$).

For a set of trades $\Xi \subseteq \Omega_i$, let $\mathcal{M}(\Xi)$ denote the set of matchings μ of elements of Ξ such that every element of Ξ belongs to exactly one pair in μ.[21] Then the valuation of intermediary i over sets of trades $\Xi \subseteq \Omega_i$ is given by:

$$u_i(\Xi) = \begin{cases} -\min_{\mu\in\mathcal{M}(\Xi)}\{c(\mu)\} & \text{if } \mathcal{M}(\Xi) \neq \varnothing \\ -\infty & \text{if } \mathcal{M}(\Xi) = \varnothing, \end{cases}$$

i.e., $u_i(\Xi)$ is equal to the cost of the cheapest way of matching all requests and inputs in Ξ if such a matching is possible, and is equal to $-\infty$ otherwise.[22] (Note that $u_i(\varnothing) = 0$.) The utility function of i over feasible sets of contracts is induced by valuation u_i in the standard way formalized in Section 2.1.

Proposition 1 (Hatfield et al. (2013, Proposition 1))**.** *Intermediary preferences are fully substitutable.*

Hatfield et al. (2013) present a rather involved proof of Proposition 1. Sun and Yang (2006) also present an elaborate proof of an analogous result for the two-sided setting (Theorem 4.1 in their paper, with the proof on pages 1397–1401). The results of the current paper allow us to construct a much simpler and shorter proof, presented in Appendix B. Proposition 1 follows as a special case of Proposition 2, which shows the full substitutability of the new class of preferences that we introduce in the next section, the class of *intermediaries with production capacity* preferences. Proposition 2, in turn, follows directly from our result on mergers of agents with fully substitutable preferences (Theorem 4 of Section 5.2).

4.2 Preferences of Intermediaries with Production Capacity

For the intermediary preferences considered in Section 4.1, the intermediary either does not need to use any of his own resources to facilitate the matches between inputs and requests, or when he does, those resources could be expressed in monetary terms: there was a cost $c_{\varphi,\psi}$

[21] Of course, $\mathcal{M}(\Xi)$ can be empty; e.g., it is empty if the number of inputs in Ξ is not equal to the number of requests, or if there are some requests in Ξ that are not compatible with any input in Ξ.

[22] Under this valuation function, any set chosen by intermediary i will contain an equal number of offers and requests. In principle, we could consider a more general (yet still fully substitutable) valuation function in which an intermediary has utility for an input that he does not resell. In that case, the intermediary may end up choosing more offers than requests.

of "preparing" input φ for request ψ. In some settings, however, we may want to consider intermediaries who need to rely on specific physical resources that they have in order to turn inputs into outputs, and it is more appropriate to think of these resources as fixed. For example, a manufacturer may have a fixed set of machines, and needs to assign a set of workers to those machines and at the same time needs to decide which outputs to produce on the machines. An agricultural firm may have a fixed set of land lots, and needs to hire workers to work on these lots, and at the same time needs to decide which outputs to produce. A steel manufacturer has access to a variety of inputs (different sources of iron ore and scrap metal) and can produce a variety of outputs (different grades and types of steel products), and needs to assign these inputs and outputs to the fixed number of steel plants that it has.

In general, the preferences of an agent with capacity constraints may not be fully substitutable. For example, consider a firm that has exactly one machine, can hire workers Ann and Bob, and has requests for outputs α and β. Suppose Ann can use the machine to produce output α (but not β), while Bob can use the machine to produce output β (but not α). In this case, the preferences of the firm are not fully substitutable: reducing a price of an input (say, Ann) may lead to the firm choosing to drop an output (β), violating Part 1 of demand language full substitutability (Definition 2). In this section, however, we identify a rich class of preferences that are fully substitutable despite the presence of capacity constraints.

Specifically, consider an intermediary i who has access to a number of inputs $Y_{\to i}$ and requests $Y_{i\to}$. Each element $(\varphi, p_\varphi) \in Y_{\to i}$ specifies the characteristics of the particular input and the price at which this input is available to intermediary i. Each element $(\psi, p_\psi) \in Y_{i\to}$ specifies the characteristics required by the contract's customer and the price that customer is willing to pay. Finally, the intermediary has a set M of *machines*; each machine $m \in M$ can be used to prepare one input for one output.

For each input φ and machine m, there is a cost $c_{\varphi,m} \in \mathbb{R} \cup \{+\infty\}$ of preparing the input to work with the machine (e.g., the cost of training a particular worker, or the cost of transporting iron ore from its source). For each machine m and each request ψ, there is a cost $c_{m,\psi} \in \mathbb{R} \cup \{+\infty\}$ of using this machine to produce the requested output (e.g., the cost of water required to produce a particular agricultural crop on a particular land lot, or the cost of transporting a batch of steel to its destination). Note that we allow both costs to take the value $+\infty$, to enable the possibility that a particular input is not compatible with a particular machine, or a particular machine is not compatible with a particular request. The total cost of preparing input φ for request ψ using machine m is thus $c_{\varphi,m} + c_{m,\psi}$. The objective of intermediary i is to match some of the inputs in $Y_{\to i}$ to some of the requests in $Y_{i\to}$, via some of the machines, in a way that maximizes his profit, $\sum_{(\varphi,m,\psi)\in\mu}(p_\psi - p_\varphi - c_{\varphi,m} - c_{m,\psi})$, where μ denotes the set of input–machine–request triples that the intermediary selects.

Formally, define a *matching*, μ, as a set of triples (φ, m, ψ) such that

1. φ is an element of $\Omega_{\to i}$,
2. m is a machine available to intermediary i,
3. ψ is an element of $\Omega_{i\to}$, and
4. each φ belongs to at most one triple in μ, each m belongs to at most one triple in μ, and each ψ belongs to at most one triple in μ.

Slightly abusing notation, let the cost of matching μ, $c(\mu)$, be equal to the sum of the costs of triples involved in μ, i.e., $c(\mu) = \sum_{(\varphi,m,\psi)\in\mu}(c_{\varphi,m} + c_{m,\psi})$.

For a set of trades $\Xi \subseteq \Omega_i$, let $\mathcal{M}(\Xi)$ denote the set of matchings μ of elements of Ξ and machines available to the intermediary, such that every element of Ξ belongs to exactly one triple in μ. Then the valuation of intermediary i over sets of trades $\Xi \subseteq \Omega_i$ is given by:

$$u_i(\Xi) = \begin{cases} -\min_{\mu\in\mathcal{M}(\Xi)}\{c(\mu)\} & \text{if } \mathcal{M}(\Xi) \neq \varnothing \\ -\infty & \text{if } \mathcal{M}(\Xi) = \varnothing, \end{cases}$$

i.e., $u_i(\Xi)$ is equal to the cost of the cheapest way of satisfying all requests in Ξ using all of the inputs in Ξ and some of the machines, if such a production plan is possible; and is equal to $-\infty$ otherwise. The utility function of intermediary i over feasible sets of contracts is induced by valuation u_i in the usual way.

Proposition 2. *Intermediary with production capacity preferences are fully substitutable.*

The intuition for the proof of Proposition 2 is as follows. First, if an intermediary i has only one machine, then his preferences are fully substitutable.[23] Next, if the intermediary has multiple machines (say, a set M of machines), he can be, in essence, viewed as a "merger" of $|M|$ single-machine agents. However, we can not "merge" the $|M|$ single-machine agents directly, as we must account for the constraints that a given input can be used by at most one machine (and, similarly, the constraints that a given request can be satisfied by at most one machine). To address these issues, we introduce a novel proof strategy: We ensure that each input and each output is only used at most once by the merged firm by adding a layer of "input dummy" firms and a layer of "request dummy" firms. Each input dummy

[23]To see this, suppose that i's set of options as a buyer expands, i.e., the hypothesis of Condition 1 of CFS. Since i chooses only one contract as a buyer, individual rationality implies that i must either choose a new contract or choose the same contract he chose before (in both cases all previously-rejected contracts are rejected); thus, the first part of Condition 1 of CFS is satisfied. Moreover, since i chooses his output contract ψ so as to maximize $p_\psi - c_{\psi,m}$, i's choice as a seller does not change, and so the second part of Condition 1 of CFS is satisfied. Similar arguments show that Condition 2 of CFS is satisfied.

firm enforces the constraint that the input corresponding to that dummy firm is used by at most one machine within the merged firm. Similarly, each request dummy firm enforces the constraint that the request corresponding to that dummy firm is fulfilled by at most one machine. The merger operation then combines these dummy firms with the single-machine firms. By Theorem 4 of Section 5.2, the preferences of this merged firm are fully substitutable, and it is clear that the preferences of the merged firm reflect the valuation u^i.

Note that while we use the "dummy firm layers and mergers" construction for the specific purpose of proving the full substitutability of intermediary with production capacity preferences, this technique may be useful more generally for incorporating various restrictions (say, incompatibility of some input trades) in agents' preferences while maintaining full substitutability, both in trading network and two-sided settings.

4.3 Discussion

The classes of preferences discussed in Sections 4.1 and 4.2 follow the tradition of constructing complex preferences out of elementary "building blocks" by combining and modifying these blocks using substitutability–preserving operations. This tradition goes back to Shapley (1962), who established the complementarity and substitutability properties of the optimal assignment problem, in which each "match" has a very simple payoff structure, but by optimizing over the space of possible overall assignments we can obtain a rich set of preferences. Sun and Yang (2006) introduced a closely related class of preferences in two-sided markets in which agents on one side (firms) have preferences over agents and objects on the other side (workers and machines) that are determined by the productivity of each worker on each machine; Sun and Yang (2006) showed that such preferences satisfy the gross substitutes and complements (GSC) condition, with workers being substitutes for one another, machines being substitutes for one another, and workers and machines being complements. The constructions of Shapley (1962) and Sun and Yang (2006) are closely related to our construction of intermediary preferences in Section 4.1.[24]

In the standard two-sided matching context, Hatfield and Milgrom (2005) further extended the construction of substitutable preferences to the class of *endowed assignment valuations*, by starting with assignment valuations of Shapley (1962) and applying the *endowment* operation to them (i.e., by allowing the optimizing agent to initially own some of the inputs; see

[24]An important substantive difference between our intermediary preferences and the classes of preferences discussed by Shapley (1962) and Sun and Yang (2006) is that we incorporate the possibility that some pairs of "inputs" and "outputs" are physically incompatible, so that they are never "matched" under any vector of prices. By contrast, in the settings of Shapley (1962) and Sun and Yang (2006), for any given worker and machine there exists a vector of prices such under which that worker and machine will be matched.

Section 5.1 for more detail).[25] Ostrovsky and Paes Leme (2015) showed that there exist substitutable preferences that cannot be represented by endowed assignment valuations, and introduced the class of *matroid-based valuations*, which is obtained by iteratively applying the endowment and merger operations to weighted-matroid valuations. As every weighted-matroid valuation is substitutable (Murota, 1996; Murota and Shioura, 1999; Fujishige and Yang, 2003), every matroid-based valuation is also substitutable. It is an open question whether every substitutable valuation is a matroid-based valuation.

Of course, by applying further transformations that preserve full substitutability, we can generate even richer classes of fully substitutable preferences than those we discussed in Sections 4.1 and 4.2. In the next section, we discuss several types of such transformations.

5 Transformations

In this section, we show that fully substitutable preferences can be transformed and combined in several economically interesting ways that preserve full substitutability. We first consider the possibility that an agent is endowed with the right to execute any trades in a given set and the possibility that an agent has an obligation to execute all trades in a given set. We also examine mergers, where the valuation function of the merged entity is constructed as the convolution of the valuation functions of the merging parties. Finally, we consider a form of limited liability, where an agent may back out of some agreed-upon trades in exchange for paying an exogenously-fixed penalty.

5.1 Trade Endowments and Obligations

Suppose an agent i is endowed with the right (but not the obligation) to execute trades in the set $\Phi \subseteq \Omega_i$ at prices p_Φ. Let

$$\hat{u}_i^{(\Phi,p_\Phi)}(\Psi) \equiv \max_{\Xi \subseteq \Phi} \left\{ u_i(\Psi \cup \Xi) + \sum_{\xi \in \Xi_{i\to}} p_\xi - \sum_{\xi \in \Xi_{\to i}} p_\xi \right\}$$

be a valuation over trades in $\Omega \smallsetminus \Phi$; $\hat{u}_i^{(\Phi,p_\Phi)}$ represents agent i having a valuation over trades in $\Omega \smallsetminus \Phi$ consistent with u_i while being endowed with the option of executing any trades in the set $\Phi \subseteq \Omega_i$ at prices p_Φ.

[25] Alternatively, the construction of Hatfield and Milgrom (2005) can be viewed as starting with elementary singleton preferences, and iteratively applying the operations of *endowment* and *merger*. Hatfield and Milgrom (2005) showed that the endowment and merger operations preserve substitutability in their context, and thus showed that all endowed assignment valuation preferences are substitutable.

Theorem 2. *If the initial preferences of agent i are fully substitutable, then the preferences induced by the valuation function $\hat{u}_i^{(\Phi,p_\Phi)}$ are fully substitutable for any $\Phi \subseteq \Omega_i$ and $p_\Phi \in \mathbb{R}^\Phi$.*

Intuitively, when we endow agent i with access to the trades in Φ at prices p_Φ, we are effectively restricting (1) the set of prices that may change and (2) the set of trades that are required to be substitutes in the demand-theoretic definition of full substitutability (Definition 2). Naturally, this process cannot *create* complementarities among trades in $\Omega \setminus \Phi$, given that under u_i these trades already are substitutes for each other *and* for the trades in Φ. Hence, $\hat{u}_i^{(\Phi,p_\Phi)}$ induces fully substitutable preferences over trades in $\Omega \setminus \Phi$.

Apart from endowments, agents may have obligations, that is, an agent i may be obliged to execute trades in some set $\Phi \subseteq \Omega_i$ at *fixed* prices p_Φ. We now show that if an agent's preferences are initially fully substitutable, they remain so when an obligation arises to execute some trades at pre-specified prices. Suppose agent i is obliged to execute trades in $\Phi \subseteq \Omega_i$ at prices p_Φ and that Φ is technologically feasible in the sense that $u_i(\Phi) \neq -\infty$. Let

$$\tilde{u}_i^{(\Phi,p_\Phi)}(\Psi) \equiv u_i(\Psi \cup \Phi) + \sum_{\varphi \in \Phi_{i\to}} p_\varphi - \sum_{\varphi \in \Phi_{\to i}} p_\varphi,$$

where $\tilde{u}_i^{(\Phi,p_\Phi)}$ represents agent i having a valuation over trades in $\Omega \setminus \Phi$ consistent with u_i while being obliged to execute all trades in the set $\Phi \subseteq \Omega_i$ at prices p_Φ.

Theorem 3. *If the initial preferences of agent i are fully substitutable, then the preferences induced by the valuation function $\tilde{u}_i^{(\Phi,p_\Phi)}$ are fully substitutable for any $\Phi \subseteq \Omega_i$ and $p_\Phi \in \mathbb{R}^\Phi$ such that $u_i(\Phi) \neq -\infty$.*

The idea of the proof is to note that the demand correspondence of agent i with valuation $\tilde{u}_i^{(\Phi,p_\Phi)}$ does not depend on prices p_Φ—changing these prices simply leads to a shift in the agent's utility function by a fixed amount. Thus, we can assume that the trades that the agent is obliged to *buy* have negative and very large (in absolute magnitude) prices, while the trades that the agent is obliged to *sell* have positive and very large prices. Under those assumptions, "obligations" become "endowments" (because the agent would voluntarily want to execute all of these trades), and thus Theorem 3 follows from Theorem 2.

Combining Theorems 2 and 3, we see that if the preferences of agent i are fully substitutable, then they remain fully substitutable when i is endowed with some trades and obliged to execute others (assuming that the obligation is technologically feasible).

5.2 Mergers

The second transformation we consider is the case when several agents merge. Given a set of agents J, we denote the set of trades that involve only agents in J as $\Omega^J \equiv \{\omega \in \Omega :$

$\{b(\omega), s(\omega)\} \subseteq J\}$. We let the *convolution* of the valuation functions $\{u_j\}_{j\in J}$ be defined as

$$u_J(\Psi) \equiv \max_{\Phi\subseteq\Omega^J}\left\{\sum_{j\in J} u_j(\Psi\cup\Phi)\right\} \tag{1}$$

for sets of trades $\Psi \subseteq \Omega \smallsetminus \Omega^J$. The convolution u_J represents a "merger" of the agents in J, as it treats the agents in J as able to execute any within-J trades costlessly.

Theorem 4. *For any set of agents $J \subseteq I$, if the preferences of each $j \in J$ are fully substitutable, then the preferences induced by the convolution u_J (defined in* (1)*) are fully substitutable.*[26]

While Theorem 4 is of independent interest, we also use it in the proof of Proposition 2.

Substitutability is not preserved following *dissolution/de-mergers*. For example, if agents i and j only trade with each other, then the preferences induced by the convolution valuation $u_{\{i,j\}}$ are trivially fully substitutable, even if the preferences of i and j are not.

Substitutability is also not preserved when merging *trades*. To see this, consider a simple economy with agents i and j and a set of trades Ω consisting of χ, φ, ψ, and ω. Agent i is the buyer in all of the trades, and agent j is the seller. Agent i's valuation is as follows:

$$u_i(\Psi) = \begin{cases} 2 & |\Psi_i| \geq 2 \\ 1 & |\Psi_i| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The preferences of i are fully substitutable. Now, consider merging the trades χ and φ into a single trade ξ. The resulting valuation function of i over the subsets of $\tilde{\Omega} \equiv (\Omega \smallsetminus \{\chi, \varphi\}) \cup \{\xi\}$ is given by

$$\tilde{u}_i(\Psi) = \begin{cases} 2 & |\Psi_i| \geq 2 \text{ or } \xi \in \Psi \\ 1 & |\Psi_i| = 1 \text{ and } \xi \notin \Psi \\ 0 & \text{otherwise.} \end{cases}$$

Valuation function $\tilde{u}_i$ is not fully substitutable. To see this, note that for price vector $p = (p_\xi, p_\psi, p_\omega) = (1.7, 0.8, 0.8)$, the unique optimal demand of agent i is $\{\psi, \omega\}$, but for price vector $p' = (p'_\xi, p'_\psi, p'_\omega) = (1.7, 1, 0.8)$, the unique optimal demand of agent i is $\{\xi\}$. That is, under price vector p', agent i no longer demands the trade ω, even though its price remains unchanged while the price of ψ increases and the price of ξ remains unchanged.

[26] We provide a direct proof of this result in Appendix B; however, it also follows as a consequence of the relationship between full substitutability and $M^\natural$-concavity (which we describe in Section 6.4) and the fact that the supremal convolution of two $M^\natural$-concave functions is $M^\natural$-concave (Murota, 2003, Theorem 6.13).

5.3 Limited Liability

The final transformation we consider is "limited liability." Suppose that after agreeing to a trade, an agent is allowed to renege on that trade in exchange for paying a fixed penalty. We show that this transformation preserves substitutability. In addition to being economically interesting, this result is also useful technically; indeed, it enables us to transform unbounded utility functions into bounded ones while preserving substitutability, which in turn simplifies the analysis in a number of settings; see, e.g., the proof of Theorem 1 in Hatfield et al. (2013).

Formally, consider a fully substitutable valuation function u_i for agent i. Take an arbitrary set of trades $\Phi \subseteq \Omega_i$, and for every trade $\varphi \in \Phi$, pick $\Pi_\varphi \in \mathbb{R}$—the penalty for reneging on trade φ. (For mathematical completeness, we allow Π_φ to be negative.) Define the modified valuation function $\hat{u}_i$ as

$$\hat{u}_i(\Psi) \equiv \max_{\Xi \subseteq \Psi \cap \Phi} \left\{ u_i(\Psi \smallsetminus \Xi) - \sum_{\varphi \in \Xi} \Pi_\varphi \right\}. \tag{2}$$

That is, under valuation $\hat{u}_i$, agent i can "buy out" some of the trades to which he has committed (provided these trades are in the set Φ of trades the agent may renege on), and pay the corresponding penalty for each trade he buys out.

Theorem 5. *For any $\Phi \subseteq \Omega_i$ and $\Pi_\Phi \in \mathbb{R}^\Phi$, if agent i has fully substitutable preferences, then the valuation function $\hat{u}_i$ with limited liability (as defined in (2)) induces fully substitutable preferences.*

A common assumption in the earlier literature on two-sided matching and exchange economies (see, e.g., Kelso and Crawford (1982) and Gul and Stacchetti (1999)) is that buyers' valuation functions are *monotonic*.[27] Intuitively, monotonicity corresponds to the special case of our setting in which an agent has free disposal, in the sense that he can renege on any trade as a buyer at no cost. More formally, if u_i is fully substitutable, then Theorem 5 implies that we can obtain a fully substitutable and monotonic valuation function $\hat{u}_i$ by allowing the agent to renege on any trade in $\varphi \in \Omega_i$ at a per-trade cost of $\Pi_\varphi = 0$.

6 Properties Equivalent to Full Substitutability

In this section, we discuss several properties of valuation functions that turn out to be equivalent to full substitutability. While these results are of independent interest, some of

[27] In the two-sided settings of Kelso and Crawford (1982) and Gul and Stacchetti (1999), buyers' valuation functions are required to be weakly increasing with respect to the set of workers/objects obtained; that is, monotonicity of the valuation function u_i requires that, for all Ξ and Ψ such that $\Xi \subseteq \Psi \subseteq \Omega_{\to i}$, we have that $u_i(\Psi) \geq u_i(\Xi)$.

them are also useful in applications. For example: The submodularity equivalence we prove in Section 6.1 is used in our proof that substitutability is preserved under trade endowments (Theorem 2). The single-improvement property, introduced in Section 6.2, is useful for efficiently computing the choices of agents with fully substitutable preferences, as it implies that local search for an optimal bundle eventually reaches a global optimum (Paes Leme, 2017). The object-language formulation of full substitutability we develop in Section 6.3 is used in showing that substitutability implies monotone–substitutability, which implies the Laws of Aggregate Supply and Demand (Theorem 10).

6.1 Submodularity of the Indirect Utility Function

A classical approach (see, e.g., the work of Gul and Stacchetti (1999) and Ausubel and Milgrom (2002)) relates substitutability of the utility function to submodularity of the indirect utility function. In particular, every (gross) substitutable utility function corresponds to a submodular indirect utility function and vice versa; here, we generalize this relationship to our setting using an argument that avoids the monotonicity conditions that Gul and Stacchetti (1999) and Sun and Yang (2009) imposed.

For price vectors $p, \bar{p} \in \mathbb{R}^{\Omega}$, let the *join* of p and $\bar{p}$, denoted $p \vee \bar{p}$, be the pointwise maximum of p and $\bar{p}$; let the *meet* of p and $\bar{p}$, denoted $p \wedge \bar{p}$, be the pointwise minimum.

Definition 4. The *indirect utility function* of agent i,

$$V_i(p) \equiv \max_{\Psi \subseteq \Omega_i} \{U_i([\Psi; p])\},$$

is *submodular* if, for all price vectors $p, \bar{p} \in \mathbb{R}^{\Omega}$, we have that

$$V_i(p \wedge \bar{p}) + V_i(p \vee \bar{p}) \leq V_i(p) + V_i(\bar{p}).$$

Just as in two-sided frameworks, fully substitutability corresponds to submodularity of the indirect utility function.

Theorem 6. *The preferences of an agent are fully substitutable if and only if they induce a submodular indirect utility function.*

6.2 The Single Improvement Property

Gul and Stacchetti (1999) first observed (in the setting of exchange economies) that substitutability is equivalent to the *single improvement property*—an agent's preferences are substitutable if and only if, when an agent does not have an optimal bundle, he can make himself better off by adding a single item, dropping a single item, or doing both. Sun and

Yang (2009) extended this result to their setting. Baldwin and Klemperer (2018) showed that in their setting the single improvement property is equivalent to requiring that agents have *complete* preferences.

Definition 5. The preferences of agent i have the *single improvement property* if for any price vector p and set of trades $\Psi \notin D_i(p)$ such that $u_i(\Psi) \neq -\infty$, there exists a set of trades Φ such that

1. $U_i([\Psi, p]) < U_i([\Phi, p])$,
2. there exists at most one trade ω such that $e_{i,\omega}(\Psi) < e_{i,\omega}(\Phi)$, and
3. there exists at most one trade ω such that $e_{i,\omega}(\Psi) > e_{i,\omega}(\Phi)$.[28]

The single improvement property says that, when an agent holds a suboptimal bundle of trades Ψ, that agent can be made be better off by

1. obtaining one item not currently held (either by making a new purchase, i.e., adding a trade in $\Omega_{\to i} \smallsetminus \Psi$, or by canceling a sale, i.e., removing a trade in $\Psi_{i\to}$),
2. relinquishing one item currently held (either by canceling a purchase, i.e., removing a trade in $\Psi_{\to i}$, or by making a new sale, i.e., adding a trade in $\Omega_{i\to} \smallsetminus \Psi$), or
3. both obtaining one item not currently held and relinquishing one item currently held.

For instance, an agent may buy one more input and commit to provide one additional output as a "single improvement."

Theorem 7. *The preferences of an agent are fully substitutable if and only if they have the single improvement property.*

6.3 Object-Language Substitutability

An alternative way of thinking about trades in our setting is to consider each trade as representing the transfer of an underlying object. Under this interpretation, an agent's preferences over trades are fully substitutable if and only if that agent's preferences over objects have the standard Kelso and Crawford (1982) property of gross substitutability.

Formally, we consider each trade $\omega \in \Omega$ as transferring an underlying object from $s(\omega)$ to $b(\omega)$; we denote this underlying object as $\mathfrak{o}(\omega)$. We call the set of all underlying objects $\mathbf{\Omega}$. Hence, after executing the set of trades $\Psi \subseteq \Omega_i$, agent i is left with both the set of objects

[28] Recall that the definition of the generalized indicator function e_i is given in Section 3.3.

corresponding to: the trades in Ψ where i is a buyer, and the trades in $\Omega_i \smallsetminus \Psi$ where i is a seller. We define the set of objects held by agent i after executing the set of trades Ψ as

$$\mathfrak{o}_i(\Psi) = \{\mathfrak{o}(\omega) : \omega \in \Psi_{\to i}\} \cup \{\mathfrak{o}(\omega) : \omega \in \Omega_{i\to} \smallsetminus \Psi_{i\to}\}.$$

Conversely, we define the trade associated with an object $\boldsymbol{\omega}$ as $\mathfrak{t}(\boldsymbol{\omega})$; note that $\mathfrak{t}(\mathfrak{o}(\omega)) = \omega$. We also define the set of trades executed by i for a given set of objects $\boldsymbol{\Psi} \subseteq \boldsymbol{\Omega}_i \equiv \{\boldsymbol{\omega} \in \boldsymbol{\Omega} : i \in \{b(\mathfrak{t}(\boldsymbol{\omega})), s(\mathfrak{t}(\boldsymbol{\omega}))\}$ as

$$\mathfrak{t}_i(\boldsymbol{\Psi}) = \{\omega \in \Omega_{\to i} : \mathfrak{o}(\omega) \in \boldsymbol{\Psi}\} \cup \{\omega \in \Omega_{i\to} : \mathfrak{o}(\omega) \in \boldsymbol{\Omega}_i \smallsetminus \boldsymbol{\Psi}\}.$$

For a partition of objects $\{\boldsymbol{\Psi}^i\}_{i\in I}$, the set of trades that implements this partition is given by

$$\bigcup_{i\in I} \mathfrak{t}_i(\boldsymbol{\Psi}^i).$$

For a set of objects $\boldsymbol{\Psi}$, we let

$$u_i(\boldsymbol{\Psi}) \equiv u_i(\mathfrak{t}_i(\boldsymbol{\Psi})) = u_i([\mathfrak{t}_i(\boldsymbol{\Psi})]_{\to i} \cup [\Omega_i \smallsetminus \mathfrak{t}_i(\boldsymbol{\Psi})]_{i\to}).$$

Using object language, we can also reformulate indicator-language full substitutability (Definition 3) to *object-language full substitutability.*

Definition 6. The preferences of agent i are *object-language fully substitutable (OFS)* if for all price vectors $p, p' \in \mathbb{R}^{\Omega}$ such that $|D_i(p)| = |D_i(p')| = 1$ and $p \leq p'$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, if $\boldsymbol{\omega} \in \mathfrak{o}_i(\Psi)$, then $\boldsymbol{\omega} \in \mathfrak{o}_i(\Psi')$ for each $\boldsymbol{\omega} \in \boldsymbol{\Omega}_i$ such that $p_{\mathfrak{t}(\boldsymbol{\omega})} = p'_{\mathfrak{t}(\boldsymbol{\omega})}$.

Under object-language full substitutability, an increase in the price of object $\boldsymbol{\psi}$ cannot decrease the agent's demand for any object $\boldsymbol{\omega} \neq \boldsymbol{\psi}$. That is, the agent's preferences over objects are grossly substitutable, in the sense of Kelso and Crawford (1982).

We can now interpret the indicator vector $e_{i,\psi}(\Psi)$ as encoding whether the object $\boldsymbol{\psi} = \mathfrak{o}(\psi)$ is transferred under Ψ:

- If $\psi \in \Psi_{\to i}$, then $\boldsymbol{\psi} \in \mathfrak{o}_i(\Psi)$ and $e_{i,\psi}(\Psi) = 1$, i.e., i obtains the object associated with ψ.
- If $\psi \in \Psi_{i\to}$, then $\boldsymbol{\psi} \notin \mathfrak{o}_i(\Psi)$ and $e_{i,\psi}(\Psi) = -1$, i.e., i gives up the object associated with ψ.
- Finally, if $\psi \notin \Psi$, then $e_{i,\psi}(\Psi) = 0$, i.e., i neither obtains nor gives up the object associated with ψ.

Moreover, object-language full substitutability is useful in our proof that fully substitutable preferences satisfy the Laws of Aggregate Supply and Demand (under quasilinear utility).

We can reformulate the definition of the single improvement property in terms of objects.

Definition 5' (Equivalent to Definition 5)**.** The preferences of agent i have the *single improvement property* if for any price vector p and set of trades $\Psi \notin D_i(p)$ such that $u_i(\Psi) \neq -\infty$, there exists a set of trades Φ such that (1) $U^i([\Psi, p]) < U^i([\Phi, p])$, (2) there exists at most one object $\boldsymbol{\varphi} \in \mathfrak{o}_i(\Phi) \smallsetminus \mathfrak{o}_i(\Psi)$, and (3) there exists at most one object $\boldsymbol{\psi} \in \mathfrak{o}_i(\Psi) \smallsetminus \mathfrak{o}_i(\Phi)$.

Using object language, we get a definition of the single improvement property that exactly matches the intuition provided on page 22. The single improvement property says that, when an agent holds a suboptimal bundle of trades Ψ, that agent can be made be better off by (1) obtaining one object $\boldsymbol{\varphi}$ not currently held, i.e., $\boldsymbol{\varphi} \notin \mathfrak{o}_i(\Psi)$, (2) relinquishing one object $\boldsymbol{\psi}$ currently held, i.e., $\boldsymbol{\psi} \in \mathfrak{o}_i(\Psi)$, or (3) both obtaining one object and relinquishing one object.

When substitutability is expressed in terms of preferences over trades, it is necessary to treat relationships between "same-side" and "cross-side" contracts differently. Both Sun and Yang (2006) and Ostrovsky (2008) introduced a concept of cross-side complementarity, which requires that agents treat buy-side contracts as complementary with sell-side contracts (as in our Definitions 1 and 2), which might suggest that there is something fundamentally different between how contracts on one side are interdependent with each other versus how contracts on different sides are interdependent. The representation of preferences in terms of object-language substitutability uncovers that cross-side complementarity is not really a complementarity condition *per se*: rather, it corresponds to an underlying substitutability condition over objects, in the sense that trades are cross-side complements if upstream trades transfer in objects that "substitute" for objects transferred out through downstream trades (and vice versa).

The formalization of substitutability in terms of preferences over objects (Definition 6) thus provides a simple and compact interpretation of full substitutability that does not require treating two sides differently: it simply says that when an agent's object opportunity set shrinks, the agent does not reduce demand for any object that remains in his opportunity set.

The preceding observations make the following result immediate.

Theorem 8. *The preferences of an agent are fully substitutable if and only if they are object-language fully substitutable.*

6.4 M$^\natural$-Concavity over Objects

Fujishige and Yang (2003) showed that gross substitutability in the Kelso and Crawford (1982) model is equivalent to a classical condition from discrete optimization theory, *M$^\natural$-concavity* (Murota (2003); see also Reijnierse et al. (2002)). In our object-language notation, the condition can be stated as follows.

Definition 7. The valuation u_i is *M$^\natural$-concave over objects* if for all $\mathbf{\Phi}, \mathbf{\Psi} \in \mathbf{\Omega}_i$, for any $\boldsymbol{\psi} \in \mathbf{\Psi}$, we have

$$u_i(\mathbf{\Psi}) + u_i(\mathbf{\Phi}) \leq \max\Big\{u_i(\mathbf{\Psi} \smallsetminus \{\boldsymbol{\psi}\}) + u_i(\mathbf{\Phi} \cup \{\boldsymbol{\psi}\}), \max_{\boldsymbol{\varphi} \in \mathbf{\Phi}} \{u_i(\mathbf{\Psi} \cup \{\boldsymbol{\varphi}\} \smallsetminus \{\boldsymbol{\psi}\}) + u_i(\mathbf{\Phi} \cup \{\boldsymbol{\psi}\} \smallsetminus \{\boldsymbol{\varphi}\})\}\Big\}.$$

A valuation function is M$^\natural$-concave if, for any sets of objects $\mathbf{\Psi}$ and $\mathbf{\Phi}$, the sum of $u_i(\mathbf{\Psi})$ and $u_i(\mathbf{\Phi})$ is weakly increased when either we move a given object $\boldsymbol{\psi}$ from $\mathbf{\Psi}$ to $\mathbf{\Phi}$ or we swap $\boldsymbol{\psi}$ for some other object $\boldsymbol{\varphi} \in \mathbf{\Phi}$.

Theorem 9. *The preferences of an agent are fully substitutable if and only if the associated valuation function is M$^\natural$-concave over objects.*

This equivalence result follows from Theorem 7 of Murota and Tamura (2003), which shows that M$^\natural$-concavity is equivalent to the single improvement property—and which in turn, by our Theorem 7, implies the equivalence between full substitutability and M$^\natural$-concavity.

7 Monotone–Substitutability and the Laws of Aggregate Supply and Demand

Hatfield and Milgrom (2005) showed that in two-sided matching markets with transfers and quasilinear utility, all fully substitutable preferences satisfy a monotonicity condition called the Law of Aggregate Demand.[29] The analogues of this condition for the current setting are the *Laws of Aggregate Supply and Demand* for trading networks (Hatfield and Kominers, 2012).

Definition 8. The preferences of agent i satisfy the *Law of Aggregate Demand* if for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $|Z^*_{\to i}| - |Z^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}|$.

The preferences of agent i satisfy the *Law of Aggregate Supply* if for all finite sets of contracts Y and Z such that $Y_{i\to} \subseteq Z_{i\to}$ and $Y_{\to i} = Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $|Z^*_{i\to}| - |Z^*_{\to i}| \geq |Y^*_{i\to}| - |Y^*_{\to i}|$.

Intuitively, the choice correspondence C_i satisfies the Law of Aggregate Demand if, whenever the set of options available to i as a buyer expands, the net demand (i.e., the difference between the number of buy-side contracts chosen and the number of sell-side

[29]In the context of two-sided matching with contracts, the Law of Aggregate Demand is essential for "rural hospitals" and strategy-proofness results (see Hatfield and Milgrom (2005) and Hatfield and Kominers (2013)).

contracts chosen) increases. Similarly, the choice correspondence C_i satisfies the Law of Aggregate Supply if, whenever the set of options available to i as a seller expands, the net supply (i.e., the difference between the number of sell-side contracts chosen and the number of buy-side contracts chosen) increases. The conditions stated in Definition 8 extend the Hatfield and Milgrom (2005) Law of Aggregate Demand (see also Alkan and Gale (2003)) to the current setting, in which each agent can be a buyer in some trades and a seller in others.

One subtle technical issue arises because choice correspondences are not necessarily single-valued in our setting. Under fully substitutable preferences, for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, if $Y^* \in C_i(Y)$ then there exists a $Z^* \in C_i(Z)$ such that $(Y_{\to i} \smallsetminus Y^*_{\to i}) \subseteq (Z_{\to i} \smallsetminus Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$ (see Appendix A). Meanwhile, when the Law of Aggregate Demand is satisfied, for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, if $Y^* \in C_i(Y)$ then there exists a $Z^* \in C_i(Z)$ such that $|Z^*_{\to i}| - |Z^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}|$. However, in principle, it may be the case that there is no Z^* that *simultaneously* satisfies the conditions for full substitutability and the Law of Aggregate Demand. Yet in some applications, it is important to have a single Z^* that simultaneously satisfies both conditions (see, e.g., Hatfield et al. (2017)). Thus we introduce the following stronger condition, called *monotone–substitutability.*

Definition 9. The preferences of agent i are *monotone–substitutable* if:

1. for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $(Y_{\to i} \smallsetminus Y^*_{\to i}) \subseteq (Z_{\to i} \smallsetminus Z^*_{\to i})$, $Y^*_{i\to} \subseteq Z^*_{i\to}$, and $|Z^*_{\to i}| - |Z^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}|$;

2. for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{\to i} = Z_{\to i}$ and $Y_{i\to} \subseteq Z_{i\to}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $(Y_{i\to} \smallsetminus Y^*_{i\to}) \subseteq (Z_{i\to} \smallsetminus Z^*_{i\to})$, $Y^*_{\to i} \subseteq Z^*_{\to i}$, and $|Z^*_{i\to}| - |Z^*_{\to i}| \geq |Y^*_{i\to}| - |Y^*_{\to i}|$.

In our setting, full substitutability implies monotone–substitutability, which in turn implies the Laws of Aggregate Supply and Demand.[30]

Theorem 10. *If the preferences of agent i are fully substitutable, then they are monotone–substitutable.*

Corollary 1. *If the preferences of agent i are fully substitutable, then they satisfy the Laws of Aggregate Supply and Demand.*

Corollary 1 generalizes Theorem 7 of Hatfield and Milgrom (2005) to our setting.

[30]Note that this result relies on our assumption that preferences are quasilinear; meanwhile, in non-quasilinear settings, it is easy to construct preferences that are fully substitutable but do not satisfy the Law of Aggregate Demand (see, e.g., Hatfield and Milgrom (2005, p. 925)).

8 Conclusion

Various forms of substitutability are essential for establishing the existence of equilibria and other useful properties in diverse settings such as matching, auctions, and exchange economies with indivisible goods. We extended earlier models' canonical definitions of substitutability to a setting in which an agent can be both a buyer in some transactions and a seller in others, and showed that all these definitions are equivalent. We introduced a new class of substitutable preferences that allows us to model intermediaries with production capacity. We proved that substitutability is preserved under economically important transformations such as trade endowments and obligations, mergers, and limited liability. We also showed that substitutability corresponds to submodularity of the indirect utility function, the single improvement property, gross substitutability under a suitable transformation (object-language full substitutability), and $M^\natural$-concavity. Finally, we showed that substitutability implies monotone–substitutability, which in turn implies the Laws of Aggregate Supply and Demand. All of our results explicitly incorporate economically important features such as indifferences, non-monotonicities, and unbounded utility functions that were not fully addressed in prior work.

In the current paper, we focused on the full substitutability of the preferences of an *individual* agent. In related work, we have explored the properties of *economies* with multiple agents whose preferences are fully substitutable. That work shows that when all agents' preferences are fully substitutable, outcomes that are *stable* (in the sense of matching theory) exist for any underlying network structure (Hatfield et al., 2013, Theorems 1 and 5). Furthermore, full substitutability of preferences guarantees that the set of stable outcomes is essentially equivalent to the set of competitive equilibria with personalized prices (Hatfield et al., 2013, Theorems 5 and 6) and to the set of *chain stable* outcomes (Hatfield et al., 2017, Theorem 1 and Corollary 1), and that all stable outcomes are in the core and are efficient (Hatfield et al., 2013, Theorem 9). Full substitutability also delineates a maximal domain for the existence of equilibria (Hatfield et al. (2013); see also Gul and Stacchetti (1999) and Yang (2017)): for any domain of preferences strictly larger than that of full substitutability, the existence of stable outcomes and competitive equilibria cannot be guaranteed.

A Definitions of Full Substitutability That Consider Multi-Valued Choices and Demands

In this Appendix, we introduce six alternative definitions of full substitutability: Definitions A.1 and A.2 are analogues of choice-language full substitutability (Definition 1); Definitions

A.3 and A.4 are analogues of demand-language full substitutability (Definition 2); and Definitions A.5 and A.6 are analogues of indicator-language full substitutability (Definition 3). In Appendix A.4, we show that Definitions A.1–A.6 are all equivalent to each other and to CFS, DFS, and IFS.

In contrast to Definitions 1, 2 and 3, which consider single-valued choices and demands, Definitions A.1–A.6 explicitly consider multi-valued correspondences and deal directly with indifferences. By explicitly accounting for indifferences and multi-valued correspondences, we directly generalize the original gross substitutability condition of Kelso and Crawford (1982) to our setting. Moreover, the conditions that explicitly account for indifferences turn out to be useful for proving various results on trading networks, as we discuss below.

A.1 Choice-Language Full Substitutability

Our next two definitions are analogues of Definition 1, explicitly considering indifferences in preferences. The first one states what happens when an agent's set of options expands, and the second one states what happens when the set of options shrinks.

Definition A.1. The preferences of agent i are *choice-language expansion fully substitutable (CEFS)* if:

1. for all finite sets of contracts $Y, Z \subseteq X$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $(Y_{\to i} \smallsetminus Y^*_{\to i}) \subseteq (Z_{\to i} \smallsetminus Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$;

2. for all finite sets of contracts $Y, Z \subseteq X$ such that $Y_{\to i} = Z_{\to i}$ and $Y_{i\to} \subseteq Z_{i\to}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $(Y_{i\to} \smallsetminus Y^*_{i\to}) \subseteq (Z_{i\to} \smallsetminus Z^*_{i\to})$ and $Y^*_{\to i} \subseteq Z^*_{\to i}$.

Definition A.2. The preferences of agent i are *choice-language contraction fully substitutable (CCFS)* if:

1. for all finite sets of contracts $Y, Z \subseteq X$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, for every $Z^* \in C_i(Z)$, there exists $Y^* \in C_i(Y)$ such that $(Y_{\to i} \smallsetminus Y^*_{\to i}) \subseteq (Z_{\to i} \smallsetminus Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$;

2. for all finite sets of contracts $Y, Z \subseteq X$ such that $Y_{\to i} = Z_{\to i}$ and $Y_{i\to} \subseteq Z_{i\to}$, for every $Z^* \in C_i(Z)$, there exists $Y^* \in C_i(Y)$ such that $(Y_{i\to} \smallsetminus Y^*_{i\to}) \subseteq (Z_{i\to} \smallsetminus Z^*_{i\to})$ and $Y^*_{\to i} \subseteq Z^*_{\to i}$.

Note that we use Y as the "starting set" in CEFS and Z as the "starting set" in CCFS to make the two definitions more easily comparable.

Furthermore, note that in Case 1 of CEFS and CCFS, requiring $Y_{\to i} \smallsetminus Y^*_{\to i} \subseteq Z_{\to i} \smallsetminus Z^*_{\to i}$ (i.e., that every buy-side contract not chosen when the smaller set Y is available is still not chosen when the larger Z is available) is equivalent to requiring that $Z^* \cap Y_{\to i} \subseteq Y^*$ (i.e., that every buy-side contract chosen when the larger set Z is available is still chosen if available when the smaller set Y is available). Similarly, in Case 2, requiring $Y_{i\to} \smallsetminus Y^*_{i\to} \subseteq Z_{i\to} \smallsetminus Z^*_{i\to}$ is equivalent to requiring that $Z^* \cap Y_{i\to} \subseteq Y^*$.

It is immediate that both Definitions A.1 and A.2 imply Definition 1 as they impose the same conditions when the choice correspondence is univalent. However, as Theorem A.1 demonstrates, imposing the conditions of Definition 1 to cases where the choice correspondence is univalent is sufficient to recover the stronger definitions discussed here.

A.2 Demand-Language Full Substitutability

Our next two definitions are analogues of Definition 2.

Definition A.3. The preferences of agent i are *demand-language expansion fully substitutable (DEFS)* if:

1. for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$ and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, for every $\Psi \in D_i(p)$ there exists $\Psi' \in D_i(p')$ such that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$;

2. for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\to}$, for every $\Psi \in D_i(p)$ there exists $\Psi' \in D_i(p')$ such that $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ and $\Psi_{\to i} \subseteq \Psi'_{\to i}$.

Definition A.4. The preferences of agent i are *demand-language contraction fully substitutable (DCFS)* if:

1. for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$ and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, for every $\Psi' \in D_i(p')$ there exists $\Psi \in D_i(p)$ such that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$;

2. for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\to}$, for every $\Psi' \in D_i(p')$ there exists $\Psi \in D_i(p)$ such that $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ and $\Psi_{\to i} \subseteq \Psi'_{\to i}$.

Note that we use p as the "starting price vector" in DEFS and p' as the "starting price vector" in DCFS. Also, in Case 1 of DEFS and DCFS, requiring $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ (i.e., any trade demanded at prices p' is still demanded when the prices of other trades rise) is equivalent to requiring that $\{\omega \in (\Omega_{\to i} \smallsetminus \Psi) : p_\omega = p'_\omega\} \subseteq \Omega_{\to i} \smallsetminus \Psi'$ (i.e., any trade not demanded at prices p is still demanded when the prices of other trades fall). Similarly, in Case 2, requiring $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ is equivalent to requiring that $\{\omega \in (\Omega_{i\to} \smallsetminus \Psi) : p_\omega = p'_\omega\} \subseteq \Omega_{i\to} \smallsetminus \Psi'$.

It is immediate that both Definitions A.3 and A.4 imply Definition 2 as they impose the same conditions when the demand correspondence is univalent. However, as Theorem A.1 demonstrates, imposing the conditions of Definition 2 to cases where the demand correspondence is univalent is sufficient to recover the stronger definitions discussed here.

As we mentioned in Footnote 13, DCFS corresponds directly to the gross substitutes and complements condition (GSC) of Sun and Yang (2006). That said, there is a subtlety in interpreting the relationship between the Sun and Yang (2006) model and ours. In the Sun and Yang (2006) model, each agent is treated as buying goods from two separate sets; goods are substitutes within each set, but complements across the two sets. As each agent in the Sun and Yang (2006) model is just a buyer, all goods are taken to have positive prices. In our framework, meanwhile, each agent is treated as the buyer of some trades (upstream) and the seller of others (downstream); we thus use the convention that prices for trades bought are positive, while prices for trades sold are negative. As both in our setting and in that of Sun and Yang (2006) prices themselves are allowed to be either positive or negative, the difference between the sign convention is immaterial: the trades available for sale and purchase in our framework correspond to the two sets of goods in the Sun and Yang (2006) model. Even so, there is a difference in interpretation between the two models: under the convention Sun and Yang (2006) have chosen, we think of the agents as buying all of the goods, whereas in our model, we think of agents more akin to intermediaries who convert inputs into outputs. (For a formal statement of an embedding of the Sun and Yang (2006) model into the trading network framework, see Hatfield et al. (2013, Section IV.B).)

A.3 Indicator-Language Full Substitutability

Our next two definitions are analogues of Definition 3.

Definition A.5. The preferences of agent i are *indicator-language increasing-price fully substitutable (IIFS)* if for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p \le p'$, for every $\Psi \in D_i(p)$ there exists $\Psi' \in D_i(p')$, such that $e_{i,\omega}(\Psi) \le e_{i,\omega}(\Psi')$ for each $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.

Definition A.6. The preferences of agent i are *indicator-language decreasing-price fully substitutable (IDFS)* if for all price vectors $p, p' \in \mathbb{R}^{\Omega}$ such that $p \leq p'$, for every $\Psi' \in D_i(p')$ there exists $\Psi \in D_i(p)$, such that $e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi')$ for each $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.

Definition A.5 considers what happens as prices rise from p to p', requiring that a trade whose price does not change that is bought/not sold by i under p is still bought/not sold by i under p'. By contrast, Definition A.6 considers what happens as prices fall from p' to p, requiring that a trade whose price does not change that is sold/not bought by i under p' is still sold/not bought by i under p.

It is immediate that both Definitions A.5 and A.6 imply Definition 3 as they impose the same conditions when the demand correspondence is univalent. However, as Theorem A.1 demonstrates, imposing the conditions of Definition 3 to cases where the demand correspondence is univalent is sufficient to recover the stronger definitions discussed here.

A.4 Equivalence Result

We now present our most general equivalence, showing that our three main substitutability concepts and the six generalizations introduced in this appendix are all equivalent. In particular, Theorem A.1 implies Theorem 1.

Theorem A.1. *The CFS, DFS, IFS, CEFS, CCFS, DEFS, DCFS, IIFS, and IDFS conditions are all equivalent.*

Proof. We assume throughout that $\Omega = \Omega_i$.[31] To prove Theorem A.1, we prove seven lemmata; we first show that all three demand language concepts of full substitutability are equivalent.

Lemma 1. *The DFS, DEFS, and DCFS conditions are all equivalent.*

Proof. It is immediate that DEFS and DCFS each imply DFS. To complete the proof, we show that DFS implies DEFS and that DFS implies DCFS.

DFS $\Rightarrow$ DEFS: We first show that Part 1 of DFS implies Part 1 of DEFS. Consider two price vectors p, p' such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$ and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, and let $\tilde{\Omega} \equiv \{\omega \in \Omega : p_\omega > p'_\omega\}$; note that $\tilde{\Omega} \subseteq \Omega_{\to i}$. Fix an arbitrary $\Psi \in D_i(p)$; we need to show that there exists a set $\Psi' \in D_i(p')$ that satisfies the requirements of Part 1 of DEFS.

Let q be given by

$$q_\omega = \begin{cases} p_\omega - \varepsilon & \omega \in \Psi_{\to i} \text{ or } \omega \in [\Omega \smallsetminus \Psi]_{i\to} \\ p_\omega + \varepsilon & \omega \in [\Omega \smallsetminus \Psi]_{\to i} \text{ or } \omega \in \Psi_{i\to} \end{cases}$$

[31] This assumption is without loss of generality, as all of the analysis here considers only the sets of trades demanded by i and, for any price vectors p and $\bar{p}$ such that $p_{\Omega_i} = \bar{p}_{\Omega_i}$, we have that $D_i(p) = D_i(\bar{p})$.

for some sufficiently small $\varepsilon > 0$. Let q' be given by

$$q'_\omega = \begin{cases} p'_\omega & \omega \in \tilde{\Omega} \\ q_\omega & \omega \in \Omega \smallsetminus \tilde{\Omega} \end{cases}$$

$$= \begin{cases} p'_\omega & \omega \in \tilde{\Omega} \\ p_\omega - \varepsilon & \omega \in \Psi_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi]_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega + \varepsilon & \omega \in [\Omega \smallsetminus \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in \Psi_{i\to} \smallsetminus \tilde{\Omega}. \end{cases}$$

$$= \begin{cases} p'_\omega & \omega \in \tilde{\Omega} \\ p'_\omega - \varepsilon & \omega \in \Psi_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi]_{i\to} \smallsetminus \tilde{\Omega} \\ p'_\omega + \varepsilon & \omega \in [\Omega \smallsetminus \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in \Psi_{i\to} \smallsetminus \tilde{\Omega}. \end{cases}$$

and let $\Psi' \in D_i(q')$. Let $\bar{q}'$ be given by

$$\bar{q}'_\omega = \begin{cases} q'_\omega - \delta & \omega \in \Psi'_{\to i} \text{ or } \omega \in [\Omega \smallsetminus \Psi']_{i\to} \\ q'_\omega + \delta & \omega \in [\Omega \smallsetminus \Psi']_{\to i} \text{ or } \omega \in \Psi'_{i\to}. \end{cases}$$

$$= \begin{cases} p'_\omega - \delta & \omega \in \Psi'_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi']_{i\to} \cap \tilde{\Omega} \\ p'_\omega + \delta & \omega \in [\Omega \smallsetminus \Psi']_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in \Psi'_{i\to} \cap \tilde{\Omega} \\ p'_\omega - \varepsilon - \delta & \omega \in [\Psi' \cap \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{i\to} \smallsetminus \tilde{\Omega} \\ p'_\omega - \varepsilon + \delta & \omega \in [\Psi \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \smallsetminus \Psi]_{i\to} \smallsetminus \tilde{\Omega} \\ p'_\omega + \varepsilon - \delta & \omega \in [\Psi' \smallsetminus \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ p'_\omega + \varepsilon + \delta & \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \cap \Psi]_{i\to} \smallsetminus \tilde{\Omega} \end{cases}$$

for some sufficiently small $\delta < \varepsilon$. Finally, let $\bar{q}$ be given by

$$\bar{q}_\omega = \begin{cases} q_\omega & \omega \in \tilde{\Omega} \\ \bar{q}'_\omega & \omega \in \Omega \smallsetminus \tilde{\Omega}. \end{cases}$$

$$= \begin{cases} p_\omega - \varepsilon & \omega \in \Psi_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi]_{i\to} \cap \tilde{\Omega} \\ p_\omega + \varepsilon & \omega \in [\Omega \smallsetminus \Psi]_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in \Psi_{i\to} \cap \tilde{\Omega} \\ p_\omega - \varepsilon - \delta & \omega \in [\Psi' \cap \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega - \varepsilon + \delta & \omega \in [\Psi \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \smallsetminus \Psi]_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega + \varepsilon - \delta & \omega \in [\Psi' \smallsetminus \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega + \varepsilon + \delta & \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \cap \Psi]_{i\to} \smallsetminus \tilde{\Omega}. \end{cases}$$

$$
= \begin{cases} q_\omega & \omega \in \Psi_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi]_{i\to} \cap \tilde{\Omega} \\ q_\omega & \omega \in [\Omega \smallsetminus \Psi]_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in \Psi_{i\to} \cap \tilde{\Omega} \\ q_\omega - \delta & \omega \in [\Psi' \cap \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{i\to} \smallsetminus \tilde{\Omega} \\ q_\omega + \delta & \omega \in [\Psi \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \smallsetminus \Psi]_{i\to} \smallsetminus \tilde{\Omega} \\ q_\omega - \delta & \omega \in [\Psi' \smallsetminus \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ q_\omega + \delta & \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \cap \Psi]_{i\to} \smallsetminus \tilde{\Omega}. \end{cases}
$$

We first show five intermediate results on the effects of our price perturbations.

Fact 1: $D_i(q) = \{\Psi\}$. We have, for any $\Phi \neq \Psi$, that[32]

$$
U_i(\Psi; q) - U_i(\Phi; q) = U_i(\Psi; p) - U_i(\Phi; p) + |\Psi \ominus \Phi|\varepsilon \geq |\Psi \ominus \Phi|\varepsilon > 0.
$$

where the equality follows from the definition of q, the first inequality follows from the fact that Ψ is optimal at p, i.e., $\Psi \in D_i(p)$, and the last inequality follows as $\Phi \neq \Psi$. Thus $D_i(q) = \{\Psi\}$.

Fact 2: $D_i(\bar{q}) = \{\Psi\}$. Consider an arbitrary $\Phi \in D_i(q)$ and an arbitrary $\Xi \notin D_i(q)$. We have that

$$
U_i([\Phi; \bar{q}]) - U_i([\Xi; \bar{q}]) \geq U_i([\Phi; q]) - U_i([\Xi; q]) - |\Phi \ominus \Xi|\delta > 0,
$$

where the first inequality follows from the definition of $\bar{q}$ and the second inequality follows as Φ is optimal at q, Ξ is not optimal at q, and δ is sufficiently small. Thus, $\Xi \notin D_i(\bar{q})$ and so $D_i(\bar{q}) \subseteq D_i(q)$. Combining this observation with Fact 1 yields $D_i(\bar{q}) = \{\Psi\}$.

Fact 3: $D_i(q') \subseteq D_i(p')$. Consider an arbitrary $\Phi \in D_i(p')$ and an arbitrary $\Xi \notin D_i(p')$. We have that

$$
U_i([\Phi; q']) - U_i([\Xi; q']) \geq U_i([\Phi; p']) - U_i([\Xi; p']) - |\Phi \ominus \Xi|\varepsilon > 0,
$$

where the first inequality follows from the definition of q' and the second inequality follows as Φ is optimal at p', Ξ is not optimal at p', and ε is sufficiently small. Thus, $\Xi \notin D_i(q')$ and so $D_i(q') \subseteq D_i(p')$.

Fact 4: $D_i(\bar{q}') \subseteq D_i(q')$. Consider an arbitrary $\Phi \in D_i(q')$ and an arbitrary $\Xi \notin D_i(q')$. We have that

$$
U_i([\Phi; \bar{q}']) - U_i([\Xi; \bar{q}']) \geq U_i([\Phi; q']) - U_i([\Xi; q']) - |\Phi \ominus \Xi|\delta > 0,
$$

where the first inequality follows from the definition of q' and the second inequality follows as Φ is optimal at q', Ξ is not optimal at q', and δ is sufficiently small. Thus, $D_i(\bar{q}') \subseteq D_i(q')$.

[32]Here, we use $\ominus$ to denote the symmetric difference between two sets, i.e., $\Psi \ominus \Phi = (\Psi \smallsetminus \Phi) \cup (\Phi \smallsetminus \Psi)$.

Fact 5: $D_i(\bar{q}') = \{\Psi'\}$**.** We have that, for any $\Phi' \neq \Psi'$,

$$U_i(\Psi'; \bar{q}') - U_i(\Phi'; \bar{q}') = U_i(\Psi'; q') - U_i(\Phi; q') + |\Psi' \ominus \Phi'|\delta \geq |\Psi' \ominus \Phi'|\delta > 0.$$

where the equality follows from the definition of $\bar{q}'$, the first inequality follows from the fact that Ψ' is optimal at q', i.e., $\Psi' \in D_i(q')$, and the last inequality follows as $\Phi' \neq \Psi'$. Thus $D_i(\bar{q}') = \{\Psi'\}$.

By Part 1 of DFS, since $D_i(\bar{q}) = \{\Psi\}$ by Fact 2 and $D_i(\bar{q}') = \{\Psi'\}$ by Fact 5, we have that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$. Thus, as $\Psi' \in D_i(p')$ by Facts 3–5, we have that Ψ' satisfies the requirements of Part 1 of DEFS.

The proof that Part 2 of DFS implies Part 2 of DEFS is analogous.

DFS $\Rightarrow$ DCFS: We first show that Part 1 of DFS implies Part 1 of DCFS. Consider two price vectors p, p' such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$ and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, and let $\tilde{\Omega} \equiv \{\omega \in \Omega : p_\omega > p'_\omega\}$; note that $\tilde{\Omega} \subseteq \Omega_{\to i}$. Fix an arbitrary $\Psi' \in D_i(p')$; we need to show that there exists a set $\Psi \in D_i(p)$ that satisfies the requirements of Part 1 of DCFS.

Let q' be given by

$$q'_\omega = \begin{cases} p'_\omega - \varepsilon & \omega \in \Psi'_{\to i} \text{ or } \omega \in [\Omega \smallsetminus \Psi']_{i\to} \\ p'_\omega + \varepsilon & \omega \in [\Omega \smallsetminus \Psi']_{\to i} \text{ or } \omega \in \Psi'_{i\to} \end{cases}$$

for some small $\varepsilon > 0$. Let q be given by

$$\begin{aligned} q_\omega &= \begin{cases} p_\omega & \omega \in \tilde{\Omega} \\ q'_\omega & \omega \in \Omega \smallsetminus \tilde{\Omega} \end{cases} \\ &= \begin{cases} p_\omega & \omega \in \tilde{\Omega} \\ p'_\omega - \varepsilon & \omega \in \Psi'_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ p'_\omega + \varepsilon & \omega \in [\Omega \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in \Psi'_{i\to} \smallsetminus \tilde{\Omega}. \end{cases} \\ &= \begin{cases} p_\omega & \omega \in \tilde{\Omega} \\ p_\omega - \varepsilon & \omega \in \Psi'_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega + \varepsilon & \omega \in [\Omega \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in \Psi'_{i\to} \smallsetminus \tilde{\Omega}. \end{cases} \end{aligned}$$

and let $\Psi \in D_i(q)$. Let $\bar{q}$ be given by

$$\bar{q}_\omega = \begin{cases} q_\omega - \delta & \omega \in \Psi_{\to i} \text{ or } \omega \in [\Omega \smallsetminus \Psi]_{i\to} \\ q_\omega + \delta & \omega \in [\Omega \smallsetminus \Psi]_{\to i} \text{ or } \omega \in \Psi_{i\to}. \end{cases}$$

$$
= \begin{cases} p_\omega - \delta & \omega \in \Psi_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi]_{i\to} \cap \tilde{\Omega} \\ p_\omega + \delta & \omega \in [\Omega \smallsetminus \Psi]_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in \Psi_{i\to} \cap \tilde{\Omega} \\ p_\omega - \varepsilon - \delta & \omega \in [\Psi' \cap \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega - \varepsilon + \delta & \omega \in [\Psi' \smallsetminus \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega + \varepsilon - \delta & \omega \in [\Psi \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \smallsetminus \Psi]_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega + \varepsilon + \delta & \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \cap \Psi]_{i\to} \smallsetminus \tilde{\Omega}. \end{cases}
$$

Finally, let $\bar{q}'$ be given by

$$
\bar{q}'_\omega = \begin{cases} q'_\omega & \omega \in \tilde{\Omega} \\ \bar{q}_\omega & \omega \in \Omega \smallsetminus \tilde{\Omega}. \end{cases}
$$

$$
= \begin{cases} p'_\omega - \varepsilon & \omega \in \Psi'_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi']_{i\to} \cap \tilde{\Omega} \\ p'_\omega + \varepsilon & \omega \in [\Omega \smallsetminus \Psi']_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in \Psi'_{i\to} \cap \tilde{\Omega} \\ p'_\omega - \varepsilon - \delta & \omega \in [\Psi' \cap \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{i\to} \smallsetminus \tilde{\Omega} \\ p'_\omega - \varepsilon + \delta & \omega \in [\Psi' \smallsetminus \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ p'_\omega + \varepsilon - \delta & \omega \in [\Psi \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \smallsetminus \Psi]_{i\to} \smallsetminus \tilde{\Omega} \\ p'_\omega + \varepsilon + \delta & \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \cap \Psi]_{i\to} \smallsetminus \tilde{\Omega}. \end{cases}
$$

$$
= \begin{cases} q'_\omega & \omega \in \Psi'_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi']_{i\to} \cap \tilde{\Omega} \\ q'_\omega & \omega \in [\Omega \smallsetminus \Psi']_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in \Psi'_{i\to} \cap \tilde{\Omega} \\ q'_\omega - \delta & \omega \in [\Psi' \cap \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{i\to} \smallsetminus \tilde{\Omega} \\ q'_\omega + \delta & \omega \in [\Psi' \smallsetminus \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ q'_\omega - \delta & \omega \in [\Psi \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \smallsetminus \Psi]_{i\to} \smallsetminus \tilde{\Omega} \\ q'_\omega + \delta & \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \cap \Psi]_{i\to} \smallsetminus \tilde{\Omega}. \end{cases}
$$

We first show five intermediate results on the effects of our price perturbations.

Fact 1: $D_i(q') = \{\Psi'\}$. We have, for any $\Phi' \neq \Psi'$, that

$$
U_i(\Psi'; q') - U_i(\Phi'; q') = U_i(\Psi'; p') - U_i(\Phi'; p') + |\Psi' \ominus \Phi'|\varepsilon \geq |\Psi' \ominus \Phi'|\varepsilon > 0.
$$

where the equality follows from the definition of q', the first inequality follows from the fact that Ψ' is optimal at p', i.e., $\Psi' \in D_i(p')$, and the last inequality follows as $\Phi' \neq \Psi'$. Thus $D_i(q') = \{\Psi'\}$.

Fact 2: $D_i(\bar{q}') = \{\Psi'\}$. Consider an arbitrary $\Phi \in D_i(q')$ and an arbitrary $\Xi \notin D_i(q')$. For δ

small enough, we have that,

$$U_i([\Phi;\bar{q}']) - U_i([\Xi;\bar{q}']) \geq U_i([\Phi;q']) - U_i([\Xi;q']) - |\Phi \ominus \Xi|\delta > 0,$$

where the first inequality follows from the definition of $\bar{q}'$ and the second inequality follows as Φ is optimal at q', Ξ is not optimal at q', and δ is sufficiently small. Thus, $\Xi \notin D_i(\bar{q}')$ and so $D_i(\bar{q}') \subseteq D_i(q')$. Combining this observation with Fact 1 yields $D_i(\bar{q}') = \{\Psi'\}$.

Fact 3: $D_i(q) \subseteq D_i(p)$**.** Consider an arbitrary $\Phi \in D_i(p)$ and an arbitrary $\Xi \notin D_i(p)$. We have that

$$U_i([\Phi;q]) - U_i([\Xi;q]) \geq U_i([\Phi;p]) - U_i([\Xi;p]) - |\Phi \ominus \Xi|\varepsilon > 0,$$

where the first inequality follows from the definition of q and the second inequality follows as Φ is optimal at p, Ξ is not optimal at p, and ε is sufficiently small. Thus, $\Xi \notin D_i(q)$ and so $D_i(q) \subseteq D_i(p)$.

Fact 4: $D_i(\bar{q}) \subseteq D_i(q)$**.** Consider an arbitrary $\Phi \in D_i(q)$ and an arbitrary $\Xi \notin D_i(q)$. We have that

$$U_i([\Phi;\bar{q}]) - U_i([\Xi;\bar{q}]) \geq U_i([\Phi;q]) - U_i([\Xi;q]) - |\Phi \ominus \Xi|\delta > 0,$$

where the first inequality follows from the definition of $\bar{q}$ and the second inequality follows as Φ is optimal at q, Ξ is not optimal at q, and ε is sufficiently small. Thus, $D_i(\bar{q}) \subseteq D_i(q)$.

Fact 5: $D_i(\bar{q}) = \{\Psi\}$**.** We have that, for any $\Phi \neq \Psi$,

$$U_i(\Psi;\bar{q}) - U_i(\Phi;\bar{q}) = U_i(\Psi;q) - U_i(\Phi;q) + |\Psi \ominus \Phi|\delta \geq |\Psi \ominus \Phi|\delta > 0.$$

where the equality follows from the definition of $\bar{q}$, the first inequality follows from the fact that Ψ is optimal at q, i.e., $\Psi \in D_i(q)$, and the last inequality follows as $\Phi \neq \Psi$. Thus $D_i(\bar{q}) = \{\Psi\}$.

By Part 1 of DFS, since $\{\Psi'\} = D_i(\bar{q}')$ by Fact 2 and $D_i(\bar{q}) = \{\Psi\}$ by Fact 5, we have that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$. Thus, as $\Psi \in D_i(p)$ by Facts 3–5, we have that Ψ satisfies the requirements of Part 1 of DCFS.

The proof that Part 2 of DFS implies Part 2 of DCFS is analogous.

This completes the proof of Lemma 1. □

We now complete the proof of Theorem A.1 by proving that DFS implies CEFS (Lemma 2), DFS implies CCFS (Lemma 3), DFS implies IIFS (Lemma 5), DFS implies IDFS (Lemma 6), CFS implies DFS (Lemma 4), and IFS implies DFS (Lemma 7), as exemplified in Figure 1.

Lemma 2. *If the preferences of agent i satisfy DFS, then they satisfy CEFS.*

$$\begin{tikzcd}
 & CEFS \arrow[dl, Rightarrow] & & IIFS \arrow[dr, Rightarrow] & \\
CFS \arrow[rr, Rightarrow, "\text{Lemma 4}"] & & DFS \arrow[ul, Rightarrow, "\text{Lemma 2}"'] \arrow[ur, Rightarrow, "\text{Lemma 5}"] \arrow[dl, Rightarrow, "\text{Lemma 3}"] \arrow[dr, Rightarrow, "\text{Lemma 6}"'] & & IFS \arrow[ll, Rightarrow, "\text{Lemma 7}"'] \\
 & CCFS \arrow[ul, Rightarrow] & & IDFS \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Figure 1: Proof strategy for Theorem A.1. Any unlabeled implication is immediate.

Proof. Consider two finite sets of contracts Y, Z such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$. Let $Y^* \in C_i(Y)$. We will show that there exists a $Z^* \in C_i(Z)$ such that $(Y_{\to i} \smallsetminus Y^*_{\to i}) \subseteq (Z_{\to i} \smallsetminus Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$.

Let

$$\tilde{Y} = Y \cup \{(\omega, M) \in X : \omega \in \Omega_{\to i}\} \cup \{(\omega, -M) \in X : \omega \in \Omega_{i\to}\}$$
$$\tilde{Z} = Z \cup \{(\omega, M) \in X : \omega \in \Omega_{\to i}\} \cup \{(\omega, -M) \in X : \omega \in \Omega_{i\to}\}$$

where M is sufficiently large so that i would never choose (ω, M) if $\omega \in \Omega_{\to i}$ or $(\omega, -M)$ if $\omega \in \Omega_{i\to}$.[33] It is immediate that $\tilde{Y}_{i\to} = \tilde{Z}_{i\to}$ and $\tilde{Y}_{\to i} \subseteq \tilde{Z}_{\to i}$. It is also immediate that $C_i(Y) = C_i(\tilde{Y})$ and $C_i(Z) = C_i(\tilde{Z})$. Let

$$q^{\tilde{Y}}_\omega = \begin{cases} \min\{p_\omega \in \mathbb{R} : \exists(\omega, p_\omega) \in \tilde{Y}\} & \omega \in \Omega_{\to i} \\ \max\{p_\omega \in \mathbb{R} : \exists(\omega, p_\omega) \in \tilde{Y}\} & \omega \in \Omega_{i\to} \end{cases}$$
$$q^{\tilde{Z}}_\omega = \begin{cases} \min\{p_\omega \in \mathbb{R} : \exists(\omega, p_\omega) \in \tilde{Z}\} & \omega \in \Omega_{\to i} \\ \max\{p_\omega \in \mathbb{R} : \exists(\omega, p_\omega) \in \tilde{Z}\} & \omega \in \Omega_{i\to}; \end{cases}$$

note that $q^{\tilde{Y}}$ and $q^{\tilde{Z}}$ are well-defined as, for every $\omega \in \Omega$, there exists a contract $(\omega, p_\omega) \in \tilde{Y} \subseteq \tilde{Z}$ by construction. Moreover, since $\tilde{Y}_{i\to} = \tilde{Z}_{i\to}$ and $\tilde{Y}_{\to i} \subseteq \tilde{Z}_{\to i}$, we have that $q^{\tilde{Y}}_\omega = q^{\tilde{Z}}_\omega$ for all $\omega \in \Omega_{i\to}$ and $q^{\tilde{Y}}_\omega \geq q^{\tilde{Z}}_\omega$ for all $\omega \in \Omega_{\to i}$.

Let $\Psi = \tau(Y^*)$; we have that $\Psi \in D_i(q^{\tilde{Y}})$. Part 1 of DEFS then implies that there exists a $\Psi' \in D_i(q^{\tilde{Z}})$ such that

$$\begin{aligned} \{\omega \in \Psi'_{\to i} : q^{\tilde{Y}}_\omega = q^{\tilde{Z}}_\omega\} &\subseteq \Psi_{\to i} \\ \Psi_{i\to} &\subseteq \Psi'_{i\to}; \end{aligned} \tag{3}$$

let $Z^* = \kappa[\Psi'; q^{\tilde{Z}}]$; note that $Z^* \in C_i(\tilde{Z}) = C_i(Z)$ as Ψ' is optimal at $q^{\tilde{Z}}$ and $q^{\tilde{Z}}_\omega$ is the best price for ω available to i from $\tilde{Z}$. Thus, we can rewrite (3) as

$$\{\omega \in \tau(Z^*)_{\to i} : q^{\tilde{Y}}_\omega = q^{\tilde{Z}}_\omega\} \subseteq \tau(Y^*)_{\to i} \tag{4}$$

[33]It is always possible to find M large enough as utility is bounded from above and $u_i(\varnothing) \in \mathbb{R}$.

$$\tau(Y^*)_{i\to} \subseteq \tau(Z^*)_{i\to};$$

If $(\omega, p_\omega) \in [Y \smallsetminus Y^*]_{\to i}$, then either:

- $\omega \notin \tau(Y^*) = \Psi$ and so either $\omega \notin \tau(Z^*)$ or $q_\omega^{\tilde{Y}} \neq q_\omega^{\tilde{Z}}$ by (4). In the former case, it is immediate that $(\omega, p_\omega) \notin Z^*_{\to i}$; in the later case, since $q_\omega^{\tilde{Y}} \geq q_\omega^{\tilde{Z}}$, we must have that $q_\omega^{\tilde{Y}} > q_\omega^{\tilde{Z}}$ and so there exists a $(\omega, \bar{p}_\omega) \in Z$ such that $\bar{p}_\omega < p_\omega$, and therefore $(\omega, p_\omega) \notin Z^*_{\to i}$.

- $\omega \in \tau(Y^*)$ but there exists $(\omega, \bar{p}_\omega) \in Y$ such that $\bar{p}_\omega < p_\omega$. In this case, $(\omega, \bar{p}_\omega) \in Z$ as $Y \subseteq Z$, and therefore $(\omega, p_\omega) \notin Z^*_{\to i}$.

Thus, $[Y \smallsetminus Y^*]_{\to i} \subseteq [Z \smallsetminus Z^*]_{\to i}$.

If $(\omega, p_\omega) \in Y^*_{i\to}$, then $\omega \in \tau(Z^*)$ by (4). Moreover, if $(\omega, p_\omega) \in Y^*_{i\to}$ then p_ω is the maximal price in Y for ω and so, as $Y^*_{i\to} = Z^*_{i\to}$, we have that p_ω is the maximal price in Z for ω. Combining these last two observations implies that $(\omega, p_\omega) \in Z^*_{i\to}$, and so $Y^*_{i\to} \subseteq Z^*_{i\to}$.

Thus, Z^* satisfies all the requirements of Part 1 of CEFS.

The proof that DFS implies Part 2 of CEFS is analogous. □

Lemma 3. *If the preferences of agent i satisfy DFS, then they satisfy CCFS.*

Proof. The proof proceeds analogously to the proof of Lemma 2. □

Lemma 4. *If the preferences of agent i satisfy CFS, then they satisfy DFS.*

Proof. We first show that Part 1 of CFS implies Part 1 of DFS. For any agent i and price vector $p \in \mathbb{R}^\Omega$, let

$$X_i(p) \equiv \{(\omega, \hat{p}_\omega) : \omega \in \Omega_{\to i} \text{ and } \hat{p}_\omega \geq p_\omega\} \cup \{(\omega, \hat{p}_\omega) : \omega \in \Omega_{i\to} \text{ and } \hat{p}_\omega \leq p_\omega\};$$

that is, $X_i(p)$ effectively denotes the set of contracts available to agent i under prices p.[34]

Let the price vectors $p, p' \in \mathbb{R}^\Omega$ be such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and $p'_\omega \leq p_\omega$ for all $\omega \in \Omega_{\to i}$; let $\{\Psi\} = D_i(p)$ and $\{\Psi'\} = D_i(p')$. Let $Y = X_i(p)$ and $Z = X_i(p')$. Clearly, $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$. Furthermore, it is immediate that $\{\kappa([\Psi; p])\} = C_i(Y)$, and similarly, $\{\kappa([\Psi; p'])\} = C_i(Z)$. Thus, Part 1 of CFS implies that

$$Y_{\to i} \smallsetminus [\kappa([\Psi; p])]_{\to i} \subseteq Z_{\to i} \smallsetminus [\kappa([\Psi'; p'])]_{\to i} \tag{5}$$

$$[\kappa([\Psi; p])]_{i\to} \subseteq [\kappa([\Psi'; p'])]_{i\to}. \tag{6}$$

From (5), we see that, if $\omega \in \tau([\kappa([\Psi'; p'])]_{\to i})$, i.e., if $\omega \in \Psi'_{\to i}$, and $p'_\omega = p_\omega$, then $(\omega, p'_\omega) \in [\kappa([\Psi; p])]_{\to i}$, and so $\omega \in \Psi_{\to i}$—in other words, $\{\omega \in \Psi'_{\to i} : p'_\omega = p_\omega\} \subseteq \Psi_{\to i}$. Furthermore, as

[34]By this, we mean that, in principle, an agent pay more for an upstream trade and receive less for a downstream trade.

$[\kappa([\Psi; p])]_{i\to} \subseteq [\kappa([\Psi'; p'])]_{i\to}$ by (6) and $p_\omega = p'_\omega$ for each $\omega \in \Omega_{i\to}$, we have that $\Psi'_{i\to} \subseteq \Psi_{i\to}$. Thus, Ψ' satisfies the requirements of Part 1 of DFS.

The proof that Part 2 of CFS implies Part 2 of DFS is analogous. $\square$

Lemma 5. *If the preferences of agent i satisfy DFS, then they satisfy IIFS.*

Proof. It is enough to show that DEFS and DCFS jointly imply IIFS, as DFS implies both DEFS and DCFS by Lemma 1. Take two price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p \leq p'$, and let $\Psi \in D_i(p)$ be arbitrary. We will show that there exists a set of trades $\Psi' \in D_i(p')$ such that $e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi')$ for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$. We let

$$p^\star_\omega = \begin{cases} p'_\omega & \omega \in \Omega_{\to i} \\ p_\omega & \omega \in \Omega_{i\to}; \end{cases}$$

thus, $p^\star_\omega = p_\omega$ for all $\omega \in \Omega_{i\to}$ and $p^\star_\omega \geq p_\omega$ for all $\omega \in \Omega_{\to i}$. Part 1 of DCFS then implies that there exists a $\Psi^\star \in D_i(p^\star)$ such that

$$\begin{aligned} \{\omega \in \Psi_{\to i} : p_\omega = p^\star_\omega\} &\subseteq \Psi^\star_{\to i} \\ \Psi^\star_{i\to} &\subseteq \Psi_{i\to}. \end{aligned} \tag{7}$$

Now, note that $p^\star_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p^\star_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\to}$. Part 2 of DEFS then implies that there exists a $\Psi' \in D_i(p')$ such that

$$\begin{aligned} \{\omega \in \Psi'_{i\to} : p^\star_\omega = p'_\omega\} &\subseteq \Psi^\star_{i\to} \\ \Psi^\star_{\to i} &\subseteq \Psi'_{\to i}. \end{aligned} \tag{8}$$

Combining (7) and (8) yields

$$\begin{aligned} \{\omega \in \Psi_{\to i} : p_\omega = p^\star_\omega\} &\subseteq \Psi^\star_{\to i} \subseteq \Psi'_{\to i} \\ \{\omega \in \Psi'_{i\to} : p^\star_\omega = p'_\omega\} &\subseteq \Psi^\star_{i\to} \subseteq \Psi_{i\to}. \end{aligned}$$

Recalling the definition of $p^\star$, we obtain

$$\begin{aligned} \{\omega \in \Psi_{\to i} : p_\omega = p'_\omega\} &\subseteq \Psi'_{\to i} \\ \{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} &\subseteq \Psi_{i\to}; \end{aligned}$$

this implies $e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi')$ for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$. $\square$

Lemma 6. *If the preferences of agent i satisfy DFS, then they satisfy ICFS.*

Proof. It is enough to show that DEFS and DCFS jointly imply IDFS, as DFS implies both DEFS and DCFS by Lemma 1. Take two price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p \leq p'$, and let

$\Psi' \in D_i(p')$ be arbitrary. We will show that there exists a set of trades $\Psi \in D_i(p)$ such that $e_{i,\omega}(\Psi) \le e_{i,\omega}(\Psi')$ for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$. Let

$$p^\star_\omega = \begin{cases} p'_\omega & \omega \in \Omega_{\to i} \\ p_\omega & \omega \in \Omega_{i\to}; \end{cases}$$

thus, $p^\star_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p^\star_\omega \le p'_\omega$ for all $\omega \in \Omega_{i\to}$. Part 2 of DCFS then implies that there exists a $\Psi^\star \in D_i(p^\star)$ such that

$$\begin{aligned} \{\omega \in \Psi'_{i\to} : p^\star_\omega = p'_\omega\} &\subseteq \Psi^\star_{i\to} \\ \Psi^\star_{\to i} &\subseteq \Psi'_{\to i}. \end{aligned} \tag{9}$$

Now, note that $p^\star_\omega = p_\omega$ for all $\omega \in \Omega_{i\to}$ and $p^\star_\omega \ge p_\omega$ for all $\omega \in \Omega_{\to i}$. Part 1 of DEFS then implies that there exists a $\Psi \in D_i(p)$ such that

$$\begin{aligned} \{\omega \in \Psi_{\to i} : p^\star_\omega = p_\omega\} &\subseteq \Psi^\star_{\to i} \\ \Psi^\star_{i\to} &\subseteq \Psi_{i\to}. \end{aligned} \tag{10}$$

Combining (9) and (10) yields

$$\begin{aligned} \{\omega \in \Psi'_{i\to} : p^\star_\omega = p'_\omega\} &\subseteq \Psi^\star_{i\to} \subseteq \Psi_{i\to} \\ \{\omega \in \Psi_{\to i} : p^\star_\omega = p_\omega\} &\subseteq \Psi^\star_{\to i} \subseteq \Psi'_{\to i}. \end{aligned}$$

Recalling the definition of $p^\star$, we obtain

$$\begin{aligned} \{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} &\subseteq \Psi_{i\to} \\ \{\omega \in \Psi_{\to i} : p_\omega = p_\omega\} &\subseteq \Psi'_{\to i}; \end{aligned}$$

this implies $e_{i,\omega}(\Psi) \le e_{i,\omega}(\Psi')$ for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$. □

Lemma 7. *If the preferences of agent i satisfy IFS, then they satisfy DFS.*

Proof. Let the price vectors $p, p' \in \mathbb{R}^\Omega$ be such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and $p'_\omega \le p_\omega$ for all $\omega \in \Omega_{\to i}$; let $\{\Psi\} = D_i(p)$ and $\{\Psi'\} = D_i(p')$. As the preferences of i satisfy the IFS condition, we have that $e_{i,\omega}(\Psi') \le e_{i,\omega}(\Psi)$ for all $\omega \in \Omega_{\to i}$ such that $p_\omega = p'_\omega$. Thus, if $p_\omega = p'_\omega$ and $\omega \in \Psi'$ then $\omega \in \Psi$ and so we have that $\{\omega \in \Psi'_{\to i} : p'_\omega = p_\omega\} \subseteq \Psi_{\to i}$. Moreover, as the preferences of i satisfy the IFS condition, we have that $e_{i,\omega}(\Psi') \le e_{i,\omega}(\Psi)$ for all $\omega \in \Omega_{i\to}$ such that $p_\omega = p'_\omega$. Thus, if $p_\omega = p'_\omega$ and $\omega \in \Psi$ then $\omega \in \Psi'$ and so, as $p_\omega = p'_\omega$ for all $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, we have that $\Psi_{\to i} \subseteq \Psi'_{\to i}$.

The proof that Part 2 of IFS implies Part 2 of DFS is analogous. □

This concludes the proof of Theorem A.1. □

B Proofs of the Results Presented in Sections 4–7

Proof of Proposition 1

Consider the intermediary i. Let Φ (with a typical element φ) denote the set of potential inputs this intermediary faces, and let Ψ (with a typical element ψ) denote the set of potential requests. The cost of using input φ to satisfy request ψ is given by $c_{\varphi,\psi}$. For convenience, when φ and ψ are incompatible, we simply say that $c_{\varphi,\psi} = +\infty$.

Let us now construct a "synthetic" agent $\hat{\imath}$ whose preferences will be identical to those of agent i, yet will be represented in the form of "intermediary with production capacity" preferences as defined in Section 4.2. The full substitutability of the preferences of intermediary i will then follow immediately from Proposition 2.

Agent $\hat{\imath}$ faces the same sets of inputs, Φ, and requests, Ψ, as agent i. Agent $\hat{\imath}$ also has $|\Phi| \times |\Psi|$ machines, indexed by pairs of inputs and requests: machine $m_{\varphi,\psi}$ "corresponds" to an input–request pair (φ, ψ). The costs of intermediary $\hat{\imath}$ are as follows (to avoid confusion, we will denote various costs of agent $\hat{\imath}$ by "$\hat{c}$" with various subindices, while the costs of agent i are denoted by "c" with various subindices):

- For input φ and machine $m_{\varphi,\psi}$ "corresponding" to input φ and some request ψ, the cost $\hat{c}_{\varphi,m_{\varphi,\psi}}$ of using input φ in machine $m_{\varphi,\psi}$ is equal to $c_{\varphi,\psi}$—the cost of using input φ to satisfy request ψ under the original cost structure of agent i.

- For any input $\varphi' \neq \varphi$ and any request ψ, the cost $\hat{c}_{\varphi',m_{\varphi,\psi}}$ is equal to $+\infty$.

- For request ψ and any machine $m_{\varphi,\psi}$ "corresponding" to request ψ and some input φ, the cost $\hat{c}_{m_{\varphi,\psi},\psi}$ of using machine $m_{\varphi,\psi}$ to satisfy request ψ is equal to 0.

- For any request $\psi' \neq \psi$ and any machine $m_{\varphi,\psi}$, the cost $\hat{c}_{m_{\varphi,\psi},\psi'}$ is equal to $+\infty$.

With this construction, the preferences of agents i and $\hat{\imath}$ over sets of inputs and requests are identical. Moreover, the preferences of agent $\hat{\imath}$ are those of "intermediary with production capacity" and are thus fully substitutable (by Proposition 2). Therefore, the "intermediary" preferences of agent i are also fully substitutable.

Proof of Proposition 2

Consider first an "intermediary with production capacity" who has exactly one machine at his disposal. It is immediate that the preferences of such an intermediary are fully substitutable.

Next, consider a general "intermediary with production capacity", i, who has a set of machines M (with a typical element m) at his disposal and faces the set of inputs Φ (with a typical element φ) and the set of potential requests Ψ (with a typical element ψ), with costs as described in Section 4.2. We will show that the preferences of intermediary i can be represented as a "merger" of several (specifically, $|M| + |\Phi| + |\Psi|$) agents with fully substitutable preferences, which by Theorem 4 will imply that the preferences of intermediary i are fully substitutable.

Specifically, consider the following set of artificial agents. First, there are $|\Phi|$ "input dummies", with a typical element $\hat{\varphi}$ for a dummy that corresponds to input φ. Second, there are $|M|$ "machine dummies", with a typical element $\hat{m}$ for a dummy that corresponds to machine m. Finally, there are $|\Psi|$ "request dummies", with a typical element $\hat{\psi}$ for a dummy that corresponds to request ψ.

Each input dummy $\hat{\varphi}$ can only buy one trade: input φ. He can also form $|M|$ trades as a seller: one trade with every machine dummy $\hat{m}$. We denote the trade between an input dummy $\hat{\varphi}$ (as the seller) and a machine dummy $\hat{m}$ (as the buyer) by $\omega_{\varphi,m}$. Likewise, each request dummy $\hat{\psi}$ can only sell one trade: request ψ. He can also form $|M|$ trades as a buyer: one trade with every machine dummy $\hat{m}$. We denote the trade between a machine dummy $\hat{m}$ (as the seller) and a request dummy $\hat{\psi}$ (as the buyer) by $\omega_{m,\psi}$. Each machine dummy can thus form $|\Phi|$ trades as the buyer (one with each input dummy) and $|\Psi|$ trades as the seller (one with each request dummy).

The preferences of the agents are as follows. Each input dummy and each request dummy has valuation 0 if the number of trades he forms as the seller is equal to the number of trades he forms as the buyer (this number can thus be equal to either 0 or 1), and $-\infty$ if these numbers are not equal. It is immediate that the preferences of input and request dummies are fully substitutable.

The preferences of each machine dummy $\hat{m}$ are as follows. If it buys no trades and sells no trades, its valuation is 0. If it buys exactly one trade, say $\omega_{\varphi,m}$ for some φ, and sells exactly one trade, say $\omega_{m,\psi}$ for some ψ, then its valuation is $-(c_{\varphi,m} + c_{m,\psi})$—the total cost of preparing input φ for request ψ using machine m in the original construction of the utility function of agent i. In all other cases (i.e., when the machine dummy buys or sells more than two trades, or when the number of trades it buys is not equal to the number of trades it sells), the valuation of the machine dummy is $-\infty$. Note that the preferences of the machine dummy are also fully substitutable.

Consider now the "synthetic" agent $\hat{\imath}$ obtained as the merger of the $|\Phi|$ input dummies, $|M|$ machine dummies, and $|\Psi|$ request dummies (see Section 5.2 for the details of the "merger" operation). By Theorem 4, the preferences of agent $\hat{\imath}$ are fully substitutable. Moreover, the valuation of agent $\hat{\imath}$ over any bundle of inputs and requests is identical to the valuation of agent i over that bundle. Thus, the preferences of agent i are fully substitutable.

Proof of Theorem 2

The indirect utility function for $\hat{u}_i^{(\Phi,p_\Phi)}$ is given by

$$
\begin{aligned}
\hat{V}_i^{(\Phi,p_\Phi)}(p_{\Omega\smallsetminus\Phi}) &\equiv \max_{\Psi\subseteq\Omega\smallsetminus\Phi}\left\{\max_{\Xi\subseteq\Phi}\left\{u_i(\Psi\cup\Xi)+\sum_{\xi\in\Xi_{\to i}}p_\xi-\sum_{\xi\in\Xi_{\to i}}p_\xi\right\}+\sum_{\psi\in\Psi_{\to i}}p_\xi-\sum_{\psi\in\Psi_{\to i}}p_\xi\right\}\\
&=\max_{\Psi\subseteq\Omega\smallsetminus\Phi}\left\{\max_{\Xi\subseteq\Phi}\left\{u_i(\Psi\cup\Xi)+\sum_{\lambda\in\Xi_{\to i}\cup\Psi_{\to i}}p_\lambda-\sum_{\lambda\in\Xi_{i\to}\cup\Psi_{i\to}}p_\lambda\right\}\right\}\\
&=\max_{\Lambda\subseteq\Omega}\left\{u_i(\Lambda)+\sum_{\lambda\in\Lambda_{\to i}}p_\lambda-\sum_{\lambda\in\Lambda_{i\to}}p_\lambda\right\}.
\end{aligned}
$$

Hence, $\hat{V}_i^{(\Phi,p_\Phi)}(p_{\Omega\smallsetminus\Phi})=V_i(p_{\Omega\smallsetminus\Phi},p_\Phi)$. Now, $V_i(p)$ is submodular over $\mathbb{R}^\Omega$ by Theorem 6. As a submodular function restricted to a subspace of its domain is still submodular, $\hat{V}_i^{(\Phi,p_\Phi)}(p_{\Omega\smallsetminus\Phi})$ is submodular over $\mathbb{R}^{\Omega\smallsetminus\Phi}$. Hence, by Theorem 6, we see that $\hat{u}_i^{(\Phi,p_\Phi)}$ is fully substitutable.

Proof of Theorem 3

Fix a set of trades $\Phi\subseteq\Omega_i$ such that $u_i(\Phi)\neq-\infty$ and a vector of prices $\overline{p}_\Phi$ for trades in Φ. Let $\tilde{D}_i$ be the demand function for trades in $\Omega\smallsetminus\Phi$ induced by $\tilde{u}_i^{\Phi,\overline{p}_\Phi}$. Fix two price vectors $p\in\mathbb{R}^{\Omega\smallsetminus\Phi}$ and $p'\in\mathbb{R}^{\Omega\smallsetminus\Phi}$ such that $|\tilde{D}_i(p)|=|\tilde{D}_i(p')|=1$, $p_\omega=p'_\omega$ for all $\omega\in\Omega_{i\to}\smallsetminus\Phi$, and $p_\omega\geq p'_\omega$ for all $\omega\in\Omega_{\to i}\smallsetminus\Phi$. Let $\Psi\in\tilde{D}_i(p)$ be the unique demanded set from $\Omega_i\smallsetminus\Phi$ at p and $\Psi'\in\tilde{D}_i(p')$ be the unique demanded set from $\Omega_i\smallsetminus\Phi$ at p'. Note that since $u_i(\Phi)\neq-\infty$, there exists a vector of prices p^*_Φ for trades in Φ such that, for all $\Xi\in D_i((p,p^*_\Phi))\cup D_i((p',p^*_\Phi))$, we have $\Phi\subseteq\Xi$. Fix an arbitrary $\Xi\in D_i((p,p^*_\Phi))$ and let $\tilde{\Psi}\equiv\Xi\smallsetminus\Phi$.

Claim 1. *We must have $\tilde{\Psi}=\Psi$.*

Proof. Suppose the contrary. Since $\tilde{\Psi}\cup\Phi=\Xi\in D_i((p,p^*_\Phi))$, we must have

$$
\begin{aligned}
u_i(\Xi)&=u_i(\tilde{\Psi}\cup\Phi)+\sum_{\psi\in\tilde{\Psi}_{i\to}}p_\psi-\sum_{\psi\in\tilde{\Psi}_{\to i}}p_\psi+\sum_{\varphi\in\Phi_{i\to}}p^*_\varphi-\sum_{\varphi\in\Phi_{\to i}}p^*_\varphi\\
&\geq u_i(\Psi\cup\Phi)+\sum_{\psi\in\Psi_{i\to}}p_\psi-\sum_{\psi\in\Psi_{\to i}}p_\psi+\sum_{\varphi\in\Phi_{i\to}}p^*_\varphi-\sum_{\varphi\in\Phi_{\to i}}p^*_\varphi. \qquad (11)
\end{aligned}
$$

The inequality (11) is equivalent to

$$
\begin{aligned}
&u_i(\tilde{\Psi}\cup\Phi)+\sum_{\psi\in\tilde{\Psi}_{i\to}}p_\psi-\sum_{\psi\in\tilde{\Psi}_{\to i}}p_\psi+\sum_{\varphi\in\Phi_{i\to}}\overline{p}_\varphi-\sum_{\varphi\in\Phi_{\to i}}\overline{p}_\varphi\\
&\qquad\geq u_i(\Psi\cup\Phi)+\sum_{\psi\in\Psi_{i\to}}p_\psi-\sum_{\psi\in\Psi_{\to i}}p_\psi+\sum_{\varphi\in\Phi_{i\to}}\overline{p}_\varphi-\sum_{\varphi\in\Phi_{\to i}}\overline{p}_\varphi. \qquad (12)
\end{aligned}
$$

However, the inequality (12) implies that $\tilde{\Psi}\in\tilde{D}_i(p)$; this contradicts the assumption that $\tilde{D}_i(p)=\{\Psi\}$ given that $\tilde{\Psi}\neq\Psi$. $\square$

The preceding claim implies that we must have $D_i((p, p^*_\Phi)) = \{\Xi\} = \{\tilde{\Psi} \cup \Phi\} = \{\Psi \cup \Phi\}$. A similar argument shows that $D_i((p', p^*_\Phi)) = \{\Psi' \cup \Phi\}$. The full substitutability of u_i then implies that $\{\psi \in \Psi'_{\to i} : p_\psi = p'_\psi\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$.

Proof of Theorem 4

We suppose, by way of contradiction, that u_J does not induce fully substitutable preferences over trades in $\Omega \smallsetminus \Omega^J$. By Corollary 1 of Hatfield et al. (2013), there exist fully substitutable preferences $\tilde{u}_i$ for the agents $i \in I \smallsetminus J$ such that no competitive equilibrium exists for the *modified economy* with set of agents $(I \smallsetminus J) \cup \{J\}$, set of trades $\Omega \smallsetminus \Omega^J$, and valuation function for agent J given by u_J.[35]

Now, we consider the *original economy* with set of agents I, set of trades Ω, valuations for $i \in I \smallsetminus J$ given by $\tilde{u}_i$, and valuations for $j \in J$ given by u_j. Let $[\Psi; p]$ be a competitive equilibrium of this economy; such an equilibrium exists by Theorem 1 of Hatfield et al. (2013).

Claim 2. $[\Psi \smallsetminus \Omega^J; p_{\Omega \smallsetminus \Omega^J}]$ *is a competitive equilibrium of the modified economy.*

Proof. It is immediate that $[\Psi \smallsetminus \Omega^J]_i \in D_i(p_{\Omega \smallsetminus \Omega^J})$ for all $i \in I \smallsetminus J$. Moreover, since Ψ is individually-optimal for each $j \in J$ in the original economy (at prices p),

$$u_j(\Psi) + \sum_{\psi \in \Psi_{j\to}} p_\psi - \sum_{\psi \in \Psi_{\to j}} p_\psi \geq u_j(\Phi) + \sum_{\varphi \in \Phi_{j\to}} p_\varphi - \sum_{\varphi \in \Phi_{\to j}} p_\varphi \tag{13}$$

for every $\Phi \subseteq \Omega$. Summing (13) over all $j \in J$ and observing that when doing so the prices paid and received by agents in J for the trades Ω^J among themselves cancel out, we obtain

$$\sum_{j \in J} u_j(\Psi) + \sum_{\psi \in [\Psi \smallsetminus \Omega^J]_{J\to}} p_\psi - \sum_{\psi \in [\Psi \smallsetminus \Omega^J]_{\to J}} p_\psi \geq \sum_{j \in J} u_j(\Phi) + \sum_{\varphi \in [\Phi \smallsetminus \Omega^J]_{J\to}} p_\varphi - \sum_{\varphi \in [\Phi \smallsetminus \Omega^J]_{\to J}} p_\psi. \square$$

The preceding claim shows that $[\Psi \smallsetminus \Omega^J; p_{\Omega \smallsetminus \Omega^J}]$ is a competitive equilibrium of the modified economy, contradicting the earlier conclusion that no competitive equilibrium exists in the modified economy. Hence, we see that u_J must be fully substitutable.

Proof of Theorem 5

The proof of this result is very close to Step 1 of the proof of Theorem 1 of Hatfield et al. (2013). The only differences are that in the Hatfield et al. (2013) results, all trades could be bought out, and the price for buying them out was set to a single large number that was

[35] To apply Corollary 1 of Hatfield et al. (2013), we must have that for every pair (i, j) of distinct agents in I, there exists a trade ω such that $b(\omega) = i$ and $s(\omega) = j$. For any pair (i, j) of distinct agents in I such that no such trade ω exists, we augment the economy by adding a trade ω and, if $i \in J$, letting $\bar{u}_i(\Psi \cup \{\omega\}) = u^i(\Psi)$ (and similarly for j). It is immediate that $\bar{u}_i$ is substitutable if and only if u_i is substitutable.

the same for all trades. By contrast, in Theorem 5 of the current paper we allow for the possibility that only a subset of trades can be bought out, and that the prices at which these trades can be bought out can be different, and need not be large. For completeness, we adapt Step 1 of the proof of Theorem 1 of Hatfield et al. (2013) to the current more general setting.

Consider the fully substitutable valuation function u_i, and take any trade $\varphi \in \Omega_{i\to} \cap \Phi$. Consider a modified valuation function u_i^φ:

$$u_i^\varphi(\Psi) = \max\{u_i(\Psi), u_i(\Psi \smallsetminus \{\varphi\}) - \Pi_\varphi\}.$$

That is, the valuation $u_i^\varphi(\Psi)$ allows (but does not require) agent i to pay Π_φ instead of executing one particular trade, φ.

Claim 3. *The valuation function u_i^φ is fully substitutable.*

Proof. Consider utility U_i^φ and demand D_i^φ corresponding to valuation u_i^φ. We show that D_i^φ satisfies the IFS condition (Definition 3). Fix two price vectors p and p' such that $p \le p'$ and $|D_i^\varphi(p)| = |D_i^\varphi(p')| = 1$. Take the unique $\Psi \in D_i^\varphi(p)$ and $\Psi' \in D_i^\varphi(p')$. We need to show that

$$e_{i,\omega}(\Psi) \le e_{i,\omega}(\Psi') \text{ for all } \omega \in \Omega_i \text{ such that } p_\omega = p'_\omega. \tag{14}$$

Let price vector q coincide with p on all trades other than φ, and set $q_\varphi = \min\{p_\varphi, \Pi_\varphi\}$. Note that if $p_\varphi < \Pi_\varphi$, then $p = q$ and $D_i^\varphi(p) = D_i(p)$. If $p_\varphi > \Pi_\varphi$, then under utility U_i^φ, agent i always wants to execute trade φ at price p_φ, and the only decision is whether to "buy it out" or not at the cost Π_φ; i.e., the agent's effective demand is the same as under price vector q. Thus, $D_i^\varphi(p) = \{\Xi \cup \{\varphi\} : \Xi \in D_i(q)\}$. Finally, if $p_\varphi = \Pi_\varphi$, then $p = q$ and $D_i^\varphi(p) = D_i(p) \cup \{\Xi \cup \{\varphi\} : \Xi \in D_i(p)\}$. We construct price vector q' corresponding to p' analogously.

Now, if $p_\varphi \le p'_\varphi < \Pi_\varphi$, then $D_i^\varphi(p) = D_i(p)$, $D_i^\varphi(p') = D_i(p')$, and thus $e_{i,\omega}(\Psi) \le e_{i,\omega}(\Psi')$ follows directly from IFS for demand D_i.

If $\Pi_\varphi \le p_\varphi \le p'_\varphi$, then (since we assumed that D_i^φ was single-valued at p and p') it has to be the case that D_i is single-valued at the corresponding price vectors q and q'. Let $\Xi \in D_i(q)$ and $\Xi' \in D_i(q')$. Then $\Psi = \Xi \cup \{\varphi\}$, $\Psi' = \Xi' \cup \{\varphi\}$, and statement (14) follows from the IFS condition for demand D_i, because $q \le q'$.

Finally, if $p_\varphi < \Pi_\varphi \le p'_\varphi$, then $p = q$, Ψ is the unique element in $D_i(p)$, and Ψ' is equal to $\Xi' \cup \{\varphi\}$, where Ξ' is the unique element in $D_i(q')$. Then for $\omega \ne \varphi$, statement (14) follows from IFS for demand D_i, because $p \le q'$. For $\omega = \varphi$, statement (14) does not need to be checked, because $p_\varphi < p'_\varphi$.

Thus, when $\varphi \in \Omega_{i\to}$, the valuation function u_i^φ is fully substitutable. The proof for the case when $\varphi \in \Omega_{\to i}$ is completely analogous. $\square$

To complete the proof, note that the valuation $\hat{u}_i(\Psi) = \max_{\Xi \subseteq \Psi \cap \Phi}\left\{u_i(\Psi \smallsetminus \Xi) - \sum_{\varphi \in \Xi} \Pi_\varphi\right\}$ can be obtained from the original valuation u_i by allowing agent i to "buy out" all of the trades in set Φ, one by one, and since the preceding claim shows that each such transformation preserves substitutability (and Ω_i is finite), the valuation function $\hat{u}_i$ is substitutable as well.

Proof of Theorem 6

We first show that if the preferences of an agent i are fully substitutable, then those preferences induce a submodular indirect utility function. It is enough to show that for any two trades $\varphi, \psi \in \Omega_i$ and any prices $p \in \mathbb{R}^\Omega$, $p_\varphi^{\text{high}} > p_\varphi$, and $p_\psi^{\text{high}} > p_\psi$ we have that[36]

$$V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}}) \geq V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi). \quad (15)$$

Suppose that $\varphi, \psi \in \Omega_{\to i}$.[37] There are three cases to consider:

Case 1: Suppose that $\varphi \notin \Phi$ for any $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi)$. Then, by individual rationality, $\varphi \notin \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi)$. Hence,

$$V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi) = 0$$

and so equation (15) is satisfied, as the left side of (15) must be non-negative.

Case 2: Suppose $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}})$. Then, by individual rationality, $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}})$. Hence,

$$V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}}) = -(p_\varphi - p_\varphi^{\text{high}}) = p_\varphi^{\text{high}} - p_\varphi$$

and so equation (15) is satisfied, as the right side of (15) is (weakly) bounded from above by $p_\varphi^{\text{high}} - p_\varphi$ (with equality in the case that φ is demanded at both $(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi)$ and $(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi))$.

Case 3: Suppose that $\varphi \in \Phi$ for some $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi)$ and $\varphi \notin \Phi$ for some $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}})$. In this case, as the preferences of i are fully substitutable, there exists a unique price $p_\varphi^{\uparrow}$ such that there exists $\Phi, \bar{\Phi} \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\uparrow}, p_\psi^{\text{high}})$ such that $\varphi \in \Phi$ and $\varphi \notin \bar{\Phi}$; note that $p_\varphi \leq p_\varphi^{\uparrow} \leq p_\varphi^{\text{high}}$. Similarly, let $p_\varphi^{\downarrow}$ be the unique price at which there exists $\Phi, \bar{\Phi} \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\downarrow}, p_\psi)$ such that $\varphi \in \Phi$ and $\varphi \notin \bar{\Phi}$; note that $p_\varphi \leq p_\varphi^{\downarrow} \leq p_\varphi^{\text{high}}$. By the definition of the utility function, $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi^{\text{high}})$ for all $\tilde{p}_\varphi < p_\varphi^{\uparrow}$, and $\varphi \notin \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi^{\text{high}})$ for all $\tilde{p}_\varphi > p_\varphi^{\uparrow}$; similarly, $\varphi \in \Phi$ for

[36]The definition of submodularity given in Definition 4 is equivalent to the pointwise definition given here; see, e.g., Schrijver (2002).

[37]The other three cases ($\varphi \in \Omega_{\to i}$ and $\psi \in \Omega_{i \to}$; $\varphi \in \Omega_{\to i}$ and $\psi \in \Omega_{i \to}$; and $\varphi, \psi \in \Omega_{i \to}$) are analogous.

all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi)$ for all $\tilde{p}_\varphi < p^{\downarrow}_\varphi$, and $\varphi \notin \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi)$ for all $\tilde{p}_\varphi > p^{\downarrow}_\varphi$.

Since the preferences of i are fully substitutable, $p^{\downarrow}_\varphi \leq p^{\uparrow}_\varphi$. Hence,

$$
\begin{aligned}
&V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p^{\text{high}}_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p^{\text{high}}_\varphi, p^{\text{high}}_\psi) \\
&\quad = V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p^{\text{high}}_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p^{\uparrow}_\varphi, p^{\text{high}}_\psi) + V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p^{\uparrow}_\varphi, p^{\text{high}}_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p^{\text{high}}_\varphi, p^{\text{high}}_\psi) \\
&\quad = -p_\varphi + p^{\uparrow}_\varphi - 0 \\
&\quad \geq -p_\varphi + p^{\downarrow}_\varphi - 0 \\
&\quad = V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p^{\downarrow}_\varphi, p_\psi) + V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p^{\downarrow}_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p^{\text{high}}_\varphi, p_\psi) \\
&\quad = V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p^{\text{high}}_\varphi, p_\psi),
\end{aligned}
$$

which is exactly (15).

Now, suppose that the preferences of i are not substitutable. We suppose moreover that the preferences of i fail the first condition of Defintion 2.[38] Hence, for some price vectors $p, p' \in \mathbb{R}^\Omega$ such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, we have that for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, either $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \not\subseteq \Psi_{\to i}$ or $\Psi_{i\to} \not\subseteq \Psi'_{i\to}$. We suppose that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \not\subseteq \Psi_{\to i}$; the latter case is analogous. Let $\varphi \in \Psi_{\to i} \smallsetminus \{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\}$. Let p^{high}_φ be a price for trade φ high enough such that φ is not demanded at either $(p^{\text{high}}_\varphi, p_{\Omega \smallsetminus \{\varphi\}})$ or $(p^{\text{high}}_\varphi, p'_{\Omega \smallsetminus \{\varphi\}})$. Hence, $V_i(p_\varphi, p'_{\Omega \smallsetminus \{\varphi\}}) - V_i(p^{\text{high}}_\varphi, p'_{\Omega \smallsetminus \{\varphi\}}) = 0$, while $V_i(p_\varphi, p_{\Omega \smallsetminus \{\varphi\}}) - V_i(p^{\text{high}}_\varphi, p_{\Omega \smallsetminus \{\varphi\}}) > 0$. Thus, we see that V_i is not submodular.

Proof of Theorem 7

The proof is an adaptation of the proof of Theorem 1 of Sun and Yang (2009) to our setting. As our model is more general than that of Sun and Yang (2009)—we do not impose either monotonicity or boundedness on the valuation functions, and we do not require that the seller values each bundle at 0 and thus sells everything that he could sell—we have to carefully ensure that the Sun and Yang (2009) approach remains valid.

We show first that IDFS and IIFS imply the single improvement property. Fix an arbitrary price vector $p \in \mathbb{R}^\Omega$ and a set of trades $\Psi \notin D_i(p)$ such that $u_i(\Psi) \neq -\infty$. Fix a set of trades $\Xi \in D_i(p)$. We focus exclusively on the trades in Ψ and Ξ by rendering all other trades that agent i is involved in irrelevant. To this end, we first define a very high price Π,

$$\Pi \equiv \max_{\substack{\Omega_1 \subseteq \Omega_i, u_i(\Omega_1) > -\infty, \\ \Omega_2 \subseteq \Omega_i, u_i(\Omega_2) > -\infty}} \{|U_i([\Omega_1; p]) - U_i([\Omega_2; p])|\} + \max_{\omega \in \Omega_i} \{|p_\omega|\} + 1,$$

[38] The case where the preferences of i fail the second condition of Defintion 2 is analogous.

and then, starting from p, we construct a preliminary price vector p' as follows:

$$p'_\omega = \begin{cases} p_\omega & \omega \in \Psi \cup \Xi \text{ or } \omega \notin \Omega_i \\ p_\omega + \Pi & \omega \in \Omega_{\to i} \smallsetminus (\Psi \cup \Xi) \\ p_\omega - \Pi & \omega \in \Omega_{i \to} \smallsetminus (\Psi \cup \Xi). \end{cases}$$

Observe that $\Psi \notin D_i(p')$ and $\Xi \in D_i(p')$. As $\Psi \neq \Xi$, we have to consider two cases (each with several subcases), which taken together will show that there exists a set of trades $\Phi' \neq \Psi$ that satisfies conditions 2 and 3 of Definition 5 and $U_i([\Phi'; p]) \geq U_i([\Psi; p])$.

Case 1: $\Xi \smallsetminus \Psi \neq \varnothing$. Select a trade $\xi_1 \in \Xi \smallsetminus \Psi$. Without loss of generality, assume that agent i is the buyer of ξ_1 (the case where i is the seller is completely analogous).

Starting from p', construct a modified price vector p'' as follows:

$$p''_\omega = \begin{cases} p'_\omega & \omega \in \Omega_i \smallsetminus ((\Xi_{\to i} \smallsetminus (\Psi_{\to i} \cup \{\xi_1\})) \cup \Psi_{i \to}) \text{ or } \omega \notin \Omega_i \\ p'_\omega + \Pi & \omega \in (\Xi_{\to i} \smallsetminus (\Psi_{\to i} \cup \{\xi_1\})) \cup \Psi_{i \to}. \end{cases}$$

First, since $\Xi \in D_i(p')$, $\xi_1 \in \Xi$, and $p'_{\xi_1} = p''_{\xi_1}$, full substitutability (Definition A.5) implies that there exists $\Xi'' \in D_i(p'')$ such that $\xi_1 \in \Xi''$. Second, observe that following the price change from p' to p'', $(\Xi''_{\to i} \smallsetminus \Psi_{\to i}) \subseteq \{\xi_1\}$ and $\Psi_{i \to} \subseteq \Xi''_{i \to}$. Thus, $\Xi''_{\to i} \smallsetminus \Psi_{\to i} = \{\xi_1\}$ and $\Psi_{i \to} \subseteq \Xi''_{i \to}$. We consider three subcases.

Subcase (a): $\Xi''_{i \to} \smallsetminus \Psi_{i \to} \neq \varnothing$. Let $\xi_2 \in \Xi''_{i \to} \smallsetminus \Psi_{i \to}$. Starting from p'', construct price vector p''' as follows:

$$p'''_\omega = \begin{cases} p''_\omega & \omega \in \Omega_i \smallsetminus ((\Xi_{i \to} \smallsetminus (\Psi_{i \to} \cup \{\xi_2\})) \cup \Psi_{\to i}) \text{ or } \omega \notin \Omega_i \\ p''_\omega - \Pi & \omega \in (\Xi_{i \to} \smallsetminus (\Psi_{i \to} \cup \{\xi_2\})) \cup \Psi_{\to i}. \end{cases}$$

First, since $\Xi'' \in D_i(p'')$, $\xi_2 \in \Xi''$, and $p''_{\xi_2} = p'''_{\xi_2}$, full substitutability (Definition A.6) implies that there exists $\Xi''' \in D_i(p''')$ such that $\xi_2 \in \Xi'''$. Second, observe that following the price change from p'' to p''', $\Psi \subseteq \Xi'''$ and $\Xi''' \smallsetminus \Psi \subseteq \{\xi_1, \xi_2\}$. Thus, $\Psi \smallsetminus \Xi''' = \varnothing$ and $\Xi''' \smallsetminus \Psi = \{\xi_1, \xi_2\}$ or $\{\xi_2\}$.

Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from the perspective of agent i the only differences from Ψ to Ξ''' are making one new sale ξ_2, i.e., $e_{i,\xi_2}(\Psi) > e_{i,\xi_2}(\Xi''')$ with $\xi_2 \in \Omega_{i \to} \smallsetminus \Psi$, and (possibly) making one new purchase ξ_1, i.e. $e_{i,\xi_1}(\Psi) < e_{i,\xi_1}(\Xi''')$ with $\xi_1 \in \Omega_{\to i} \smallsetminus \Psi$.

Subcase (b): $\Xi''_{i \to} \smallsetminus \Psi_{i \to} = \varnothing$ and $\Psi_{\to i} \smallsetminus \Xi''_{\to i} \neq \varnothing$. Let $\psi \in \Psi_{\to i} \smallsetminus \Xi''_{\to i}$. Starting from p'', construct price vector p''' as follows:

$$p'''_\omega = \begin{cases} p''_\omega & \omega \in \Omega_i \smallsetminus ((\Xi_{i\to} \smallsetminus \Psi_{i\to}) \cup (\Psi_{\to i} \smallsetminus \{\psi\})) \text{ or } \omega \notin \Omega_i \\ p''_\omega - \Pi & \omega \in (\Xi_{i\to} \smallsetminus \Psi_{i\to}) \cup (\Psi_{\to i} \smallsetminus \{\psi\}). \end{cases}$$

First, since $\Xi'' \in D_i(p'')$, $\psi \notin \Xi''$, and $p''_\psi = p'''_\psi$, by full substitutability (Definition A.6) implies that there exists $\Xi''' \in D_i(p''')$ such that $\psi \notin \Xi'''$. Second, observe that following the price change from p'' to p''', $\Psi \smallsetminus \Xi''' \subseteq \{\psi\}$ and $\Xi''' \smallsetminus \Psi \subseteq \{\xi_1\}$. Thus, $\Psi \smallsetminus \Xi''' = \{\psi\}$ and $\Xi''' \smallsetminus \Psi = \{\xi_1\}$ or $\varnothing$.

Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \le U_i([\Xi''', p'''])$. Furthermore, observe that from agent i's perspective the only differences from Ψ to Ξ''' are canceling one purchase ψ, i.e., $e_{i,\psi}(\Psi) > e_{i,\psi}(\Xi''')$ with $\psi \in \Psi_{\to i}$, and (possibly) making one new purchase ξ_1, i.e., $e_{i,\xi_1}(\Psi) < e_{i,\xi_1}(\Xi''')$ with $\xi_1 \in \Omega_{\to i} \smallsetminus \Psi$.

Subcase (c): $\Xi'' = \Psi \cup \{\xi_1\}$. Let $p''' = p''$ and $\Xi''' = \Xi''$. Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \le U_i([\Xi''', p'''])$. Furthermore, from agent i's perspective the only difference from Ψ to Ξ''' is making a new purchase ξ_1, i.e., $e_{i,\xi_1}(\Psi) < e_{i,\xi_1}(\Xi''')$ with $\xi_1 \in \Omega_{\to i} \smallsetminus \Psi$.

Case 2: $\Xi \smallsetminus \Psi = \varnothing$ and $\Psi \smallsetminus \Xi \neq \varnothing$. Select a trade $\psi_1 \in \Psi \smallsetminus \Xi$. Without loss of generality, assume that agent i is a buyer in ψ_1 (the case where i is a seller is completely analogous).

Starting from p', construct price vector p'' as follows:

$$p''_\omega = \begin{cases} p'_\omega & \omega \in \Omega_i \smallsetminus (\Psi_{\to i} \smallsetminus \{\psi_1\}) \text{ or } \omega \notin \Omega_i \\ p'_\omega - \Pi & \omega \in \Psi_{\to i} \smallsetminus \{\psi_1\}. \end{cases}$$

First, since $\Xi \in D_i(p')$, $\psi_1 \notin \Xi$, and $p'_{\psi_1} = p''_{\psi_1}$, full substitutability (Definition A.6) implies that there exists $\Xi'' \in D_i(p'')$ such that $\psi_1 \notin \Xi''$. Second, observe that following the price change from p' to p'', $\Xi'' \subseteq \Psi$ and $\Psi_{\to i} \smallsetminus \Xi''_{\to i} \subseteq \{\psi_1\}$. Thus, $\Psi_{\to i} \smallsetminus \Xi''_{\to i} = \{\psi_1\}$ and $\Xi'' \subseteq \Psi$. We consider two subcases.

Subcase (a): $\Psi_{i\to} \smallsetminus \Xi''_{i\to} \neq \varnothing$. Let $\psi_2 \in \Psi_{i\to} \smallsetminus \Xi''_{i\to}$. Starting from p'', construct price vector p''' as follows:

$$p'''_\omega = \begin{cases} p''_\omega & \omega \in \Omega_i \smallsetminus (\Psi_{i\to} \smallsetminus \{\psi_2\}) \text{ or } \omega \notin \Omega_i \\ p''_\omega + \Pi & \omega \in \Psi_{i\to} \smallsetminus \{\psi_2\}. \end{cases}$$

First, since $\Xi'' \in D_i(p'')$, $\psi_2 \notin \Xi''$, and $p''_{\psi_2} = p'''_{\psi_2}$, full substitutability (definition A.5) implies that there exists $\Xi''' \in D_i(p''')$ such that $\psi_2 \notin \Xi'''$. Second, observe that following the price change from p'' to p''', $\Xi''' \subseteq \Psi$ and $\Psi \smallsetminus \Xi''' \subseteq \{\psi_1, \psi_2\}$. Thus, $\Xi''' \smallsetminus \Psi = \varnothing$ and $\Psi \smallsetminus \Xi''' = \{\psi_1, \psi_2\}$ or $\{\psi_2\}$.

Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from agent i's perspective the only differences from Ψ to Ξ''' are canceling one sale ψ_2, i.e., $e_{i,\psi_2}(\Psi) < e_{i,\psi_2}(\Xi''')$ with $\psi_1 \in \Omega_{i\to} \smallsetminus \Psi$, and (possibly) canceling one purchase ψ_1, i.e., $e_{i,\psi_1}(\Psi) > e_{i,\psi_1}(\Xi''')$ with $\psi_1 \in \Psi_{\to i}$.

Subcase (b): $\Xi'' = \Psi \smallsetminus \{\psi_1\}$. In this subcase, let $p''' = p''$ and $\Xi''' = \Xi''$. Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, from the perspective of agent i, the only difference from Ψ to Ξ''' is canceling purchase ψ_1, i.e., $e_{i,\psi_1}(\Psi) < e_{i,\psi_1}(\Xi''')$ with $\psi_1 \in \Omega_{\to i} \smallsetminus \Psi$.

Taking together all the final statements from each subcase of Cases 1 and 2, if we take $\Phi' \equiv \Xi'''$, we obtain that we always have a price vector p''' and the sets Ψ and Φ' that satisfy conditions (2) and (3) of Definition 5. Moreover, since we always have $\Phi \in D_i(p''')$, $U_i([\Phi', p''']) \geq U_i([\Psi, p'''])$.

Next, we show that $U_i([\Phi', p''']) - U_i([\Psi, p''']) \geq 0$ implies $U_i([\Phi', p]) \geq U_i([\Psi, p])$. First, observe that when taking the difference the prices of all trades $\omega \in \Phi' \cap \Psi$ cancel each other out. Thus, replacing the prices p'''_ω with p_ω for all trades $\omega \in \Phi' \cap \Psi$ leaves the difference unchanged. Second, observe that in all previous subcases, the construction of p''' implies that for all $\omega \in ((\Psi \smallsetminus \Phi') \cup (\Phi' \smallsetminus \Psi))$, $p_\omega = p'''_\omega$. Combining the two observations above, $U_i([\Phi', p''']) - U_i([\Psi, p''']) = U_i([\Phi', p]) - U_i([\Psi, p])$, and therefore $U_i([\Phi', p]) \geq U_i([\Psi, p])$.

We now show that there exists a set of trades Φ that satisfies all conditions of Definition 5. Since $\Psi \notin D_i(p)$, $V_i(p) > U_i([\Psi; p])$. Since i's utility is continuous in prices, there exists $\varepsilon > 0$ such that $V_i(q) > U_i([\Psi; q])$ where q is defined as follows:

$$q_\omega = \begin{cases} p_\omega + \varepsilon & \omega \in (\Omega_{\to i} \smallsetminus \Psi_{\to i}) \cup \Psi_{i\to} \\ p_\omega - \varepsilon & \omega \in (\Omega_{i\to} \smallsetminus \Psi_{i\to}) \cup \Psi_{\to i}. \end{cases}$$

Our arguments above imply that there exists a set of trades $\Phi \neq \Psi$ such that $U_i([\Phi; q]) \geq U_i([\Psi; q])$. Using the construction of q, we obtain $U_i([\Phi; p]) - U_i([\Psi; p]) = U_i([\Phi; q]) - U_i([\Psi; q]) + \varepsilon|(\Psi \smallsetminus \Phi) \cup (\Phi \smallsetminus \Psi)| > U_i([\Phi; q]) - U_i([\Psi; q]) \geq 0$. Thus, $U_i([\Phi; p]) > U_i([\Psi; p])$. This completes the proof that IDFS and IIFS imply the single improvement property.

We now show that the single improvement property implies full substitutability DCFS. More specifically, we will establish that single improvement implies the first condition of Definition A.4; the proof that the second condition is also satisfied uses an analogous argument.

Let $p \in \mathbb{R}^\Omega$ and $\Psi \in D_i(p)$ be arbitrary. It is sufficient to establish that for any $p' \in \mathbb{R}^\Omega$ such that $p'_\psi > p_\psi$ for some $\psi \in \Omega_{\to i}$ and $p'_\omega = p_\omega$ for all $\omega \in \Omega \setminus \{\psi\}$, there exists a set of trades $\Psi' \in D_i(p')$ that satisfies the first condition of Definition A.4.

Fix one $p' \in \mathbb{R}^\Omega$ that satisfies the conditions mentioned in the previous paragraph and let $\psi \in \Omega_{\to i}$ be the one trade for which $p'_\psi > p_\psi$. Note that if either $\psi \notin \Psi$ or $\Psi \in D_i(p')$, there is nothing to show. From now on, assume that $\psi \in \Psi$ and $\Psi \notin D_i(p')$.

For any $\varepsilon > 0$ define a price vector $p^\varepsilon \in \mathbb{R}^\Omega$ by setting $p^\varepsilon_\psi = p_\psi + \varepsilon$ and $p^\varepsilon_\omega = p_\omega$ for all $\omega \in \Omega \setminus \{\psi\}$. Let $\Delta \equiv \max\{\varepsilon : \Psi \in D_i(p^\varepsilon)\}$. Note that Δ is well defined since i's utility function is continuous in prices. Also, given that $\Psi \notin D_i(p')$, we must have $\Delta < p'_\psi - p_\psi$.

Next, for any integer n, define a price vector $p^n \in \mathbb{R}^\Omega$ by setting $p^n_\psi = p_\psi + \Delta + \frac{1}{n}$ and $p^n_\omega = p_\omega$ for all $\omega \in \Omega \setminus \{\psi\}$. By the definition of Δ we must have $\Psi \notin D_i(p^n)$ for all $n > 0$. By the single improvement property, this implies that for all $n > 0$, there exists a set of trades Φ^n such that the following conditions are satisfied: (i) $U_i([\Psi, p^n]) < U_i([\Phi^n, p^n])$, (ii) there exists at most one trade ω such that $e_{i,\omega}(\Psi) < e_{i,\omega}(\Phi^n)$, and (iii) there exists at most one trade ω such that $e_{i,\omega}(\Psi) > e_{i,\omega}(\Phi^n)$.

Note that we must have $\psi \notin \Phi^n$ for all $n \geq 1$. This follows since for any $n \geq 1$ and any set of trades Φ such that $\psi \in \Phi$, $U_i([\Phi; p^n]) = U_i([\Phi; p]) - \Delta - \frac{1}{n} \leq U_i([\Psi; p]) - \Delta - \frac{1}{n} = U_i([\Psi; p^n])$ given that $\Psi \in D_i(p)$.

Conditions 2 and 3 imply that for all $n > 0$, we must have $\{\omega \in \Psi_{\to i} : p'_\omega = p_\omega\} = \{\omega \in \Psi_{\to i} : p^n_\omega = p_\omega\} \subseteq \Phi^n_{\to i}$ and $\Phi^n_{i\to} \subseteq \Psi_{i\to}$.

Since the set of trades is finite, it is without loss of generality to assume that there is a set of trades $\Phi^* \in \Omega_i$ and an integer $\bar{n}$ such that $\Phi^n = \Phi^*$ for all $n \geq \bar{n}$. Since i's utility function is continuous with respect to prices and $p^n \to p^\Delta$, we must have $U_i([\Phi^*; p^\Delta]) \geq U_i([\Psi; p^\Delta])$. Since $\Psi \in D_i(p^\Delta)$, this implies $\Phi^* \in D_i(p^\Delta)$. Since $\Delta < p'_\psi - p_\psi$ and V_i is decreasing in the prices of trades for which i is a buyer, we must have $V_i(p^\Delta) \geq V_i(p')$. Since $\psi \notin \Phi^*$, we have that $U_i([\Phi^*; p']) = U_i([\Phi^*; p^\Delta]) = V_i(p^\Delta)$. Hence, $\Phi^* \in D_i(p')$ and setting $\Psi' \equiv \Phi^*$ yields a set that satisfies the first condition of Definition A.4.

Proof of Theorem 10

We assume throughout that $\Omega = \Omega_i$ (and so $X = X_i$); this is without loss of generality as all of the analysis here considers only the sets of contracts demanded by i and, for any sets of contracts Y and Z such that $Y_i = Z_i$, we have that $Y^* \in C_i(Y)$ if and only if $Y^* \in C_i(Z)$.

Step 1: We show first that full substitutability implies monotone–substitutability for opportunity sets such that the choice correspondence is single-valued. That is, we will show for all finite sets of contracts Y and Z such that $|C_i(Y)| = |C_i(Z)| = 1$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$, for the unique $Y^* \in C_i(Y)$ and the unique $Z^* \in C_i(Z)$, we have $|Z^*_{\to i}| - |Y^*_{\to i}| \geq |Z^*_{i\to}| - |Y^*_{i\to}|$.

Fix a fully substitutable valuation function u_i for agent i. Consider two finite sets of contracts Y and Z such that $|C_i(Y)| = |C_i(Z)| = 1$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$. Assume that for any $\omega \in \Omega_{i\to}$, if $(\omega, p_\omega) \in Y_{i\to}$ and $(\omega, p'_\omega) \in Y_{i\to}$, then $p_\omega = p'_\omega$; this is without loss of generality as for a given trade $\omega \in \Omega_{i\to}$, agent i, as a seller, will only choose a contract with the highest price available for that trade, and thus we can ignore all other contracts involving that trade.

Let $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$. Define a modified valuation $\tilde{u}_i$ on $\tau(Z_i)$ for agent i by setting, for each $\Psi \subseteq \tau(Z_i)$,

$$\tilde{u}_i(\Psi) = u_i(\Psi_{\to i} \cup (\tau(Z) \smallsetminus \Psi)_{i\to}).$$

For all feasible $W \subseteq Z$, let

$$\tilde{U}_i(W) = \tilde{u}_i(\tau(W)) + \sum_{(\omega,p_\omega)\in(Z\smallsetminus W)_{i\to}} p_\omega - \sum_{(\omega,p_\omega)\in W_{\to i}} p_\omega,$$

and let $\tilde{C}_i$ denote the choice correspondence over Z associated with $\tilde{U}_i$. By construction,

$$\tilde{u}_i(\Psi) = u_i(\tilde{\mathfrak{o}}_i(\Psi)), \tag{16}$$

where here the object operator $\tilde{\mathfrak{o}}$ is defined with respect to the underlying set of trades $\tau(Z)$:

$$\tilde{\mathfrak{o}}_i(\Psi) = \{\mathfrak{o}(\omega) : \omega \in \Psi_{\to i}\} \cup \{\mathfrak{o}(\omega) : \omega \in \tau(Z) \smallsetminus \Psi_{i\to}\}.$$

As the preferences of i are fully substitutable, the restriction of those preferences to $\tau(Z)$ is fully substitutable, as well.[39] Thus, the restriction of i's preferences to $\tau(Z)$ is object-language fully substitutable and so $\tilde{u}_i$ satisfies the gross substitutability condition of Kelso and Crawford (1982) over objects.

Now, we must have $\tilde{C}_i(Y) = \{Y^*_{\to i} \cup (Z \smallsetminus Y^*)_{i\to}\}$ and $\tilde{C}_i(Z) = \{Z^*_{\to i} \cup (Z \smallsetminus Z^*)_{i\to}\}$. As we assume quasilinearity, the Law of Aggregate Demand for two-sided markets applies to $\tilde{C}_i$ (by Theorem 7 of Hatfield and Milgrom (2005)). As $Y \subseteq Z$, this implies that $|Z^*_{\to i} \cup (Z \smallsetminus Z^*)_{i\to}| \geq |Y^*_{\to i} \cup (Z \smallsetminus Y^*)_{i\to}|$; this inequality is equivalent to $|Z^*_{\to i}| - |Z^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}|$, which is precisely the Law of Aggregate Demand. We also have that $Y_{i\to} \smallsetminus Y^*_{\to i} \subseteq Z_{i\to} \smallsetminus Z^*_{\to i}$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$, as the preferences of i are fully substitutable. Thus, the preferences of i satisfy the requirements of Part 1 of Definition 9 when the choice correspondence is single-valued.

The proof that i's preferences satisfy the requirements of Part 2 of Definition 9 is analogous.

Step 2: We now use Step 1 to show that full substitutability implies monotone–substitutability. For this step, let

$$\hat{u}(\Psi; Y) \equiv u_i(\Psi) - \sum_{\psi\in\Psi_{\to i}} \inf\{p_\psi : (\psi, p_\psi) \in Y\} + \sum_{\psi\in\Psi_{i\to}} \sup\{p_\psi : (\psi, p_\psi) \in Y\},$$

where we take $\inf \varnothing = \infty$ and $\sup \varnothing = -\infty$; that is, $\hat{u}(\Psi; Y)$ is the utility that i obtains from the set of trades Ψ and both paying, for each trade in $\Psi_{\to i}$, the lowest price corresponding to a contract in Y and receiving, for each trade in $\Psi_{i\to}$, the highest price corresponding to a contract in Y.

[39]To see the full substitutability of $\tilde{U}_i$, note that the full substitutability of the restriction of U_i to any subset of X follows immediately from the fact that the preferences of i satisfy CFS.

We also extend the operator τ to sets of sets of contracts, so that $\tau(\mathcal{Y}) = \cup_{Y \in \mathcal{Y}}\{\tau(Y)\}$ for any $\mathcal{Y} \subseteq \wp(X)$.

Finally, it is helpful to define an operator which, given a set of available contracts W, makes each trade in $\tau(W')$ slightly more appealing to i relative to W' and each trade not in $\tau(W')$ slightly less appealing to i relative to W'. Let

$$r(W; W', \varepsilon) \equiv \{(\omega, p_\omega - \varepsilon) \in X : (\omega, p_\omega) \in W'_{\to i}\} \cup \{(\omega, p_\omega + \varepsilon) \in X : (\omega, p_\omega) \in [W \smallsetminus W']_{\to i}\}$$
$$\cup \{(\omega, p_\omega + \varepsilon) \in X : (\omega, p_\omega) \in W'_{i\to}\} \cup \{(\omega, p_\omega - \varepsilon) \in X : (\omega, p_\omega) \in [W \smallsetminus W']_{i\to}\}.$$

The r function here allows us to perturb sets of contracts so as to obtain unique choices, similar to the methods used to prove Lemma 1.

Observation 1. *For all sets of contracts $W, Y, Z \subseteq X$ such that $Y \subseteq Z$, we have that $r(Y; W, \varepsilon) \subseteq r(Z; W, \varepsilon)$ for all $\varepsilon > 0$.*

Now, we consider two finite sets of contracts Y and Z such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$. Fix an arbitrary $Y^* \in C_i(Y)$; we need to show that there exists a set $Z^* \in C_i(Z)$ that satisfies the requirements of Part 1 of Definition 9. Let $\hat{Z}^* \in C_i(r(Z; Y^*, \varepsilon))$.

We first show five intermediate results on the effects of our price perturbations, where we $\varepsilon > 0$ to be sufficiently small and $\delta > 0$ to be sufficiently small given ε.

Fact 1: $C_i(r(Y; Y^*, \varepsilon)) = \{r(Y^*; Y^*, \varepsilon)\}$**.** For any feasible $W \subseteq Y$ such that $W \neq Y^*$,[40,41]

$$U_i(r(Y^*; Y^*, \varepsilon)) - U_i(r(W; Y^*, \varepsilon)) = U_i(Y^*) - U_i(W) + |Y^* \ominus W|\varepsilon$$
$$\geq |Y^* \ominus W|\varepsilon > 0,$$

where the equality follows from the definition of r, the first inequality follows from the fact that Y^* is optimal at Y (i.e., $Y^* \in C_i(Y)$) and the second inequality follows from the fact that $W \neq Y^*$. Thus, we see that $C_i(r(Y; Y^*, \varepsilon)) = \{r(Y^*; Y^*, \varepsilon)\}$, as desired.

Fact 2: $\tau(C_i(r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta))) \subseteq \tau(C_i(r(Y; Y^*, \varepsilon)))$**.** Consider an arbitrary $\Phi \in \tau(C_i(r(Y; Y^*, \varepsilon)))$ and an arbitrary $\Xi \notin \tau(C_i(r(Y; Y^*, \varepsilon)))$. For ε small enough, we have that,

$$\hat{u}(\Phi; r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta)) - \hat{u}(\Xi; r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta))$$
$$\geq \hat{u}(\Phi; r(Y; Y^*, \varepsilon)) - \hat{u}(\Xi; r(Y; Y^*, \varepsilon)) - |\Phi \ominus \Xi|\delta > 0,$$

where the first inequality follows from the definition of r and the second inequality follows as Φ is associated with an optimal set of contracts at $r(Y; Y^*, \varepsilon)$, Ξ is not associated with an optimal

[40]Note that (feasible) subsets of Y and (feasible) subsets of $r(Y; Y^*, \varepsilon)$ are in a one-to-one correspondence.
[41]Here, we use $\ominus$ to denote the symmetric difference between two sets, i.e., $W \ominus W' = (W \smallsetminus W') \cup (W \smallsetminus W')$.

set of contracts at $r(Y; Y^*, \varepsilon)$, and δ is sufficiently small. Thus, $\Xi \notin \tau(C_i(r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta)))$ and so $\tau(C_i(r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta))) \subseteq \tau(C_i(r(Y; Y^*, \varepsilon)))$.

Fact 3: $\tau(C_i(r(Z; Y^*, \varepsilon))) \subseteq \tau(C_i(Z))$**.** Consider an arbitrary $\Phi \in \tau(C_i(Z))$ and an arbitrary $\Xi \notin \tau(C_i(Z))$. For ε small enough, we have that

$$\hat{u}(\Phi; r(Z; Y^*, \varepsilon)) - \hat{u}(\Xi; r(Z; Y^*, \varepsilon)); Y^*, \varepsilon) \geq \hat{u}(\Phi; Z) - \hat{u}(\Xi; Z) - |\Phi \ominus \Xi|\varepsilon > 0,$$

where the first inequality follows from the definition of r and the second inequality follows as Φ is associated with an optimal set of contracts at Z, Ξ is not associated with an optimal set of contracts at Z, and ε is sufficiently small. Thus, $\Xi \notin \tau(C_i(r(Z; Y^*, \varepsilon)))$ and so $\tau(C_i(r(Z; Y^*, \varepsilon))) \subseteq \tau(C_i(Z))$.

Fact 4: $\tau(C_i(r(r(Z; Y^*, \varepsilon)); \hat{Z}^*, \delta))) \subseteq \tau(C_i(r(Z; Y^*, \varepsilon)))$**.** Consider an arbitrary $\Phi \in \tau(C_i(r(Z; Y^*, \varepsilon)))$ and an arbitrary $\Xi \notin \tau(C_i(r(Z; Y^*, \varepsilon)))$. For δ small enough, we have that

$$\begin{aligned} &\hat{u}(\Phi; r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta)) - \hat{u}(\Xi; r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta)) \\ &\quad \geq \hat{u}(\Phi; r(Z; Y^*, \varepsilon)) - \hat{u}(\Xi; r(Z; Y^*, \varepsilon)) - |\Phi \ominus \Xi|\delta > 0, \end{aligned}$$

where the first inequality follows from the definition of r and the second inequality follows as Φ is associated with an optimal set of contracts at $r(Z; Y^*, \varepsilon)$, Ξ is not associated with an optimal set of contracts at $r(Z; Y^*, \varepsilon)$, and δ is sufficiently small. Thus, $\Xi \notin \tau(C_i(r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta)))$ and so we have that $\tau(C_i(r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta))) \subseteq \tau(C_i(r(Z; Y^*, \varepsilon)))$.

Fact 5: $C_i(r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta)) = \{r(\hat{Z}^*; \hat{Z}^*, \delta)\}$**.** We have that for any feasible $W \subseteq r(Z; Y^*, \varepsilon)$ such that $W \neq \hat{Z}^*$,[42]

$$\begin{aligned} U_i(r(\hat{Z}^*; \hat{Z}^*, \delta)) - U_i(r(W; \hat{Z}^*, \delta)) &= U_i(\hat{Z}^*) - U_i(W) + |\hat{Z}^* \ominus W|\delta \\ &\geq |\hat{Z}^* \ominus W|\delta > 0 \end{aligned}$$

where the equality follows from the definition of r, the first inequality follows from the fact that $\hat{Z}^*$ is optimal at $r(Z; Y^*, \varepsilon)$, i.e., $\hat{Z}^* \in C_i(r(Z; Y^*, \varepsilon))$, and the last inequality follows as $W \neq \hat{Z}^*$. Thus $C_i(r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta)) = \{r(\hat{Z}^*; \hat{Z}^*, \delta)\}$.

Combining Facts 1 and 2 shows that there is a unique element of $\tau(C_i(r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta)))$ and, since $r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta)$ is a finite set, there must therefore exist a unique

$$\tilde{Y}^* \in C_i(r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta)).$$

Fact 5 shows that $\tilde{Z}^* \equiv r(\hat{Z}^*; \hat{Z}^*, \delta)$ is the unique element of $C_i(r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta))$. Thus, as $[r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta)]_{\to i} \subseteq [r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta)]_{\to i}$ by Observation 1 (as $Y_{\to i} \subseteq Z_{\to i}$) and

[42] Note that there is a natural one-to-one correspondence between (feasible) subsets of Z, (feasible) subsets of $r(Z; Y^*, \varepsilon)$, and (feasible) subsets of $r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta)$.

$[r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)]_{i\to} = [r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)]_{i\to}$ (as $Y_{i\to} = Z_{i\to}$), Step 1 of the proof implies

$$|\tilde{Z}^*_{\to i}| - |\tilde{Z}^*_{i\to}| \geq |\tilde{Y}^*_{\to i}| - |\tilde{Y}^*_{i\to}| \tag{17a}$$

$$[r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)]_{\to i} \smallsetminus \tilde{Y}^*_{\to i} \subseteq [r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)]_{\to i} \smallsetminus \tilde{Z}^*_{\to i} \tag{17b}$$

$$\tilde{Y}^*_{i\to} \subseteq \tilde{Z}^*_{i\to}. \tag{17c}$$

Each contract (ω,p_ω) in $\tilde{Y}^*_{\to i}$ has the property that p_ω is the minimal price associated with ω among all prices associated with ω by some contract in $r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)$ as $\tilde{Y}^*$ is optimal at $r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)$. Similarly, each contract (ω,p_ω) in $\tilde{Z}^*_{\to i}$ has the property that p_ω is the minimal price associated with ω among all prices associated with ω by some contract in $r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)$ as $\tilde{Z}^*$ is optimal at $r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)$. Moreover, each contract $(\omega,p_\omega) \in \tilde{Y}^*_{i\to}$ has the property that p_ω is the maximal price associated with ω among all contracts associated with ω in $r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)$, as $\tilde{Y}^*$ is optimal at $r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)$. Similarly, each contract $(\omega,p_\omega) \in \tilde{Z}^*_{i\to}$ has the property that p_ω is the maximal price associated with ω among all contracts associated with ω in $r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)$, as $\tilde{Z}^*$ is optimal at $r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)$. We thus rewrite (17b) and (17c) (while maintaining (17a)) as

$$|\tilde{Z}^*_{\to i}| - |\tilde{Z}^*_{i\to}| \geq |\tilde{Y}^*_{\to i}| - |\tilde{Y}^*_{i\to}| \tag{18a}$$

$$\left[\left\{\begin{array}{c}(\omega,p_\omega) \in r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta): \\ \omega \notin \tau(\tilde{Y}^*_{\to i}) \text{ or} \\ \exists(\omega,\bar{p}_\omega) \in r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta) \\ \text{such that } \bar{p}_\omega < p_\omega\end{array}\right\}\right]_{\to i} \subseteq \left[\left\{\begin{array}{c}(\omega,p_\omega) \in r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta): \\ \omega \notin \tau(\tilde{Z}^*_{\to i}) \text{ or} \\ \exists(\omega,\bar{p}_\omega) \in r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta) \\ \text{such that } \bar{p}_\omega < p_\omega\end{array}\right\}\right]_{\to i} \tag{18b}$$

$$\left[\left\{\begin{array}{c}(\omega,p_\omega) \in r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta): \\ \omega \in \tau(\tilde{Y}^*_{\to i}) \text{ and} \\ \nexists(\omega,\bar{p}_\omega) \in r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta) \\ \text{such that } \bar{p}_\omega < p_\omega\end{array}\right\}\right]_{\to i} \subseteq \left[\left\{\begin{array}{c}(\omega,p_\omega) \in r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta): \\ \omega \in \tau(\tilde{Z}^*_{i\to}) \text{ and} \\ \nexists(\omega,\bar{p}_\omega) \in r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta) \\ \text{such that } \bar{p}_\omega > p_\omega\end{array}\right\}\right]_{i\to}. \tag{18c}$$

Combining Facts 1 and 2 yields that $\tau(Y^*) = \tau(\tilde{Y}^*)$, implying that $|Y^*_{\to i}| = |\tilde{Y}^*_{\to i}|$ and $|Y^*_{i\to}| = |\tilde{Y}^*_{i\to}|$, and so we have

$$|\tilde{Z}^*_{\to i}| - |\tilde{Z}^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}| \tag{19a}$$

$$\left[\left\{\begin{array}{c}(\omega,p_\omega) \in r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta): \\ \omega \notin \tau(Y^*_{\to i}) \text{ or} \\ \exists(\omega,\bar{p}_\omega) \in r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta) \\ \text{such that } \bar{p}_\omega < p_\omega\end{array}\right\}\right]_{\to i} \subseteq \left[\left\{\begin{array}{c}(\omega,p_\omega) \in r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta): \\ \omega \notin \tau(\tilde{Z}^*_{\to i}) \text{ or} \\ \exists(\omega,\bar{p}_\omega) \in r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta) \\ \text{such that } \bar{p}_\omega < p_\omega\end{array}\right\}\right]_{\to i} \tag{19b}$$

$$\left[\left\{\begin{array}{c}(\omega,p_\omega)\in r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta):\\ \omega\in\tau(Y^*_{i\to})\text{ and}\\ \nexists(\omega,\bar{p}_\omega)\in r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)\\ \text{such that }\bar{p}_\omega>p_\omega\end{array}\right\}\right]_{i\to}\subseteq\left[\left\{\begin{array}{c}(\omega,p_\omega)\in r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta):\\ \omega\in\tau(\tilde{Z}^*_{i\to})\text{ and}\\ \nexists(\omega,\bar{p}_\omega)\in r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)\\ \text{such that }\bar{p}_\omega>p_\omega\end{array}\right\}\right]_{i\to}. \tag{19c}$$

Similarly, combining Facts 3–5 yields that there exists $Z^*\in C_i(Z)$ such that $\tau(Z^*)=\tau(\tilde{Z}^*)$, implying $|Z^*_{\to i}|=|\tilde{Z}^*_{\to i}|$ and $|Z^*_{i\to}|=|\tilde{Z}^*_{i\to}|$, and so we have

$$|Z^*_{\to i}|-|Z^*_{i\to}|\geq|Y^*_{\to i}|-|Y^*_{i\to}| \tag{20a}$$

$$\left[\left\{\begin{array}{c}(\omega,p_\omega)\in r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta):\\ \omega\notin\tau(Y^*_{\to i})\text{ or}\\ \exists(\omega,\bar{p}_\omega)\in r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)\\ \text{such that }\bar{p}_\omega<p_\omega\end{array}\right\}\right]_{\to i}\subseteq\left[\left\{\begin{array}{c}(\omega,p_\omega)\in r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta):\\ \omega\notin\tau(Z^*_{\to i})\text{ or}\\ \exists(\omega,\bar{p}_\omega)\in r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)\\ \text{such that }\bar{p}_\omega<p_\omega\end{array}\right\}\right]_{\to i} \tag{20b}$$

$$\left[\left\{\begin{array}{c}(\omega,p_\omega)\in r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta):\\ \omega\in\tau(Y^*_{i\to})\text{ and}\\ \nexists(\omega,\bar{p}_\omega)\in r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)\\ \text{such that }\bar{p}_\omega>p_\omega\end{array}\right\}\right]_{i\to}\subseteq\left[\left\{\begin{array}{c}(\omega,p_\omega)\in r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta):\\ \omega\in\tau(Z^*_{i\to})\text{ and}\\ \nexists(\omega,\bar{p}_\omega)\in r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)\\ \text{such that }\bar{p}_\omega>p_\omega\end{array}\right\}\right]_{i\to}. \tag{20c}$$

We have, by (20c) that, if $\omega\in\tau(Y^*_{i\to})$, then $\omega\in\tau(Z^*_{i\to})$; moreover, since $Y_{i\to}=Z_{i\to}$ by assumption, the set of prices corresponding to a given $\omega\in\Omega_{i\to}$ is the same in Y and Z. We thus rewrite (20c) (while maintaining (20a)) and (20b)) as,

$$|Z^*_{\to i}|-|Z^*_{i\to}|\geq|Y^*_{\to i}|-|Y^*_{i\to}| \tag{21a}$$

$$\left[\left\{\begin{array}{c}(\omega,p_\omega)\in r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta):\\ \omega\notin\tau(Y^*_{\to i})\text{ or}\\ \exists(\omega,\bar{p}_\omega)\in r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)\\ \text{such that }\bar{p}_\omega<p_\omega\end{array}\right\}\right]_{\to i}\subseteq\left[\left\{\begin{array}{c}(\omega,p_\omega)\in r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta):\\ \omega\notin\tau(Z^*_{\to i})\text{ or}\\ \exists(\omega,\bar{p}_\omega)\in r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)\\ \text{such that }\bar{p}_\omega<p_\omega\end{array}\right\}\right]_{\to i} \tag{21b}$$

$$\left[\left\{\begin{array}{c}(\omega,p_\omega)\in Y:\\ \omega\in\tau(Y^*_{i\to})\text{ and}\\ \nexists(\omega,\bar{p}_\omega)\in Y\\ \text{such that }\bar{p}_\omega>p_\omega\end{array}\right\}\right]_{i\to}\subseteq\left[\left\{\begin{array}{c}(\omega,p_\omega)\in Z:\\ \omega\in\tau(Z^*_{i\to})\text{ and}\\ \nexists(\omega,\bar{p}_\omega)\in Z\\ \text{such that }\bar{p}_\omega>p_\omega\end{array}\right\}\right]_{i\to}. \tag{21c}$$

We have, by (21b) that, if $\omega\notin\tau(Y^*_{\to i})$, then $\omega\notin\tau(Z^*_{\to i})$; moreover, since $Y_{\to i}\subseteq Z_{\to i}$ by assumption, the set of prices available for a given $\omega\in\Omega_{\to i}$ is larger in Y than in Z. We thus rewrite (21b) (while maintaining (21a) and (21c)) as

$$|Z^*_{\to i}|-|Z^*_{i\to}|\geq|Y^*_{\to i}|-|Y^*_{i\to}| \tag{22a}$$

$$\left[\left\{\begin{array}{c}(\omega,p_\omega)\in Y:\\ \omega\notin\tau(Y^*_{\to i})\text{ or}\\ \exists(\omega,\bar{p}_\omega)\in Y\\ \text{such that }\bar{p}_\omega<p_\omega\end{array}\right\}\right]_{\to i}\subseteq\left[\left\{\begin{array}{c}(\omega,p_\omega)\in Z:\\ \omega\notin\tau(Z^*_{\to i})\text{ or}\\ \exists(\omega,\bar{p}_\omega)\in Z\\ \text{such that }\bar{p}_\omega<p_\omega\end{array}\right\}\right]_{\to i} \tag{22b}$$

$$\left[\left\{\begin{array}{c}(\omega, p_\omega) \in Y : \\ \omega \in \tau(Y^*_{i\to}) \text{ and} \\ \nexists(\omega, \bar{p}_\omega) \in Y \\ \text{such that } \bar{p}_\omega > p_\omega\end{array}\right\}\right]_{i\to} \subseteq \left[\left\{\begin{array}{c}(\omega, p_\omega) \in Z : \\ \omega \in \tau(Z^*_{i\to}) \text{ and} \\ \nexists(\omega, \bar{p}_\omega) \in Z \\ \text{such that } \bar{p}_\omega > p_\omega\end{array}\right\}\right]_{i\to}. \tag{22c}$$

We rewrite this expression as

$$|Z^*_{\to i}| - |Z^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}|$$
$$[Y \smallsetminus Y^*]_{\to i} \subseteq [Z \smallsetminus Z^*]_{\to i}$$
$$[Y^*]_{i\to} \subseteq [Z^*]_{i\to}.$$

Thus the preferences of i satisfy the requirements of Part 1 of Definition 9.

The proof that the preferences of i satisfy the requirements of Part 2 is analogous.

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