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Abstract

We consider several notions of setwise stability for many-to-many matching markets with contracts and provide an analysis of the relations between the resulting sets of stable allocations for general, substitutable, and strongly substitutable preferences. Apart from obtaining "set inclusion results" on all three domains, we introduce weak setwise stability as a new stability concept and prove that for substitutable preferences the set of pairwise stable matchings is nonempty and coincides with the set of weakly setwise stable matchings. For strongly substitutable preferences the set of pairwise stable matchings coincides with the set of setwise stable matchings.

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1 Introduction

We consider a general class of two-sided many-to-many matching markets, so-called *matching markets with contracts* (Roth, 1984b). Closely related markets are many-to-one matching markets with contracts (*e.g.*, Hatfield and Milgrom, 2005) and many-to-many matching markets (*e.g.*, Echenique and Oviedo, 2006; Konishi and Ünver, 2006; Sotomayor, 1999). In a matching market with contracts, agents do not only choose partners (as, for instance, in the marriage market introduced by Gale and Shapley, 1962) or partners and wages (see Kelso and Crawford, 1982),[1] but they can agree on further characteristics of the relationship, *e.g.,* students choose a medical and a surgical internship in the British internship program (see Konishi and Ünver, 2006), the members of a couple choose contracts with distinct job profiles for each other (see Hatfield and Kojima, 2008), or firms choose among different packages of spectrum frequencies from the same seller in bandwidth auctions (see Hatfield and Milgrom, 2005). Clearly, all of the previously studied matching markets can be considered to be special matching markets with contracts, but Hatfield and Milgrom (2005) demonstrate that some results that do hold for these matching markets without contracts – *e.g.,* the rural hospital theorem (see Roth, 1996) or the existence of a one-sided strategy-proof mechanism (see

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[1] Roth and Sotomayor (1990) give a comprehensive and complete survey of these and related two-sided matching models up to 1990.

Dubins and Freedman, 1981; Roth, 1996) – do not straightforwardly generalize to matching markets with contracts. If matching markets with couples are interpreted as matching markets with contracts as discussed in Hatfield and Kojima (2008), then Klaus and Klijn (2005) also show how standard results for matching markets without contracts might not extend to matching markets with contracts. On the other hand, Hatfield and Milgrom (2005) show that if preferences are substitutable and the "law of aggregate demand" holds, then the above mentioned results do extend to matching markets with contracts.

Throughout the article, without loss of generality, we model matching markets with contracts as trading platforms where buyers and sellers interact: there is a set of bilateral contracts between buyers and sellers that specify the trading conditions (*e.g.*, the set of items that are bought/sold, delivery dates, prices, service agreements, *etc.*). Moreover, buyers as well as sellers can trade with more than one agent on the other side of the market at the same time. The agents' strict preferences over (feasible, legal, *etc.*) sets of contracts or allocations completes the description of a many-to-many matching market with contracts.

Stability is the central solution requirement derived from empirical as well as theoretical studies (*e.g.*, Roth, 1982, 1984a, 1991). Loosely speaking, an allocation of contracts is "stable" if it is individually rational [no buyer or seller would prefer to cancel some of her contracts] and satisfies a no blocking requirement. There are various ways for a set of agents to block a given allocation of contracts:

- *pairwise blocking*: a buyer and a seller would like to add a new joint contract or replace a previous joint contract while possibly canceling other contracts;
- *setwise blocking*: a set of buyers and sellers would like to implement a new set of contracts among themselves while possibly canceling other contracts.

In the case of setwise blocking, the blocking requirement can be refined by imposing extra conditions on when a set of agents is allowed to block a given allocation of contracts:

- *strongly setwise blocking* describes the minimal requirement for any setwise blocking (it is the benchmark without extra conditions): the agents in the blocking coalition should be better off with the new sets of contracts they receive through blocking;
- *setwise blocking*: the set of new contracts is a strong setwise block and individually rational;
- *weakly setwise blocking*: the set of new contracts is a setwise block, but blocking can only occur if agents in the blocking coalition receive their best set of contracts from the new and the previous contracts.

Clearly, the various setwise blocking concepts differ with respect to the admissibility of a blocking coalition. While strong setwise blockings only require an improvement upon the status quo, setwise blockings also require that the blocking allocation is individually rational, *i.e.*, none of the members of the blocking coalition has an incentive to cancel contracts unilaterally. Finally, weakly setwise blockings require that there is no conflict in the blocking coalition in the sense that all members of the blocking coalition obtain their best set of contracts among their original contracts and the new contracts of the blocking coalition. Weak setwise stability is a new stability concept that adds a robustness requirement to setwise blockings. It excludes that blocking agents cancel some of the contracts they were supposed to implement or keep some of the contracts they were supposed to cancel. We will demonstrate that weak setwise stability bridges the gap between pairwise stability and previously considered setwise stability notions.

Anticipating future conflict within the blocking coalition and thereby imposing an additional requirement on admissible blockings has also been the motivation of credible group stability (see Konishi and Ünver, 2006) and the bargaining set (see Echenique and Oviedo, 2006). In Appendix A we show that there is no logical relationship between weak setwise stability and credible group stability or the bargaining set. In particular, weak setwise stability is not a weaker concept than credible group stability or the bargaining set.[2]

For many matching models various of the blocking notions coincide (see for instance Hatfield and Milgrom's, 2005, many-to-one matching markets with contracts). Sotomayor (1999) demonstrates that for many-to-many matching markets notions of pairwise and setwise stability indeed differ.[3]

As Echenique and Oviedo (2006, p. 233) point out, "Many-to-many matching markets are understood less well than many-to-*one* markets ...". In our opinion, one of the reasons is that different stability notions are used in various papers while their relationships are not well understood. After a description of the model, we therefore first introduce pairwise stability and the various notions of setwise stability as implied by the blocking requirements listed before. Then, the set of pairwise (weakly setwise, *etc.*) stable allocations equals the set of all allocations that cannot be pairwise (weakly setwise, *etc.*) blocked.

We then analyze the relations between the resulting sets of stable allocations for various standard preference domains and obtain the following set inclusion results:

- *general preferences:* strongly setwise stable allocations $\subseteq$ setwise stable allocations $\subseteq$ weakly setwise stable allocations $\subseteq$ pairwise stable allocations;

- *substitutable preferences:* strongly setwise stable allocations $\subseteq$ setwise stable allocations $\subseteq$ weakly setwise stable allocations $=$ pairwise stable allocations;

- *strongly substitutable preferences:* strongly setwise stable allocations $\subseteq$ setwise stable allocations $=$ weakly setwise stable allocations $=$ pairwise stable allocations.

Moreover, we provide matching markets for which all set inclusion results are tight (Examples 1, 3, and 4). Here, the introduction of contracts allows for more compact examples compared to the existing literature on many-to-many matching without contracts (*e.g.*, Echenique and Oviedo, 2006; Konishi and Ünver, 2006).

[2]We demonstrate in Appendix A that weak setwise stability is weaker than credible group stability and the bargaining set if preferences are substitutable, and that the set of weakly setwise stable allocations equals the set of credibly group stable allocations and the bargaining set if preferences are strongly substitutable.

[3]Note that for many-to-many matching markets also the core no longer coincides with the set of (setwise) stable matchings. Echenique and Oviedo (2006, Example 2.1) demonstrate that the core can be a problematic solution concept for many-to-many matching markets because agents may well have an incentive to unilaterally cancel contracts in a core allocation. We therefore follow the literature on matching (see, *e.g.*, Roth and Sotomayor, 1990; Sotomayor, 1999; Echenique and Oviedo, 2006) and focus exclusively on setwise blocking concepts instead of core concepts.

Apart from surveying set inclusion results on all three domains, we prove that for substitutable preferences the set of pairwise stable matchings is nonempty and coincides with the set of weakly setwise stable matchings. Hence weakly setwise stable matchings do exist for substitutable preferences, and weak setwise stability is indeed a setwise stability concept *in between* the familiar pairwise stability and previously considered setwise stability notions. For strongly substitutable preferences the set of pairwise stable matchings coincides with the set of setwise stable matchings.

2 Many-to-Many Matching Markets with Contracts

2.1 Buyers, Sellers, and Contracts

We consider a model, in which buyers and sellers are matched to each other (alternatively, we could model many-to-many matching markets with contracts as job-matching markets or matching markets where clients are assigned to consultancy firms). Let B denote the finite *set of buyers*, S the finite *set of sellers*, and $N = B \cup S$ the *set of agents*. By b we denote a generic buyer, by s a generic seller, and by i, j generic agents.

We model the typical features of a many-to-many matching market with contracts by assuming that each buyer can buy from several sellers and each seller can sell to several buyers. A *(bilateral) contract* specifies a trade between one buyer and one seller and further terms of trade such as, for instance, the set and quantity of items sold, the price, postal and handling fees, delivery time, guarantees for the product and delivery, and service agreements. Formally the *set of contracts* is described by a set X in connection with a mapping $\mu = (\mu_B, \mu_S) : X \to B \times S$ that specifies the bilateral structure of each contract. So, for any contract $x \in X$, $\mu(x) = (b, s)$ means that contract x is established between buyer b and seller s. Note that for two contracts $x, x' \in X, x \neq x'$, with $\mu(x) = \mu(x')$, x and x' specify different contract terms between the same buyer and seller. By $X_i = \{x \in X \mid \mu(x) = (i, s) \text{ or } \mu(x) = (b, i)\}$ we denote the set of contracts that involve agent i.

If all sellers offer the same set of contract specifications K to all buyers, then the set of contracts X can be represented as a Cartesian product $X = B \times S \times K$. An example of such a contract specification K would be a price scale that all sellers of a standardized product employ. However, note that sellers may not necessarily use the same contract specification: sellers may sell different types and numbers of products and even if they sell the same standardized products, a seller who is further away may have to charge higher shipment costs than a seller who is located closer to the buyer. For each agent it is always possible to reject any set of contracts, that is to not buy or sell certain items. We refer to the specific situation in which an agent does not buy or sell any item as a *null contract*, denoted by $\emptyset$.

As a special case consider pure matching contracts, *i.e.*, any contract only consists of a match between a buyer and a seller such that $X = \{(b, s) \in B \times S\}$. This restricts our model to many-to-many matching markets as analyzed in Sotomayor (1999), Echenique and Oviedo (2006), and Konishi and Ünver (2006).

2.2 Buyers' and Sellers' Preferences

No buyer b is allowed to have more than one contract with a certain seller at the same time (buying several items from one seller is summarized in a single contract).

Therefore, we define the set of feasible sets of contracts for buyer b by $\mathcal{X}_b := \{X' \subseteq X_b \mid \text{ for all } s \in S, |X' \cap X_s| \leq 1\}$. Symmetrically to the buyers, no seller s is allowed to have more than one contract with a buyer at the same time (selling several items to the same buyer is summarized in a single contract). The set of feasible sets of contracts for seller s is $\mathcal{X}_s := \{X' \subseteq X_s \mid \text{ for all } b \in B, |X' \cap X_b| \leq 1\}$.

Note that for each agent i the null contract is always feasible, *i.e.*, $\emptyset \in \mathcal{X}_i$, and that $X' \in \mathcal{X}_i$ implies for all $Y' \subseteq X'$, $Y' \in \mathcal{X}_i$. Each agent i has a total (linear) order over sets of feasible contracts in $\mathcal{X}_i$ represented by a preference relation R_i.[4] Given $X', Y' \in \mathcal{X}_i$, $X' \, P_i \, Y'$ means that agent i strictly prefers the set of contracts X' to the set of contacts Y'; $X' \, R_i \, Y'$ means that $X' \, P_i \, Y'$ or $X' = Y'$ and that agent i weakly prefers the set of contracts X' to the set of contacts Y'. We denote the set of all possible total orders for agent i by $\mathcal{R}_i$. Since preference relation $R_i \in \mathcal{R}_i$ is a total order, it induces a well-defined *choice correspondence* $C_i : 2^X \to \mathcal{X}_i$ that assigns to each set of contracts $X' \subseteq X$ agent i's most preferred feasible set of contracts available for her in $X' \cup \{\emptyset\}$, *i.e.*, for all $X' \subseteq X$, $C_i(X') \in \mathcal{X}_i$, $C_i(X') \subseteq X' \cup \{\emptyset\}$, and there is no $Y \subseteq X'$, $Y \in \mathcal{X}_i$, with $Y \, P_i \, C_i(X')$.

After having introduced the domain of general preferences, we will introduce two domains that have played an important role in two-sided matching, namely substitutability and strong substitutability: see for instance Echenique and Oviedo (2006) who also use these domains to analyze the relation between various solution concepts such as (setwise) stable allocations and core(-like) solutions.

2.3 Substitutability

Loosely speaking, an agent has substitutable preferences if she does not consider complementarities in the sets of contracts. To be precise, the condition for substitutable preferences states that if a contract is chosen by an agent from some set of contracts, then that contract is still chosen by the agent from a smaller set of contracts that include it. Formally, agent i's preferences R_i are *substitutable* if

(SUB) for all sets of contracts $X' \subseteq Y' \subseteq X$: $X' \cap C_i(Y') \subseteq C_i(X')$.

Equivalently one can formulate substitutability as follows (Hatfield and Milgrom, 2005). If a contract is not chosen by an agent from some set of contracts, then that contract is still not chosen by the agent from a larger set of contracts. For any set of contracts $X' \subseteq X$, $NC_i(X') := X' \setminus C_i(X')$ denotes the set of all contracts that are not chosen from set X' by choice correspondence C_i. One can easily prove that condition (SUB) is equivalent to the following condition (SUB').

(SUB') For all sets of contracts $X' \subseteq Y' \subseteq X$: $NC_i(X') \subseteq NC_i(Y')$.

2.4 Strong Substitutability

Essentially, the strong substitutable preference condition (Echenique and Oviedo, 2006) states that if a contract is chosen by an agent from some set of contracts, then that contract is still chosen by the agent from a worse set of contracts that include it. We first have to extend

[4] In other words, R_i represents a binary relation that satisfies *antisymmetry* (for all $X', Y' \in \mathcal{X}_i$, if $X' \, R_i \, Y'$ and $Y' \, R_i \, X'$, then $X' = Y'$), *transitivity* (for all $X', Y', Z' \in \mathcal{X}_i$, if $X' \, R_i \, Y'$ and $Y' \, R_i \, Z'$, then $X' \, R_i \, Z'$), and *comparability* (for all $X', Y' \in \mathcal{X}_i$, $X' \, R_i \, Y'$ or $Y' \, R_i \, X'$).

Echenique and Oviedo's definition of strong substitutability for many-to-many matching markets to many-to-many matching markets with contracts.

Let i be an agent who participates in a many-to-many matching market. Recall that in such a "pure" matching market exactly one contract between any buyer and any seller exists. Hence, all subsets of X_i are automatically feasible and one can easily extend agent i's preferences to all subsets of X: for $X', Y' \subseteq X$, $(X' \cap X_i)\, P_i\, (Y' \cap X_i)$ implies $X'\, P_i\, Y'$.

When allowing for different contracts between buyers and sellers, not all subsets of X_i are feasible anymore and the agent's strict preferences over feasible contract sets cannot be straightforwardly extended to strict preferences over all subsets of X_i. Thus, while Echenique and Oviedo (2006) can simply use the strict preference relation of an agent to compare two sets of contracts X' and Y', we have to use the agent's choice function to avoid having to extend preferences to compare sets of contracts that are not feasible.[5]

Formally, agent i's preferences R_i are *strongly substitutable* if

(SSUB) for all sets of contracts $X', Y' \subseteq X$ such that $C_i(Y') P_i C_i(X')$: $X' \cap C_i(Y') \subseteq C_i(X')$.

For many-to-many matching markets, our definition coincides with that of Echenique and Oviedo (2006). Equivalently one can formulate strong substitutability as follows. If a contract is not chosen by an agent from some set of contracts, then that contract is still not chosen by the agent from a set of contracts she considers better according to her choice function. One can easily prove that condition (SSUB) is equivalent to the following condition (SSUB').

(SSUB') For all sets of contracts $X', Y' \subseteq X$ such that $C_i(Y')\, P_i\, C_i(X')$: $NC_i(X') \cap Y' \subseteq NC_i(Y')$.

It is clear that if preferences are strongly substitutable, then they are also substitutable. Suppose, for instance, that agents are not capacity constrained in the sense that they prefer to sign an additional contract if this contract is individually rational (*i.e.,* agents have separable preferences). Then, agents have strongly substitutable preferences. In the presence of capacity constraints or quota, however, strong substitutability is rather restrictive (and stronger than responsiveness) as indicated by Echenique and Oviedo (2006, Example 6.8). For a further discussion of strong substitutability see Echenique and Oviedo (2006, Section 6.3).

2.5 Allocations in Many-to-Many Matching Markets with Contracts

Since the set of contracts X and the set of agents N remain fixed throughout this study, we denote a *(many-to-many) matching market (with contracts)* by a *preference profile* $R = (R_i)_{i \in N}$. The *set of all preference profiles* is denoted by $\mathcal{R}$.

For any set of contracts $A \subseteq X$ and any agent i we denote by $A_i = A \cap X_i$ all contracts in A that involve agent i. An *allocation* is a set of contracts $A \subseteq X$ such that **(i)** for any buyer b, $[A_b \in \mathcal{X}_b]$ and $[x \in A_b \cap X_s$ implies $x \in A_s]$ and **(ii)** for any seller s, $[A_s \in \mathcal{X}_s]$ and $[x \in A_s \cap X_b$ implies $x \in A_b]$.

We denote the set of allocations by $\mathcal{A}$. Clearly, all preference relations R_i induce *weak preferences over allocations* in a natural way. We use the same notation for preferences over feasible contract sets and allocations: for all agents $i \in N$ and allocations $A, A' \in \mathcal{A}$, $A\, R_i\, A'$ if and only if $A_i\, R_i\, A'_i$.

[5] We could also use the following preference extension: for $X', Y' \subseteq X$, $C_i(X')\, R_i\, C_i(Y')$ implies $X'\, R_i\, Y'$.

3 Pairwise and Setwise Stability

As described in the Introduction, an important criterion for an allocation to be accepted as final outcome is stability. Next, we introduce various notions of stability.

3.1 Individual Rationality and Pairwise Stability

First, since the matching markets we consider here are based on voluntary participation, a necessary condition for allocation A to be stable is individual rationality:

At any allocation A each agent i who is assigned a set of contracts $A_i \neq \emptyset$ can reject some or all contracts in A_i. Thus, an allocation A is *individually rational* for matching market $R \in \mathcal{R}$ if for all $i \in N$ and $X' \subseteq A_i$, $A_i \, R_i \, X'$. Alternatively, A is individually rational if

(IR) for all $i \in N$, $C_i(A) = A_i$.

By $IR(R) \subseteq \mathcal{A}$ we denote the *set of individually rational allocations* for matching market R.

The notion of individual rationality we use here is the same as in Echenique and Oviedo (2006) or Hatfield and Milgrom (2005). Konishi and Ünver (2006) refer to individual rationality as "individual stability."

We continue with a weak notion of stability — pairwise stability — that plays a central role in all previous articles on (many-to-many) matching (*e.g.*, Roth, 1984b; Echenique and Oviedo, 2006).

Let $b \in B$ and $s \in S$, and consider an individually rational allocation A. We define *pairwise blocking of an allocation A by buyer b and seller s* as follows.

Assume that no $y \in A$ such that $\mu(y) = (b, s)$ exists (buyer b and seller s do not have a contract at A). Then, b and s can block allocation A if there is a contract $x \in X$, $\mu(x) = (b, s)$, such that b and s strictly prefer adding x to their respective sets of contracts A_b and A_s while possibly canceling some contracts at the same time.

Assume that $y \in A$ such that $\mu(y) = (b, s)$ (buyer b and seller s have a contract at A). Then, b and s can block allocation A if there is a contract $x \in X \setminus A$, $\mu(x) = (b, s)$, such that b and s strictly prefer replacing y by x in their respective sets of contracts A_b and A_s while possibly canceling some contracts at the same time.

A buyer b and a seller s *pairwise block an allocation A* if

(PB) there exists a contract $x \in X \backslash A$, $\mu(x) = (b, s)$, such that $x \in C_b(A \cup x)$ and $x \in C_s(A \cup x)$ — buyer b and seller s would like to implement contract x while possibly canceling contracts in A.

An allocation A is *pairwise stable* if it is individually rational (IR) and no buyer and seller can pairwise block it [not (PB)]. By $PS(R) \subseteq \mathcal{A}$ we denote the *set of pairwise stable allocations* for matching market R.

Next, we introduce various notions of setwise stability.

3.2 Strong Setwise Stability

Let $B' \subseteq B$, $S' \subseteq S$, $N' = B' \cup S'$, and assume that allocation A is individually rational. We define *strong setwise blocking of an allocation A by the set of agents N'* as follows.

Each member of the blocking coalition N' can add contracts with members of N' or replace existing contracts with members of N' while possibly canceling other contracts. If all members of N' can obtain a better set of contracts by adding, replacing, and/or canceling contracts as described above, then they can strongly setwise block allocation A.

A set of agents $N' = B' \cup S'$ *strongly setwise blocks an allocation A* if

(SSB) there exists a set of contracts $X' \in \mathcal{A}$ such that
(1) for all $x \in X' \setminus A$, $\mu(x) \in B' \times S'$
— new contracts are among the members of the blocking coalition only,
(2) for all $i \in N'$, $X'_i \, P_i \, A_i$
— all members of the blocking coalition receive a better set of contracts, and
(3) for all $j \in N \setminus N'$, $X'_j \subseteq A_j$
— agents outside the blocking coalition do not receive new contracts, but possibly some of their contracts are canceled by members of the blocking coalition.

An allocation A is *strongly setwise stable* if it is individually rational (IR) and no set of agents can strongly setwise block it [not (SSB)]. By $SSS(R) \subseteq \mathcal{A}$ we denote the *set of strongly setwise stable allocations* for matching market R.

Note that from the definition of strongly setwise blocking (SSB) we can easily construct a corresponding "blocking allocation $A' \in \mathcal{A}$" defined as follows:

(i) for all $i \in N'$, $A'_i = X'_i$
— members of the blocking coalition receive their "blocking contract sets" X'_i, and
(ii) for all $j \in N \setminus N'$,
$A'_j = X'_j \cup \{x \in A_j \mid [\mu(x) = (j,s) \text{ and } s \in N \setminus N'] \text{ or } [\mu(x) = (b,j) \text{ and } b \in N \setminus N']\}$
— agents outside the blocking coalition receive all previous contracts with agents outside the blocking coalition and all previous contracts with the members of the blocking coalition that were not canceled.

If an allocation A' is such that some $N' = B' \cup S'$ can strongly setwise block allocation A with $X' = \cup_{i \in N'} A'_i$, then we say that allocation A is strongly setwise blocked via allocation A'.

Our definition of strong setwise stability corresponds to Konishi and Ünver's (2006) group stability.

The requirements of a strongly setwise block are minimal requirements for a blocking coalition to form. We next discuss two weaker setwise stability notions that are obtained by adding extra blocking conditions.

3.3 Setwise Stability

If we require that for all members of the blocking coalition the new set of contracts that is obtained by blocking (the new contracts with members of the blocking coalition and the old contracts that are kept with agents outside the blocking coalition) is preferred to the old set of contracts and individually rational, then we obtain the weaker notion of setwise blocking as, for instance, used in Echenique and Oviedo (2006).

A set of agents $N' = B' \cup S'$ *setwise blocks an allocation* A if

(SB) there exists a set of contracts $X' \in \mathcal{A}$ such that
(1) for all $x \in X' \setminus A$, $\mu(x) \in B' \times S'$
— new contracts are among the members of the blocking coalition only,
(2) for all $i \in N'$, $X'_i \, P_i \, A_i$ and $X'_i = C_i(X'_i)$,
— all members of the blocking coalition receive a better and individually rational set of contracts, and
(3) for all $j \in N \setminus N'$, $X'_j \subseteq A_j$
— agents outside the blocking coalition do not receive new contracts, but possibly some of their contracts are canceled by members of the blocking coalition.

An allocation A is *setwise stable* if it is individually rational (IR) and no set of agents can setwise block it [not (SB)]. By $SS(R) \subseteq \mathcal{A}$ we denote the *set of setwise stable allocations* for matching market R.

Note that from the definition of setwise blocking (SB) we can similarly as in Section 3.2 construct a corresponding blocking allocation $A' \in \mathcal{A}$ and define setwise blocking of allocation A via allocation A'.

Our definition of setwise stability corresponds to Echenique and Oviedo's (2006) setwise stability. It is closely related to Konishi and Ünver's (2006) and Sotomayor's (1999) setwise stability. The difference is that we only require individual rationality for the blocking coalition while Konishi and Ünver (2006) and Sotomayor (1999) require that a setwise blocking results in an individually rational blocking allocation (hence individual rationality has to hold for all agents and not only the agents who block). For substitutable preferences these two notions of setwise stability coincide, but one can easily show that for general preferences our setwise stability is stronger than Konishi and Ünver's (2006) and Sotomayor's (1999) setwise stability – our setwise blocking notion admits more setwise blockings and therefore in comparison fewer setwise stable allocations result.

3.4 Weak Setwise Stability

Next, we further strengthen the setwise blocking assumption, and thereby weaken setwise stability, by requiring that all members of the blocking coalition obtain their best set of contracts among their original contracts and the new contracts of the blocking coalition. Intuitively, such a blocking does not create any tension within the blocking coalition and thereby incorporates a very basic notion of farsightedness or credibility of a coalitional agreement (e.g., no member of the blocking coalition is tempted to reestablish a canceled contract).

A set of agents $N' = B' \cup S'$ *weakly setwise blocks an allocation* A if

(WSB) there exists a set of contracts $X' \in \mathcal{A}$ such that
(1) for all $x \in X' \setminus A$, $\mu(x) \in B' \times S'$
— new contracts are among the members of the blocking coalition only,
(2) for all $i \in N'$, $X'_i \neq A_i$ and $X'_i = C_i(A \cup X')$,
— all members of the blocking coalition receive their best set of contracts among their original contracts and the new contracts of the blocking coalition, and
(3) for all $j \in N \setminus N'$, $X'_j \subseteq A_j$
— agents outside the blocking coalition do not receive new contracts, but possibly some of their contracts are canceled by members of the blocking coalition.

An allocation A is *weakly setwise stable* if it is individually rational (IR) and no set of agents can weakly setwise block it [not (WSB)]. By $WSS(R) \subseteq \mathcal{A}$ we denote the *set of weakly setwise stable allocations* for matching market R.

Note that from the definition of weakly setwise blocking (WSB) we can similarly as in Section 3.2 construct a corresponding blocking allocation $A' \in \mathcal{A}$ and define weakly setwise blocking of allocation A via allocation A'.

Weak setwise-stability is a new stability concept that, as we will see in Section 4.2 (Theorem 2), bridges the gap between pairwise stability and previously considered setwise stability notions (see Appendix A for a discussion of weak setwise stability in relation to credible group stability and the bargaining set).

4 Relations between Stability Notions

4.1 Stability in General

We now show that on the unrestricted preference domain there is a set inclusion relation between all stability notions which is a straightforward implication of the logical relation between the blocking concepts. Furthermore, we provide a matching market for which all set inclusion relations are strict and cite a matching market without pairwise stable allocations.

Theorem 1. ***The "Onion Structure of Stable Allocations" - General Preferences***

(i) *For all matching markets $R \in \mathcal{R}$, $SSS(R) \subseteq SS(R) \subseteq WSS(R) \subseteq PS(R)$.*

(ii) *There exist matching markets $R \in \mathcal{R}$ such that $SSS(R) \subsetneq SS(R) \subsetneq WSS(R) \subsetneq PS(R)$.*

(iii) *There exist matching markets $R \in \mathcal{R}$ such that $PS(R) = \emptyset$.*

Proof. **(i)** Consider matching market $R \in \mathcal{R}$. We assume, without loss of generality, that allocations A that are blocked are individually rational.

By the definitions of strongly setwise blocking (SSB) and setwise blocking (SB) it is clear that any allocation A that is setwise blocked is also strongly setwise blocked. Hence, $SSS(R) \subseteq SS(R)$.

Assume that allocation A is weakly setwise blocked (WSB), *i.e.*, there exists a coalition $N' = B' \cup S'$ and a set of contracts $X' \in \mathcal{A}$ such that for all $x \in X' \setminus A$, $\mu(x) \in B' \times S'$, for all $i \in N'$, $X'_i \neq A_i$ and $X'_i = C_i(A \cup X')$, and for all $j \in N \setminus N'$, $X'_j \subseteq A_j$.

Note that $X'_i \neq A_i$ and $X'_i = C_i(A \cup X')$ imply that $X'_i \; P_i \; A_i$. Furthermore, since $X'_i \subseteq (A \cup X')$, $X'_i = C_i(A \cup X')$ implies $X'_i = C_i(X'_i)$. So, any allocation A that is weakly setwise blocked is also setwise blocked. Hence, $SS(R) \subseteq WSS(R)$.

Next, assume that allocation A is pairwise blocked (PB), *i.e.*, there exists a contract $x \in X \setminus A$ such that $\mu(x) = (b, s)$ and $x \in C_b(A \cup x)$ and $x \in C_s(A \cup x)$. Define $B' := \{b\}$, $S' := \{s\}$, $N' := \{b, s\}$, $X'_b := C_b(A \cup x)$, $X'_s := C_b(A \cup x)$, and $X' := X'_b \cup X'_s$. Then, $X' \setminus A = \{x\}$, for $i \in N'$, $X'_i = C_i(A \cup x)$, and for all $j \in N \setminus N'$, $x \notin X'_j$ (and therefore $X'_j \subseteq A_j$).

Thus, $X' \subseteq X$ is such that for all $x \in X' \setminus A$, $\mu(x) \in B' \times S'$, for all $i \in N'$, $X'_i \neq A_i$ and $X'_i = C_i(A \cup X')$, and for all $j \in N \setminus N'$, $X'_j \subseteq A_j$ and N' weakly setwise blocks (WSB) allocation A. So, any allocation A that is pairwise blocked is also weakly setwise blocked. Hence, $WSS(R) \subseteq PS(R)$.

(ii) With Example 1 we introduce a matching market $R \in \mathcal{R}$ such that $SSS(R) \subsetneq SS(R) \subsetneq WSS(R) \subsetneq PS(R)$.

(iii) Roth and Sotomayor (1990, Example 2.7) presented a matching market (without contracts) to demonstrate that the set of pairwise stable matchings of a many-to-one matching market can be empty. □

Example 1. ***A "Stable Matching Onion" for General Preferences***
We consider a matching market with two buyers and two sellers and a set of contracts $X = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n\}$. Table 1 indicates the bilateral structure of contracts in X. Table 2 first lists agents' preferences in its columns, *e.g.*, buyer b_1's preferences are such that $ej \; P_{b_1} \; m \; P_{b_1} \; im \; P_{b_1} \; ij \; P_{b_1} \; ef \; P_{b_1} \; ab \; P_{b_1} \; \emptyset \; P_{b_1} \ldots$, where "…" represents any ordering of the remaining feasible sets of contracts. Second, we list the following allocations for matching market R in the rows of Table 2: $A = \{a, b, c, d\}$, $B = \{e, f, g, h\}$, $C = \{i, j, k, l\}$, $D = \{i, m, n, l\}$, and $E = \{e, j\}$. Finally, Figure 1 illustrates that for matching market R we have $SSS(R) \subsetneq SS(R) \subsetneq WSS(R) \subsetneq PS(R)$: it is easy to check that $IR(R) = \{A, B, C, E\}$. Furthermore,

- allocation A is pairwise stable, but weakly setwise blocked via allocation B;
- allocation B is weakly setwise stable, but setwise blocked via allocation C, which is not a weakly setwise block because $C_{b_1}(e, f, i, j) = ej \neq ij$;
- allocation C is setwise stable, but strongly setwise blocked via allocation D, which is not a setwise block because $C_{b_1}(i, m) = m \neq im$;
- allocation E is strongly setwise stable.

x	a	b	c	d	e	f	g
$\mu(x)$	(b_1, s_1)	(b_1, s_2)	(b_2, s_1)	(b_2, s_2)	(b_1, s_1)	(b_1, s_2)	(b_2, s_1)
x	h	i	j	k	l	m	n
$\mu(x)$	(b_2, s_2)	(b_1, s_1)	(b_1, s_2)	(b_2, s_1)	(b_2, s_2)	(b_1, s_2)	(b_2, s_1)

Table 1: Example 1 – the bilateral structure of contracts.

allocation	buyer b_1	buyer b_2	seller s_1	seller s_2
E	ej			
	m	n		
D	im	nl	in	ml
C	ij	kl	ik	jl
B	ef	gh	eg	fh
A	ab	cd	ac	bd
E			e	j
	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
	$\ldots$	$\ldots$	$\ldots$	$\ldots$

Table 2: Example 1 – preferences and allocations A, B, C, D, E.

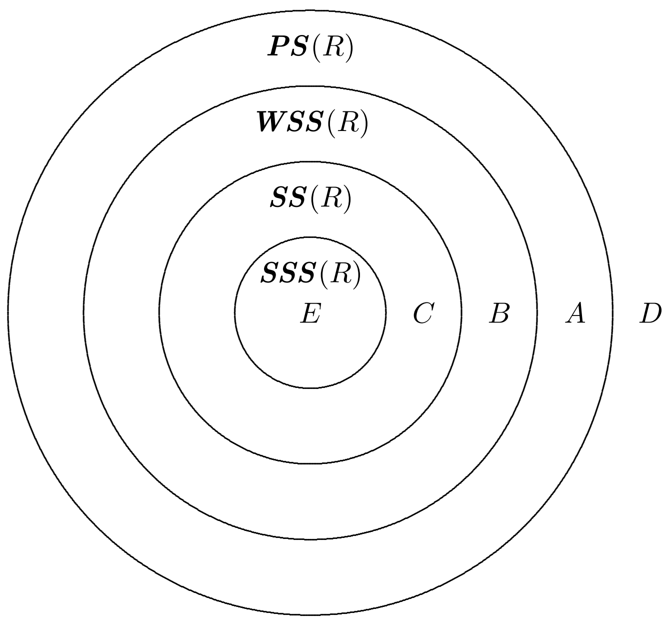

Figure 1: Example 1 – a "stable matching onion" and allocations A, B, C, D, E.

4.2 Stability and Substitutability

In his seminal paper Roth (1984b) proved that the set of pairwise stable matchings is nonempty for substitutable preferences. Here, we prove that weak setwise stability and pairwise stability coincide. Hence, for substitutable preferences we have found a setwise blocking concept that induces a nonempty set of stable allocations. Furthermore, we provide a matching market with substitutable preferences for which the remaining set inclusion relations are strict and cite a matching market with substitutable preferences without setwise stable allocations.

Theorem 2. ***The "Onion Structure of Stable Allocations" - Substitutable Preferences***

(i) *For matching markets with substitutable preferences an allocation is weakly setwise stable if and only if it is pairwise stable.*

(ii) *For matching markets with substitutable preferences the set of pairwise (weakly setwise) stable matchings is nonempty.*

Hence, (i) and (ii) imply that for all matching markets with substitutable preferences $R \in \mathcal{R}$,

$$SSS(R) \subseteq SS(R) \subseteq WSS(R) = PS(R) \neq \emptyset.$$

(iii) *There exist matching markets with substitutable preferences $R \in \mathcal{R}$ such that*

$$SSS(R) \subsetneq SS(R) \text{ and } SS(R) \subsetneq WSS(R) = PS(R).$$

(iv) *There exist matching markets with substitutable preferences $R \in \mathcal{R}$ such that $SS(R) = \emptyset$.*

Proof. **(i)** Let R be substitutable. By Theorem 1 (i), $WSS(R) \subseteq PS(R)$. It remains to prove that $PS(R) \subseteq WSS(R)$.

Assume that allocation A is weakly setwise blocked (WSB) (and, without loss of generality, individually rational), *i.e.*, there exists a coalition $N' = B' \cup S'$ and a set of contracts $X' \in \mathcal{A}$ such that for all $x \in X' \setminus A$, $\mu(x) \in B' \times S'$, for all $i \in N'$, $X'_i \neq A_i$ and $X'_i = C_i(A \cup X')$, and for all $j \in N \setminus N'$, $X'_j \subseteq A_j$. Then, there exist $b \in B'$, $s \in S'$, and $x \in X' \setminus A$ such that $\mu(x) = (b, s)$.

Let $i \in \{b, s\}$ and assume that $x \notin C_i(A \cup x)$ or, equivalently, $x \in NC_i(A \cup x)$. Note that $(A \cup x) \subseteq (A \cup X'_i)$. Then, by substitutability (SUB'), $x \in NC_i(A \cup X'_i)$. But then, since $x \in X'_i$, $X'_i \neq C_i(A \cup X'_i)$; a contradiction. Thus, $x \in C_b(A \cup x)$ and $x \in C_s(A \cup x)$ and allocation A is pairwise blocked (PB). So, any allocation A that is weakly setwise blocked is also pairwise blocked. Hence, $PS(R) \subseteq WSS(R)$.

Note that substitutability is a crucial assumption for our proof of the identity of $WSS(R)$ and $PS(R)$. A simple example of a situation where $WSS(R) \neq PS(R)$ is Example 2.

(ii) Roth (1984b, Theorem 1) proved for many-to-many matching markets with contracts and substitutable preferences that the set of pairwise stable matchings is nonempty.

(iii) With Example 3 we introduce a matching market with substitutable preferences $R \in \mathcal{R}$ such that $SSS(R) \subsetneq SS(R)$. With Example 4 we introduce a matching market with substitutable preferences $R \in \mathcal{R}$ such that $SS(R) \subsetneq WSS(R) = PS(R)$.[6]

(iv) Blair (1988, Example 2.6) presented a matching market with substitutable preferences (without contracts) to demonstrate that the core of a many-to-many matching market with substitutable preferences can be empty. It is easy to check that for this example (see also Roth and Sotomayor, 1990, Example 6.9), $SS(R) = \emptyset$. □

[6] In their Example 4, Konishi and Ünver (2006) provide a 16-agent many-to-many matching market (without contracts) that also exhibits $SS(R) \subsetneq PS(R)$.

Example 2. $PS(R) \neq WSS(R)$ ***if preferences of one agent are not substitutable.*** We consider a matching market with two buyers and two sellers and a set of contracts $X = \{a, b, c\}$. Table 3 indicates the bilateral structure of contracts in X. Table 4 first lists agents' preferences in its columns. Note that b_1's preferences are not substitutable while all other agent's preferences are. Second, we list the following allocations for matching market R in the rows of Table 4: $A = \{c\}$ and $B = \{a, b\}$. Obviously, allocation A is pairwise stable. However, it can be weakly setwise blocked by $\{b_1, s_1, s_2\}$ who agree upon allocation B. Hence, $PS(R) = \{A, B\} \neq \{B\} = WSS(R)$.

x	a	b	c
$\mu(x)$	(b_1, s_1)	(b_1, s_2)	(b_2, s_1)

Table 3: Example 2 – the bilateral structure of contracts.

allocation	buyer b_1	buyer b_2	seller s_1	seller s_2
B	ab		a	b
A	$\emptyset$	c	c	$\emptyset$
	a	$\emptyset$	$\emptyset$	
	b		ac	

Table 4: Example 2 – preferences and allocations A and B.

Example 3. $SSS(R) \subsetneq SS(R)$ ***for substitutable preferences.*** We consider a matching market with two buyers and two sellers and a set of contracts $X = \{a, b, c, d, e, f\}$. Table 5 indicates the bilateral structure of contracts in X. Table 6 first lists agents' substitutable preferences in its columns. Second, we list the following allocations for matching market R in the rows of Table 6: $A = \{a, b\}$, $B = \{c, d, e, f\}$, and $C = \{c, f\}$. Finally, Figure 2 illustrates that for matching market R we have $SSS(R) \subsetneq SS(R) = WSS(R) = PS(R)$. To be specific,

- allocation A is pairwise and (weakly) setwise stable, but strongly setwise blocked via allocation B, which is not a setwise block because $C_{b_1}(c, d) = c$;
- allocation C is strongly setwise stable.

x	a	b	c	d	e	f
$\mu(x)$	(b_1, s_1)	(b_2, s_2)	(b_1, s_1)	(b_1, s_2)	(b_2, s_1)	(b_2, s_2)

Table 5: Example 3 – the bilateral structure of contracts.

allocation	buyer b_1	buyer b_2	seller s_1	seller s_2
C	c	f		
			e	d
B	cd	ef	ce	df
A	a	b	a	b
	d	e		
C			c	f
	ad	eb	ae	bd
	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 6: Example 3 – preferences and allocations A, B, and C.

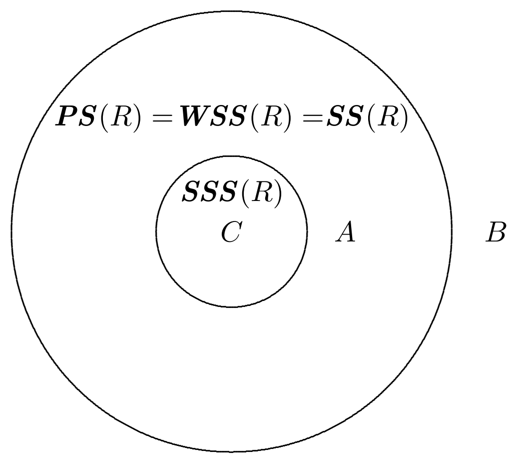

Figure 2: Example 3 – $SSS(R) \subsetneq SS(R) (= WSS(R) = PS(R))$ for substitutable preferences.

Example 4. $SS(R) \subsetneq WSS(R) = PS(R)$ ***for substitutable preferences***
We consider a matching market with two buyers and two sellers and a set of contracts $X = \{a, b, c, d, e, f, g, h\}$. Table 7 indicates the bilateral structure of contracts in X. Table 8 first lists agents' substitutable preferences in its columns. Second, we list the following allocations for matching market R in the rows of Table 8: $A = \{a, b, c, d\}$ and $B = \{e, f, g, h\}$. Finally, Figure 3 illustrates that for matching market R we have $SS(R) \subsetneq WSS(R) = PS(R)$. To be specific,

- allocation A is pairwise and weakly setwise stable but not setwise stable (B setwise blocks allocation A but is not a weakly setwise block because *e.g.*, $C_{b_1}(abef) = af \neq ef$).
- allocation B is setwise stable.

x	a	b	c	d	e	f	g	h
$\mu(x)$	(b_1, s_1)	(b_1, s_2)	(b_2, s_1)	(b_2, s_2)	(b_1, s_1)	(b_1, s_2)	(b_2, s_1)	(b_2, s_2)

Table 7: Example 4 – the bilateral structure of contracts.

allocation	buyer b_1	buyer b_2	seller s_1	seller s_2
	af	dg	ce	bh
B	ef	gh	eg	fh
A	ab	cd	ac	bd
	be	ch	ag	df
	f	g	e	h
	b	c	a	d
	a	d	c	b
	e	h	g	f
	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 8: Example 4 – substitutable preferences, and allocations A and B.

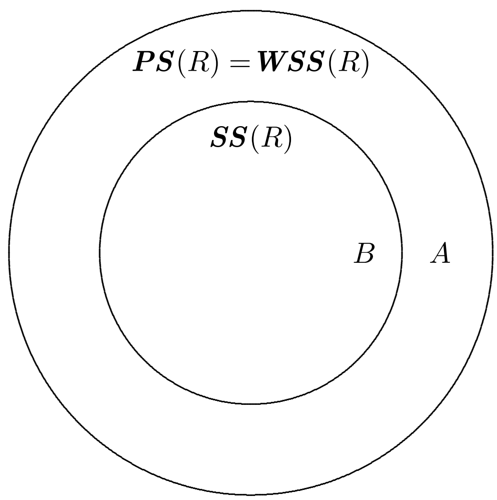

Figure 3: Example 4 – $SS(R) \subsetneq WSS(R) = PS(R)$ for substitutable preferences.

Remark 1. *Setwise Stability in Roth (1984b)*
Roth's (1984b) definition of (setwise) stability does not coincide with the notions introduced in Section 3. Roth also strengthened the (setwise) blocking condition and thereby weakened (setwise) stability. He required that all members of the blocking coalition obtain a subset of their best set of contracts among their original contracts and the new contracts of the blocking coalition (we provide the formal definition and analysis in Appendix B). In Appendix B we show that there is a set inclusion relation between all stability notions introduced so far if instead of setwise stability we consider Roth stability. On the other hand, Roth stability and setwise stability are not logically related. Hence, Roth (1984b) and Sotomayor (1999) refer to different stability notions when they use the term (setwise) stability. Furthermore, for substitutable preferences, Roth (1984b) proved that Roth stability coincides with pairwise stability (Roth, 1984b, Lemma 2) and that the set of pairwise stable matchings is always nonempty (Roth, 1984b, Theorem 1).

4.3 Stability and Strong Substitutability

If one market-side has strong substitutable preferences and the other market-side has substitutable preferences, setwise stability, weak setwise stability, and pairwise stability coincide. Echenique and Oviedo (2006, Theorem 6.1, Proposition 6.3, and Theorem 6.4) proved this result for many-to-many matching markets. Since we consider many-to-many matching markets with contracts, our result implies theirs as a special case (in addition, here we give a direct and shorter proof). Since strong substitutability implies substitutability, Theorem 2 implies that the set of setwise stable matchings is nonempty. Furthermore, we provide a matching market with strongly substitutable preferences for which the set of strongly setwise stable matchings is a strict subset of the set of setwise stable matchings and a matching market with strongly substitutable preferences without strongly setwise stable allocations.

Theorem 3. ***The "Onion Structure of Stable Allocations" - Strongly Substitutable Preferences***

(i) *Suppose sellers (buyers) have strong substitutable preferences and buyers (sellers) have substitutable preferences. Then, an allocation is setwise stable if and only if it is pairwise stable. Hence, for all such matching markets $R \in \mathcal{R}$, $SS(R) = WSS(R) = PS(R) \neq \emptyset$.*

(ii) *There exist matching markets with strongly substitutable preferences $R \in \mathcal{R}$ such that $SSS(R) \subsetneq SS(R)$.*

(iii) *There exist matching markets with strongly substitutable preferences $R \in \mathcal{R}$ such that $SSS(R) = \emptyset$.*

Proof. **(i)** By Theorem 1, $SS(R) \subseteq WSS(R) \subseteq PS(R)$. We complete the proof by showing that $PS(R) \subseteq SS(R)$.

Assume that allocation A is setwise blocked (SB) (and, without loss of generality, individually rational), *i.e.*, there exists a coalition $N' = B' \cup S'$ and a set of contracts $X' \in \mathcal{A}$ such that for all $x \in X' \setminus A$, $\mu(x) \in B' \times S'$, for all $i \in N'$, $X'_i \, P_i \, A_i$ and $X'_i = C_i(X'_i)$, and for all $j \in N \setminus N'$, $X'_j \subseteq A_j$. Let $b \in B'$. Then, by individual rationality of A and $X'_b \, P_b \, A_b$, $C_b(X' \cup A) \nsubseteq A$ and there exist $x \in X' \setminus A$ with $\mu(x) = (b, s)$ such that $x \in C_b(X' \cup A)$. By substitutability of buyer's preferences (SUB), $x \in C_b(A \cup \{x\})$.

Assume that $x \notin C_s(A \cup x)$ or, equivalently, $x \in NC_s(A \cup x)$. Hence, $C_s(A \cup x) = C_s(A) = A_s$. Since $x \in X' \setminus A$, $X' \cup x = X'$ and $C_s(X' \cup x) = C_s(X') = X'_s$. Thus, $X'_s \, P_s \, A_s$ implies $C_s(X' \cup x) \, P_s \, C_s(A \cup x)$. By strong substitutability of seller's preferences (SSUB'), $x \in NC_s(X' \cup x)$; contradicting $C_s(X' \cup x) = X'_s$. Hence, $x \in C_s(A \cup x)$.

Thus, $x \in C_b(A \cup x)$ and $x \in C_s(A \cup x)$ and allocation A is pairwise blocked (PB). So, any allocation A that is setwise blocked is also pairwise blocked. Hence, $PS(R) \subseteq SS(R)$.

Note that strong substitutability is a crucial assumption for our proof of the identity of $SS(R)$ and $PS(R)$. A simple example (with substitutable preferences) of a situation where $SS(R) \neq PS(R)$ is Blair (1988, Example 2.6).

(ii) Preferences in Example 3 are actually strongly substitutable. Hence, this example is a matching market with strongly substitutable preferences $R \in \mathcal{R}$ such that $SSS(R) \subsetneq SS(R)$.

(iii) With Example 5 we introduce a matching market with strongly substitutable preferences $R \in \mathcal{R}$ such that $SSS(R) = \emptyset$. □

Example 5. ***Strongly Substitutable Preferences and an Empty Set of Strongly Setwise Stable Matchings***

We consider a matching market with three buyers and three sellers and a set of contracts $X = \{a, b, c, d, e, f, h, i\}$. Table 9 indicates the bilateral structure of contracts in X. Table 10 lists agents' strongly substitutable preferences in its columns. To check that there exists no strongly setwise stable allocation, we start with the observation that no allocation that assigns all agents a set of contracts below their respective (complete) set of contracts (*i.e.,* allocation X) can be strongly setwise stable (agents can use all their contracts to block). Note that for instance $A = \{a, d, g\}$ is pairwise (and weakly setwise) stable; the only feasible blocking (X) is not individually rational. Next, no allocation that assigns an agent her (complete) set of contracts can be strongly setwise stable since any such agent would be better off by canceling one of the contracts. Thus, at least one agent has to receive her best set of contracts. For instance, assume that there exists a strongly setwise stable allocation B at which buyer b_1 obtains contracts a and b. Then, seller s_2 has to obtain only contract b (no other contract set containing b is individually rational for seller s_2). But then agents b_2 and s_2 could block allocation B using contract d. So, no such strongly setwise stable allocation B at which buyer b_1 obtains her best set of contracts exists. Similarly, we can prove that no strongly setwise stable allocation B at which agent $i \in \{b_2, b_3, s_1, s_2, s_3\}$ obtains her best set of contracts exists. Hence, no allocation can be strongly setwise stable.

x	a	b	c	d	e
$\mu(x)$	(b_1, s_1)	(b_1, s_2)	(b_1, s_3)	(b_2, s_2)	(b_2, s_3)
x	f	g	h	i	
$\mu(x)$	(b_2, s_1)	(b_3, s_3)	(b_3, s_1)	(b_3, s_2)	

Table 9: Example 5 – the bilateral structure of contracts.

buyer b_1	buyer b_2	buyer b_3	seller s_1	seller s_2	seller s_3
ab	de	gh	af	di	cg
abc	def	ghi	afh	bdi	cge
a	d	g	a	d	g
ac	df	gi	ah	bd	eg
bc	ef	hi	fh	bi	ce
b	e	h	f	i	c
c	f	i	h	b	e
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 10: Example 5 – preferences.

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Appendix

A Credible Group Stability and the Bargaining Set

In this Appendix we investigate the relation between weak setwise stability and credible group stability as introduced by Konishi and Ünver (2006) and the bargaining set as discussed in Echenique and Oviedo (2006). Similarly to weak setwise stability, both concepts add additional robustness conditions to setwise blockings to avoid future conflict within the blocking coalition. To simplify the comparison, we follow Konishi and Ünver (2006) and Echenique and Oviedo (2006) in considering many-to-many matching markets with contracts that only fix a match between buyers and sellers. Hence, throughout this appendix we set the set of contracts that involve buyer b to $X_b = \{(b, s) \mid s \in S\}$ and the set of contracts that involve seller s to $X_s = \{(b, s) \mid b \in B\}$. A feasible set of contracts for agent i can therefore simply be interpreted as a set of agents of the other market side (*i.e.,* a set of "partners") and an allocation is simply a matching as defined for many-to-many matching markets (see, for instance, Konishi and Ünver, 2006, p. 61).

A.1 Credible Group Stability

Credible group stability as introduced in Konishi and Ünver (2006) strengthens the strong setwise blocking condition and thereby weakens strong setwise stability. It requires that a set of contracts (*i.e.,* partners) agreed upon by a blocking coalition is pairwise stable within the members of the blocking coalition assuming that outsiders are passive players. Formally, let $N' = B' \cup S'$ strongly setwise block an allocation A with the set of contracts $X' \in \mathcal{A}$. Denote the resulting allocation by A'. For $i \in N'$, by $\bar{A}_i = \{(b, s) \in A \mid \{b, s\} \cap N' = i\}$ we denote the set of contracts in A that involve agent i but no other agent of the blocking coalition. The strong setwise blocking is called *executable* if for all $i \in N'$, $C_i(A'_i \cup \bar{A}_i) = A'_i$ and for all $i, j \in N'$ with $(i, j) \notin A'$, $[(i, j) \in C_i(A'_i \cup \bar{A}_i \cup \{(i, j)\})$ implies $(i, j) \notin C_j(A'_j \cup \bar{A}_j \cup \{(i, j)\})]$ and $[(i, j) \in C_j(A'_j \cup \bar{A}_j \cup \{(i, j)\})$ implies $(i, j) \notin C_i(A'_i \cup \bar{A}_i \cup \{(i, j)\})]$.

An allocation A is *credibly group stable* if it is immune to any executable strong setwise blocking. By $CGS(R) \subseteq \mathcal{A}$ we denote the *set of credibly group stable allocations* for matching market R.

A.2 The Bargaining Set

The bargaining set as analyzed in Echenique and Oviedo (2006) strengthens the setwise blocking condition and thereby weakens setwise stability. It requires that a set of contracts (*i.e.,* partners) agreed upon by a blocking coalition is a setwise stable allocation within the members of the blocking coalition. Formally, let $N' = B' \cup S'$ setwise block an allocation A with the set of contracts $X' \in \mathcal{A}$. Denote the resulting allocation by A'. Then, the setwise block has a *counterobjection* if there exists a set of agents $N'' \subseteq N'$ and a set of contracts $X'' \in \mathcal{A}$ such that N'' setwise blocks the allocation A' with X''.

An allocation A is in the *bargaining set* if it is individually rational and every setwise blocking of A has a counterobjection. By $B(R) \subseteq \mathcal{A}$ we denote the *barganing set* for matching market R.

A.3 Results for General Preferences

We show that there is no logical relationship between the set of weakly setwise stable matchings and the set of credibly group stable matchings or the bargaining set. In particular, weak setwise stability is not a weaker concept than credible group stability or the bargaining set.

Proposition 1. ***Comparison for General Preferences***

(i) *There exist matching markets $R \in \mathcal{R}$ such that*

$$WSS(R) \supsetneq CGS(R) \text{ and } WSS(R) \supsetneq B(R).$$

(ii) *There exist matching markets $R \in \mathcal{R}$ such that*

$$WSS(R) \subsetneq CGS(R) \text{ and } WSS(R) \subsetneq B(R).$$

Proof. **(i)** Konishi and Ünver (2006, Example 4) present a matching market (without contracts) with 16 agents (eight of each market side). The agents of one market side have responsive preferences (which implies that they have substitutable preferences) and the agents of the other market side have substitutable preferences. In the example, two matchings (denoted by μ and μ') are pairwise stable and the set of all agents strongly setwise blocks μ by μ' in an executable way. Hence, μ is pairwise stable but not credibly group stable. μ' cannot be strongly setwise blocked and is therefore credibly group stable. As μ' is the only credibly group stable matching, the example provides a matching market R with $PS(R) \supsetneq CGS(R)$. By Theorem 2 (i), for substitutable preferences $PS(R) = WSS(R)$. Hence Konishi and Ünver (2006, Example 4) also provides a matching market for which $WSS(R) \supsetneq CGS(R)$. Moreover, the set of all agents setwise blocks μ by μ' and μ' is setwise stable and therefore has no counterobjection. Hence, μ is not in the bargaining set. As μ' is the only matching which cannot be setwise blocked without counterobjection, the example provides a matching market R with $PS(R) \supsetneq B(R)$ and thereby $WSS(R) \supsetneq B(R)$.

(ii) With Example 6 we introduce a matching market $R \in \mathcal{R}$ such that $WSS(R) \subsetneq CGS(R)$ and $WSS(R) \subsetneq B(R)$. □

Example 6. $WSS(R) \subsetneq CGS(R)$ ***and*** $WSS(R) \subsetneq B(R)$ ***for General Preferences***
We consider a matching market with four buyers and six sellers. With some abuse of notation, Tables 11 and 12 list buyers' and sellers' preferences in its columns as preferences over sets of agents on the other side of the market (*e.g.*, for b_1 we list $\{s_3, s_5\}$ as the best set of contracts instead of $\{(b_1, s_3), (b_1, s_5)\}$). A and A' are the only individually rational allocations. Allocation A is pairwise stable and can only be (setwise) blocked by N and the allocation A'. Since for all $i \in N$, $A'_i = C_i(A \cup A')$, N also weakly setwise blocks A by A'. Allocation A' is not pairwise stable and can only be blocked by $\{b_1, s_5\}$. Hence, by Theorem 1 (i), A' is not weakly setwise stable. Thus, the set of weakly setwise stable allocations is empty.

Next, A can only be strongly setwise blocked by A'. However, b_1 and s_5 are members of blocking coalition N and are not matched under A' but $(b_1, s_5) \in C_{s_5}(A'_{s_5} \cup \{(b_1, s_5)\})$ and $(b_1, s_5) \in C_{b_1}(A'_{b_1} \cup \{(b_1, s_5)\})$. Hence, the only strong setwise blocking is not executable and A is credibly group stable. Finally observe that A can only be setwise blocked by A'. However, b_1 and s_5 are members of the blocking coalition and pairwise block A'. Hence, the only setwise block of A has a counterobjection and A is in the bargaining set.

allocation	buyer b_1	buyer b_2	buyer b_3	buyer b_4
	$\{s_3, s_5\}$			
A'	$\{s_3, s_4\}$	$\{s_1, s_2\}$	$\{s_2, s_4, s_5\}$	$\{s_1, s_3, s_6\}$
A	$\{s_1, s_2\}$	$\{s_3, s_4\}$	$\{s_1, s_3\}$	$\{s_2, s_4, s_5\}$
	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
	$\dots$	$\dots$	$\dots$	$\dots$

Table 11: Example – buyer's preferences over sets of sellers and allocations A and A'.

allocation	seller s_1	seller s_2	seller s_3	seller s_4	seller s_5	seller s_6
					$\{b_1\}$	
A'	$\{b_2, b_4\}$	$\{b_2, b_3\}$	$\{b_1, b_4\}$	$\{b_1, b_3\}$	$\{b_3\}$	$\{b_4\}$
A	$\{b_1, b_3\}$	$\{b_1, b_4\}$	$\{b_2, b_3\}$	$\{b_2, b_4\}$	$\{b_4\}$	
	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

Table 12: Example – seller's preferences over sets of buyers and allocations A and A'.

A.4 Results for Substitutable Preferences

We now show that for substitutable preferences there is a set inclusion relation between the stability notions discussed in the main part of the paper and the set of credibly group stable allocations and the bargaining set.

Proposition 2. ***Comparison for Substitutable Preferences***

(i) *For all matching markets $R \in \mathcal{R}$ with substitutable preferences,*

$$SS(R) \subseteq CGS(R) \subseteq WSS(R) = PS(R) \text{ and } SS(R) \subseteq B(R) \subseteq WSS(R) = PS(R).$$

(ii) *There exist matching markets $R \in \mathcal{R}$ with substitutable preferences such that*

$$WSS(R) \supsetneq CGS(R) \text{ and } WSS(R) \supsetneq B(R).$$

Proof. **(i)** Recall from Theorem 2 (i) that, for substitutable preferences, $WSS(R) = PS(R)$.

Konishi and Ünver (2006, Proposition 1) proved that $CGS(R) \subseteq PS(R)$ for substitutable preferences and $SS(R) \subseteq CGS(R)$ follows from the definition of credible group stability by strengthening the strong setwise blocking notion (see also Konishi and Ünver, 2006, p. 64).

To see that $B(R) \subseteq PS(R)$ we show that an allocation A that is not pairwise stable is also not in the bargaining set. Assume that allocation A is pairwise blocked. Then, A is either not individually rational or there exists a pair of buyers and sellers $\{b, s\}$ that pairwise blocks allocation A. Suppose A is not individually rational for agent i. Hence, $\{i\}$ can (setwise) block allocation A by canceling all individually irrational contracts. Then, the resulting allocation is individually rational for i such that the setwise blocking by $\{i\}$ we have considered does not have a counterobjection. Now suppose A is individually rational and there exists a pair of buyers and sellers $\{b, s\}$ that pairwise blocks allocation A. Hence, $\{b, s\}$ can (setwise) block allocation A by matching and by canceling all individually irrational contracts. Then, the resulting allocation is individually rational for b and s and cannot be blocked by $\{b, s\}$ such that the setwise blocking we have considered does not have a counterobjection.

To summarize, in both cases we considered, A not being pairwise stable implies that A is not in the bargaining set. Finally, by definition, every setwise blocking without counterobjection is also a setwise blocking. Hence, $SS(R) \subseteq B(R)$.

(ii) Recall from the proof of Proposition 1 (i) that Konishi and Ünver's (2006) Example 4 presents a matching market with substitutable preferences for which $WSS(R) \supsetneq CGS(R)$ and $WSS(R) \supsetneq B(R)$. □

A.5 Results for Strongly Substitutable Preferences

For strongly substitutable preferences, the following stability notions discussed in the main part of the paper coincide with credible group stability and the bargaining set.

Proposition 3. ***Comparison for Strongly Substitutable Preferences***
For all matching markets $R \in \mathcal{R}$ with strongly substitutable preferences,

$$SS(R) = B(R) = CGS(R) = WSS(R) = PS(R).$$

Proof. Recall from Proposition 2 (i) that for (strongly) substitutable preferences, $SS(R) \subseteq CGS(R) \subseteq WSS(R) = PS(R)$ and $SS(R) \subseteq B(R) \subseteq WSS(R) = PS(R)$. By Theorem 3 (i), for strongly substitutable preferences, $SS(R) = PS(R)$. Hence, for strongly substitutable preferences, $SS(R) = B(R) = CGS(R) = WSS(R) = PS(R)$. □

B Roth's (1984b) (Setwise) Stability

Roth's (1984b) definition of (setwise) stability does not coincide with the notions introduced in Section 3. Roth also strengthened the (setwise) blocking condition and thereby weakened (setwise) stability. He required that all members of the blocking coalition obtain a subset of their best set of contracts among their original contracts and the new contracts of the blocking coalition.

A set of agents $N' = B' \cup S'$ *Roth blocks an allocation* A if
(RSB) there exists a set of contracts $Y' \in \mathcal{A}$ such that
(1) $\cup_{x \in Y'} \mu(x) = B' \times S'$ and
(2) for all $i \in N'$, $Y'_i \subseteq C_i(A \cup Y')\ P_i\ A_i$.

An allocation A is *Roth stable* if it is individually rational (IR) and no set of agents can Roth block allocation A [not (RSB)]. By $RSS(R) \subseteq \mathcal{A}$ we denote the *set of Roth stable allocations* for matching market R.

We now show that there is a set inclusion relation between all stability notions introduced so far if instead of setwise stability we consider Roth stability. On the other hand, Roth stability and setwise stability are not logically related. Hence, Roth (1984b) and Sotomayor (1999) refer to different stability notions when they use the term (setwise) stability.

Proposition 4. ***Roth Stability and General Preferences***

(i) *For all matching markets $R \in \mathcal{R}$, $SSS(R) \subseteq RSS(R) \subseteq WSS(R) \subseteq PS(R)$.*

(ii) *There exist matching markets $R \in \mathcal{R}$ such that $SS(R) \subsetneq RSS(R)$.*

(iii) *There exist matching markets $R \in \mathcal{R}$ such that $RSS(R) \subsetneq SS(R)$.*

Proof. **(i)** Consider matching market $R \in \mathcal{R}$.

By the definitions of strongly setwise blocking (SSB) and Roth setwise blocking (RSB) it is clear that any allocation A that is Roth setwise blocked is also strongly setwise blocked. Hence, $SSS(R) \subseteq RSS(R)$.

Assume that allocation A is weakly setwise blocked (WSB), *i.e.*, there exists a coalition $N' = B' \cup S'$ and a set contracts $X' \in \mathcal{A}$ such that for all $x \in X' \setminus A$, $\mu(x) \in B' \times S'$, for all $i \in N'$, $X'_i \neq A_i$ and $X'_i = C_i(A \cup X')$, and for all $j \in N \setminus N'$, $X'_j \subseteq A_j$. Let $Y' := X' \setminus A$. Then, for all $i \in N'$, $Y'_i \subseteq C_i(A \cup Y')\ P_i\ A_i$. So, any allocation A that is weakly setwise blocked is also Roth setwise blocked. Hence, $RSS(R) \subseteq WSS(R)$.

(ii) For Example 4, $SS(R) \subsetneq WSS(R) = RSS(R)$.

(iii) With Example 7 we introduce a matching market $R \in \mathcal{R}$ such that $RSS(R) \subsetneq SS(R)$. □

Example 7. $RSS(R) \subsetneq SS(R)$ ***for General Preferences***
We consider a matching market with three buyers and two sellers and a set of contracts $X = \{a, b, c, d, e, f, g, h, i, j\}$. Table 13 indicates the bilateral structure of contracts in X. Table 14 first lists agents' preferences in its columns. Second, we indicate the unique individually rational allocation for matching market R in the second row of Table 14: $A = \{a, b, d, e, i, j\}$. Obviously, allocation A is setwise stable and $SS(R) = \{A\}$. However, it can be Roth setwise blocked by $\{b_1, b_2, b_3, s_1, s_2\}$ and $Y' = \{c, f, g, h\}$. Hence, $RSS(R) = \emptyset \subsetneq \{A\} = SS(R)$. Note, that the Roth block Y' is not an individually rational allocation.

x	a	b	c	d	e
$\mu(x)$	(b_1, s_1)	(b_1, s_2)	(b_1, s_2)	(b_2, s_1)	(b_2, s_2)
x	f	g	h	i	j
$\mu(x)$	(b_2, s_1)	(b_3, s_1)	(b_3, s_2)	(b_3, s_1)	(b_3, s_1)

Table 13: Example 7 – the bilateral structure of contracts.

allocation	buyer b_1	buyer b_2	buyer b_3	seller s_1	seller s_2
	c	f	gh	afg	ceh
A	ab	de	ij	adi	bej
	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
	...	...	...	...	...

Table 14: Example 7 – preferences and allocation A.

Remark 2. ***Roth Stability and Substitutable Preferences***
For substitutable preferences, Roth (1984b) proved that Roth stability coincides with pairwise stability (Roth, 1984b, Lemma 2) and that the set of pairwise stable matchings is always nonempty (Roth, 1984b, Theorem 1).