Over-reaction to Demand Changes due to Subjective and Quantitative Forecasting

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Abstract

In this paper, we study managers’ errors in decision making for inventory replenishment and how these errors affect their inventory system. In particular, primarily for its expected relationship with the bullwhip effect, we focus on the error of the over-reaction to demand changes and a common contributor of decision making biases: forecasting of demand. By over-reaction we mean that the manager over (under) orders when seeing a change in demand. We show that our representative manifestations of forecasting - managers’ subjective response to demand signals and the use of simple quantitative forecasting techniques - share similar consequences: both can result in an increase in internal costs and in the uncertainty and volatility of the system’s replenishment orders. Further results of this paper provide argument and thus incentive for mitigating the bullwhip effect by relating it to decision making that would help reduce costs for the manager as well.

Keywords: forecasting, inventory management, bullwhip effect, behavioral operations

1 Introduction

The recent supply chain management research has been fruitful. It expanded the scope of classic multi-echelon inventory theory by incorporating more realistic considerations such as competition, decentralized ownership, information asymmetries and incentive issues. As a result, it has provided much insight on how to improve the efficiency of supply chains through coordination, information sharing and effective competition. However, most of this research inherits the long tradition of assuming

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that the decision makers are completely rational, that is, that they always optimize when managing resources. Supply chains are however incredibly complex with numerous moving parts and activities. In response to the complexity of the supply chains, many managers must limit the problem space, whether consciously or unconsciously, by selecting a limited set of inputs to inform their decisions, and use simplifying heuristics (shortcuts) when making these decisions, Kahneman, Slovic and Tversky (1982), Conlisk (1996), Oliva and Watson (2004). Thus while managers generally try to optimize, they do not necessarily reach the best possible decisions all the time due to their limited information processing and decision-making capabilities.

In this paper, we study managers’ systematic deviations in inventory replenishments from their best possible decision and how these deviations affect their systems. In this study, we focus on one type of error: the over-reaction to demand changes and a common contributor of decision making biases: forecasting of demand. By over-reaction we mean that the manager over (under) orders when seeing a change in demand and the benchmark for this over-reaction is the manager’s best possible decision. We consider two forecasting approaches which can manifest this error within the appropriate decision making setting: managers’ subjective response to demand signals and the use of simple quantitative forecasting techniques. Although our examination of the settings resulting in this error is not exhaustive, we believe that our results generalize to analogous settings where the bias may not stem from forecasting demand or the particulars of the inventory system may not exactly match ours.

We examine this error because of i) its expected relationship to the bullwhip effect which is considered a major driver/symptom of impaired supply chain performance, Sterman (1989), Naish (1994), Lee, Padmanabhan and Whang (1997), Paik and Bagchi (2007), ii) the real world occurrences of orders especially of upstream suppliers confirming the error Kelly (1995), iii) the academic support for the assumptions of the models used to study the error, Chen et al. (2000), Lawrence et. al. (2006), and iv) the pervasiveness of the related decision making biases and settings that we examine, Kahneman, Slovic and Tversky (1982), Dalrymple (1987), Armstrong (2001), Sanders and Manrodt (1994, 2003), Massey and Wu (2005), Kaipia, Korhonen and Hartiala (2006).

In their seminal article, Lee Padmanabhan and Whang propose optimally responding to demand signals as one cause of the bullwhip effect. They model this cause using stationary demand with positive serial correlation. As we emphasize in section 3, this optimal response implies the best possible reaction to demand changes for the manager. However real world experiences, e.g., in Motorola’s supply chain, see Kelly, and simulations, Sterman; and Paik and Bagchi, suggest that, as a cause of the bullwhip effect, responding to demand signals can be broadened beyond considering only optimal responses. Broadening the range of behavioral responses to contexts, as suggested here by our focus
on errors, can contribute to our understanding of this cause of the bullwhip effect. Furthermore, it holds potential for revealing additional insights into the nature of managing our inventory systems in a world where errors are commonplace.

We study forecasting as the contributor for biases which lead to our error of over-reacting to demand changes because i) it is a common place activity in many firms and thus provides a viable source of decision making biases, Dalrymple (1987) and Sanders and Manrodt (1994, 2003) and ii) there has been much research especially in the past 25 years on both subjective and quantitative forecasting to provide a sufficient theoretical grounding for modeling subjective forecasting and for choosing representative quantitative models, Armstrong (1985 & 2001), Lawrence et. al. (2006). Consider subjective forecasting, which surveys show characterize most of the forecasts in organizations, Dalrymple (1987) and Sanders and Manrodt (1994, 2003). Attributing spurious correlation to data or events - the illusory correlations bias - is a systematic human bias documented by both experimental psychology literature (cf. Edwards 1961; Gilovitch, Vallone and Tversky 1985) and forecasting experiment studies (cf. Lawrence and O'Connor 1992; O'Connor, Remus and Griggs 1993; and Remus, O'Connor and Griggs 1995). The forecasting experiment studies in particular provide evidence of humans perceiving correlation in non-correlated sequences and as a result reading stronger signals into current data about future patterns. O'Connor, Remus and Griggs state: “Our analysis has demonstrated that people were trying to read too much signal into a series as it changed. As a consequence, they over-reacted to each new value of the series as it was revealed to them.” (p. 170). Lawrence and O'Connor provide specifications for a model of such subjective forecasting of which we make use in this paper. In their study the authors found that subjects’ forecasts could be modeled using exponential smoothing or anchoring and adjustment, where the anchor point corresponded to the long term average of the stationary series. Such a specification matches very closely the model for subjective forecasting we use in the paper.

In order to study the error of over-reaction to demand changes we need to pair our forecasting approaches to the appropriate inventory system. We are able to pair the same inventory system to both approaches to generate our error. The most significant aspect of the inventory system in this respect is the nature of the underlying demand process facing the inventory system. We assume that demand is a stationary auto-regressive stochastic process of degree 1 (AR(1)) and as such it can be formulated as the weighted average of the most previous demand and the long run mean of the demand process. The weight on the previous demand process is the serial correlation of the demand process. The manager exhibiting subjective forecasting can be modeled as responding to a similar AR(1) process but with a serial correlation that is different from that of the demand process as in Lawrence and O'Connor.
In our model, with positive (negative) serial correlation, an over-estimation (under-estimation) of the serial correlation of the demand process leads to over-reaction to demand changes. We show that the bullwhip effect exists if and only if the estimated serial correlation is positive and is increasing in the estimated serial correlation. Thus for positive serial correlation, an over-reaction causes a more severe bullwhip effect than would have been caused by using the optimal policy, in addition to an internal cost increase. As significant, is that for negative serial correlation, a bullwhip effect is still introduced to the orders if the estimated serial correlation is positive. These results complement those of Kahn (1987), Lee, Padmanabhan and Whang (1997) and Gilbert (2005), who have shown that the optimal response to AR(1) demand leads to the bullwhip effect and thus only occurs if the serial correlation is positive. We further compare the expected internal cost increases caused by under-estimating and over-estimating the correlation for the case of positive correlation, and find that there is a preference for the former.

At first glance, moving averages seem appropriate for forecasting a serially correlated demand. Surveys conducted by Dalrymple (1987) and Sanders and Manrodt (1994, 2003) have also shown that the moving average is the most popular quantitative forecasting technique for short to medium range forecasting of product lines and families. However an analysis of orders based on using the moving average for forecasting shows that orders react to the change in demand between the current demand and the demand at the end of the moving average’s rolling period, and as shown by Chen et al. (2000) can cause the bullwhip effect. This reaction to demand changes is increasing in the leadtime and decreasing in the number of periods used for the moving average. Within our inventory system then, the moving average with a low number of periods and long leadtimes could lead to over-reaction to demand changes for positive serial correlation (or extreme under-reaction for negative serial correlation) which we confirm. We also show that unless the demand correlation is extremely strong and positive, (a required strength that is increasing in the leadtime,) using moving average forecasting does not help reduce the uncertainty of the demand over the lead time for the buyer but increases it; and the non-stationary base stock policy based on moving average forecasts is even more costly than the stationary base stock policy based on the (long run) mean demand over the lead time. Consequently, the system would be better off both internally and externally without the use of the moving average and the attendant over-reaction to demand changes.

Our work is not the first to explore irrational or boundedly rational behavior in supply chain management. The Beer Game has proved a fruitful site for this exploration. With the bullwhip effect as the observed deviation from optimal, the research here has concentrated on discovering the biases in decision making which, coupled with the game’s setting, results in the error. Sterman (1989)
observes that the bullwhip effect seen in the Beer Game is caused by players’ misperceptions on their inventory status resulting from forgetting the inventory in the pipeline. For the same observed behavior, Oliva and Goncalves (2007) propose a more sophisticated behavioral rationale in which actors are informationally constrained but process their available informational cues reasonably well. Croson and Donohue (2003 and 2006) and Croson et al. (2007) build on Sterman by confirming that behavioral factors still exist under different treatments of game conditions including when the knowledge of the demand distribution is public. Croson et al. argue as contributor to the bullwhip effect, the presence of a coordination risk within the beer game based on the uncertainty surrounding the performance of other members of the chain. Other research which examine biases within supply chain management include Schweitzer and Cachon (2001) who provide experimental evidence of bias in the use of the newsvendor model.

The literature focusing directly on the effect of errors on supply chain performance is smaller. Chen (1999) conducted numerical studies on the impact of irrational behavior related to Sterman’s observations in the context of a decentralized multi-stage supply chain. He considers the scenario where local managers of the chains use base stock policies, whose base stock levels are incorrectly based on their net inventory levels rather than their inventory positions. His results show that such mismanagement, especially at the downstream stages, could be very costly. Porteus (2000) and Watson and Zheng (2005) examine the design of coordination schemes for serial supply chains emphasizing the features of their designs that mitigate the effect on supply chain performance of errors in replenishment decisions. Su (2008) proposes a decision framework of bounded rationality, applies it to the newsvendor model identifying systematic biases, and investigates the impact of these biases on inventory settings such as supply chain contracting, the bullwhip effect, and inventory pooling. They find that incorporating decision noise and optimization error yields results that are consistent with some anomalies highlighted by recent experimental findings.

The rest of this paper is organized as follows. In Section 2, we describe analytical the error of over-reaction to demand changes and the inventory system in which we situate our decision making biases. In Sections 3 and 4, we analyze models for the effects of our two approaches to forecasting separately and conclude in section 5.

2 The inventory setting

In this section we describe the inventory system used to study over-reacting to demand changes. We then provide an analytical description of the error of over-reacting to demand changes and provide arguments supporting the use of the AR(1) process as the model for demand.
2.1 The Inventory System

Throughout this paper, we consider the following single location inventory system facing AR(1) demand. Let $D_t$ be the demand in period $t$. The demand process can be written as

$$D_t = \alpha D_{t-1} + (1 - \alpha) \mu + \sqrt{1 - \alpha^2} \sigma \epsilon_t,$$  

(1)

where $\alpha (-1 < \alpha < 1)$, $\mu (\geq 0)$ and $\sigma (\geq 0)$ are constants and $\epsilon_t (t = 1, 2, \cdots)$ are independent standard normal random variables. The normality assumption on $\epsilon_t$ is for analytical convenience. The mean $\mu$ is assume to be much larger than $\sigma$ so that the probability of having negative demand is negligible. The constant $\alpha$ is also the serial correlation of demand.

The system replenishes its inventory from an outside supplier with infinite supply. In each period, replenishment orders are placed and delivered at the beginning of the period; demands occur and are filled during the period. Orders take $l$ periods to deliver, where $l$ is a constant (integer). The system incurs linear inventory carrying costs and shortage penalty costs, which are charged at the end of the period. Let $h$ and $p$ be the inventory holding and shortage penalty costs per unit per period respectively. The manager of the system is incentivized to minimize the long-run average system-wide cost. In addition, we assume that cost free returns of inventory to the supplier are allowed, i.e., we allow the order quantity to be negative.

The above described inventory system is the platform for our models throughout this paper. For easy reference, we refer to it as system $\Gamma$.

2.2 Orders as Reaction to Demand Changes

In this subsection we show that for general models of stationary demand, orders can be interpreted as including a reaction to demand changes, and then provide justification for assuming for our system $\Gamma$ that demand follows a stationary AR(1) stochastic process. For representing general stationary and non-stationary stochastic demand processes, we can consider autoregressive integrated moving average (ARIMA) models, see Box et. al. (1994). In particular, stationary demand can be represented by an auto-regressive moving average model, ARMA($p,q$), with $p$ the number of auto-regressive terms and $q$ the number of moving average terms as follows:

$$D_t = \mu + \sum_{i=1}^{p} \phi_i (D_{t-i} - \mu) + \sum_{i=1}^{q} -\theta_i a_{t-i},$$  

(2)

where $\mu$ is the level or mean of demand, $\phi_i$ and $\theta_i$ are constants and $a_t$ is a noise series of independent and identically distributed random variables with mean 0 and variance $\sigma_a^2$. For convenience, these ARMA models can be represented more concisely and are more easily manipulated mathematically by
using a backshift operator. Let $B$ denote the time-series backshift operator such that $BD_t = D_{t-1}$ and $B^n D_t = D_{t-n}$. The stationary series (2) can then be written using the backshift operator as follows:

$$\phi (B) (D_t - \mu) = \theta (B) a_t,$$

where $\phi (B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p$ and $\theta (B) = 1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q$. Non-stationary demand can be modeled by assuming that differencing of demand results in a stationary series. The notation $\nabla$ can be used to denote the difference operation such that $\nabla D_t = (1 - B) D_t$ and $\nabla^d D_t = (1 - B)^d D_t$ where $\nabla^d = (1 - B)^d$ is the polynomial resulting from raising $1 - B$ to the $d$th power. Assuming the $d$th difference of a non-stationary demand series is stationary, the resulting ARIMA$(p, d, q)$ series can be expressed as follows:

$$\varphi (B) (D_t - \mu) = \theta (B) a_t$$

(3)

where $\varphi (B) = \phi (B) \nabla^d$ is a polynomial of order $p + d$.

For convenience, if we assume zero leadtime for an inventory system like system $\Gamma$, then for non-stationary demand which follows the ARIMA$(p, d, q)$ process (3), the optimal myopic policy has order upto target

$$z^*_t = \mu + (1 - \varphi (B)) (D_t - \mu) + (\theta (B) - 1) a_t + K,$$

where $K$ is a constant. The orders based on this optimal myopic policy can be expressed as

$$y_t = D_{t-1} + z^*_t - z^*_{t-1}
= D_{t-1} - \left( \nabla^d - 1 \right) (D_t - D_{t-1}) - (\varphi (B) - 1) \nabla^d (D_t - D_{t-1}) + (\theta (B) - 1) (a_t - a_{t-1})$$

Here orders can be broken down into a replenishment portion which is equal to demand and an adjustment, if necessary, given any change in inventory order upto targets. This adjustment comprises multiple reactions to changes in demand captured by the term $(\nabla^d - 1) (D_t - D_{t-1})$, along with multiple reactions to changes in differences of demand captured by the term $(\varphi (B) - 1) \nabla^d (D_t - D_{t-1})$. When $d = 0$, that is for stationary demand, we have

$$y_t = D_{t-1} - (\varphi (B) - 1) (D_t - D_{t-1}) + (\theta (B) - 1) (a_t - a_{t-1})$$

where adjustments comprise multiple reactions to changes in demand captured by the term $(\varphi (B) - 1) (D_t - D_{t-1})$ but do not comprise reaction to changes in differences of demand. In order to focus primarily on the effect of over-reacting to demand changes we need then to focus on stationary demand. Furthermore, orders are stationary if and only if demand is stationary see Watson (2008) and stationary order processes facilitate the study of variance properties for orders and thus insights.
on the bullwhip effect. Thus a focus on reaction to demand changes with a view to developing insights for the bullwhip effect is intrinsically related to assuming as we do that demand is stationary.

Using then a general form for orders below (4), we can describe analytically the error of over-reacting to demand changes. We rewrite the expression for orders as follows:

\[ y_t = D_{t-1} + \sum_{i=1}^{k} \eta_i (D_{t-i} - D_{t-i-1}) + W_t, \]  

(4)

where \( W_t \) is a stochastic process that may be correlated with \( D_t \). In (4), \( \eta_i \) can be considered the reaction to the change in demand as implied by \( (D_{t-i} - D_{t-i-1}) \). Assume that \( \eta^*_i \) is the best possible reaction to a change in demand \( (D_{t-i} - D_{t-i-1}) \). If \( \eta^*_i \geq 0 \) then \( \eta_i > (\eta^*_i) \) would imply an over-reaction (under-reaction) to a change in demand as the reaction amplifies the response to the demand change. For similar reasons, if \( \eta^*_i < 0 \) then \( \eta_i < (\eta^*_i) \) would imply an over-reaction (under-reaction) to a change in demand.

For our setting involving subjective forecasting, we provide the following justifications for using the stationary AR(1) process to model demand:

1. Forecasting studies have shown that subjective forecasting can be modeled using these ARMA processes which increases the realism of our model assumptions, Lawrence and O’Connor, Lawrence et. al..

2. Choosing demand with fewer variables to be estimated allows us to concentrate on the effect of over-reacting to demand changes and reduce the effect of other errors. In particular here, using the AR(1) model versus an ARMA\((p,q)\) allows for the mis-estimation of fewer parameters of the demand which would introduce additional errors.

3. The difference in serial correlation between the model for subjective forecasting and that of demand modeled as an AR(1) process results in the over-reaction or under-reaction to demand changes. This feature lends parsimony to our model.

4. The resulting focus on serial correlation of our model is behaviorally pertinent. With its basis in cause and effect, correlation as a relationship which can be exploited for perceived improved accuracy is appealing to forecasters. However subjectively quantifying serial correlation is a much more difficult task for individuals than say estimating averages, (Lawrence and Makridakis 1989; and Lawrence and O’Connor) and thus interpreting the operational consequences for inventory management of serial correlation is generally difficult. Furthermore, the use of correlation for predictive activity tends to be abused rather than avoided, (Edwards; and Gilovitch, Vallone, and Tversky).

For our setting involving the use of the moving average, we keep the assumption of demand modeled as a stationary AR(1) demand process with the following justification:
1. Again choosing demand with fewer variables to be estimated allows us to concentrate on the effect of over-reacting to demand changes and reduce the effect of other errors.

2. Moving averages seem appropriate for forecasting a serially correlated demand.

3. Our results enable us to build on the results from Chen et. al. who also use the same assumption for demand.

4. Since we do not attempt to exhaust all settings in which our error occurs, we can focus on discussing over-reaction to demand changes with respect to one demand process and choose to build on the analysis of the AR(1) process attempted for the setting involving subjective forecasting.

3 Over-reaction to Demand Changes due to Subjective Forecasting

Our model in this section assumes that the manager of system $\Gamma$ subjectively misestimates the serial correlation $\alpha$ of the demand process and then manages the inventory system in response to his “perceived” leadtime demand as opposed to the true leadtime demand process. We first provide the optimal policy for our system $\Gamma$ as benchmark. We then provide the analytical description of the over-reacting manager for subjective forecasting. We then examine the implications for the bullwhip effect and for the internal inventory costs of our over-reacting managers.

3.1 Optimal Policy as Benchmark

We briefly characterize the optimal policy for system $\Gamma$ to provide the appropriate benchmark for our over-reacting manager. These results first appeared in Lee, Padmanabhan and Whang; and Lee, So and Tang (2000). To minimize the cost of system $\Gamma$, the manager should be concerned with the leadtime demand. Let $D[t, t + l] = D_t + D_{t+1} + \cdots + D_{t+l}$, which can be expressed as,

$$D[t, t + l] = \gamma_l D_{t-1} + (l + 1 - \gamma_l) \mu + \Psi_l^t,$$

where $\gamma_l = \sum_{i=1}^{l+1} \alpha^i$ with $\gamma_{-1} = 0$ and $\Psi_l^t = \sqrt{1-\alpha^2} \sigma \sum_{i=0}^{l} (1 + \gamma_{i-1}) \epsilon_{t+i-1}$. It is easy to show that the variance of $\Psi_l^t$ is $(l + 1 + 2 \sum_{i=0}^{l} \gamma_{i-1}) \sigma^2 - \gamma_l^2$ and that the long run variance of (5) is $(l + 1 + 2 \sum_{i=0}^{l} \gamma_{i-1}) \sigma^2$. Let $V_l = l + 1 + 2 \sum_{i=0}^{l} \gamma_{i-1}$. The long run variance of the leadtime demand is then $V_l \sigma^2$. The leadtime demand can be written as the following equivalent stochastic process

$$D[t, t + l] = \gamma_l D_{t-1} + (l + 1 - \gamma_l) \mu + \left(\sqrt{V_l - \gamma_l^2}\right) \sigma \varepsilon_t,$$

where $\varepsilon_t \sim N(0, 1)$.

With the full backlogging assumption, the expected cost incurred in period $t + l$ will be $G_l(z) = E\{h(z - D[t, t + l])^+ + p(z - D[t, t + l])^-\}$, where $z$ is the target for inventory position and $(x)^+=\begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$.
max\{x, 0\} and \((x)^- = \max\{-x, 0\}\). Assuming that cost free returns of inventory are allowed, setting 
\[ z^*_t = \max\{-x, 0\} - (x) \] 
is always feasible, and therefore optimal where
\[ z^*_t = \gamma_l D_{t-1} + (l + 1 - \gamma_l) \mu + \sqrt{V_l - \gamma_l^2 \sigma k^*}, \tag{7} \]
with \(k^* = \Phi^{-1}\left(\frac{p}{p+n}\right)\) where \(\Phi\) is the cdf. of the standard normal random variable.

Let \(y_t\) be the order placed in period \(t\). Under the myopic policy, we have
\[ y_t = D_{t-1} + z^*_t - z^*_{t-1} \]
\[ = D_{t-1} + \gamma_l (D_{t-1} - D_{t-2}). \tag{8} \]

Note that the optimal order quantity is a function of \(\alpha\) only, and independent of the cost parameters.

3.2 Subjective Forecasting Based Inventory Policy

As mentioned earlier, our model in this section assumes that the manager of system \(\Gamma\) subjectively misestimates the serial correlation \(\alpha\) of the demand process and then manages the inventory system in response to his “perceived” leadtime demand as opposed to the true leadtime demand process. More specifically, we assume that with a misestimated correlation \(\tilde{\alpha}\), the manager’s perceived leadtime demand is
\[ \tilde{D}[t, t + l] = \tilde{\gamma}_l D_{t-1} + (l + 1 - \tilde{\gamma}_l) \mu + \left(\sqrt{V_l - \tilde{\gamma}_l^2}\right) \sigma \varepsilon_t, \tag{9} \]
where \(\tilde{\gamma}_l = \sum_{i=1}^{l+1} \tilde{\alpha}^i\) with \(\tilde{\gamma}_{-1} = 0\). The reader should note that the manager’s perceived leadtime demand (9) differs from the true leadtime demand (6) in that \(\alpha\) is replaced by \(\tilde{\alpha}\) and as a result implies a single period demand process where the forecast for next period \(E \tilde{D}_t\) is given by
\[ E \tilde{D}_t = \tilde{\alpha} D_{t-1} + \mu (1 - \tilde{\alpha}). \tag{10} \]

We refer to \(\tilde{\alpha}\) as the perceived serial correlation of the manager. Note that the long run mean and variance of (6) are equal to that of (9). (We further assume that \(\tilde{\alpha}\) is not too large so that \(V_l \geq \tilde{\gamma}_l^2\), holds to ensure that the implied variance of the i.i.d. portion of demand remains positive.) It is important to note here the similarities between (10) and the findings from Lawrence and O’Connor. In their time series forecasting study, the authors found that subjects’ forecasts could be modeled using exponential smoothing or anchoring and adjustment where the anchor point corresponded to the long term average of the stationary series. The RHS of (10) clearly captures this model.
In response to this perceived leadtime demand process (9), the manager chooses the “optimal” order upto level \( \hat{z}_t^* \) in period \( t \) where

\[
\hat{z}_t^* = \hat{\gamma}_t D_{t-1} + (l + 1 - \hat{\gamma}_t) \mu + \left( \sqrt{V_l - \hat{\gamma}_t^2} \right) \sigma k^* .
\]

The order quantity in period \( t, \tilde{y}_t, \) is (see (8))

\[
\tilde{y}_t = D_{t-1} + \hat{\gamma}_t (D_{t-1} - D_{t-2}) .
\] (11)

Similar to what was said with respect to \( \gamma_l \) in Section 2, \( \hat{\gamma}_l \) here represents the manager’s reaction level to demand changes. When \( \alpha > 0 \), then \( \gamma_l > 0 \) and thus it is natural to say that the manager is over-reacting when he adds or reduces by an amount that is greater than \( \gamma_l (D_{t-1} - D_{t-2}) \) respectively. Since \( \hat{\gamma}_l \) is an increasing function of \( \hat{\alpha} \), over-estimating (under-estimating) the positive serial correlation corresponds to an over-reaction (under-reaction) to the demand change. Conversely when \( \alpha < 0 \), then \( \gamma_l < 0 \) and under-estimating (over-estimating) the negative serial correlation corresponds to an over-reaction (under-reaction) to the demand change.

### 3.3 Effect on Order Variability

In this subsection, we show an over-reaction to demand changes for \( \alpha \geq 0 \) increases both the “traditional” bullwhip effect and a measure of the uncertainty of orders referred to as the bullwhip effect modulo full information. We also complement Lee, Padmanabhan and Whang and their discussion on demand signaling as the cause of the bullwhip effect by showing that even for \( \alpha < 0 \) where the optimal policy does not result in a bullwhip effect, an estimate of positive serial correlation does. Thus the discussion of demand signaling as a cause of the bullwhip effect should not only include optimally responding to demand signals driven by positive correlation, but also include behavior that implies the inaccurate perception of such demand signals.

#### 3.3.1 The Traditional Bullwhip Effect

The “traditional” bullwhip effect refers to the phenomenon where orders to the supplier (or production by the supplier) exhibit larger variance than sales at the buyer (i.e. variance amplification as one moves upstream in a supply chain). Using orders from the optimal policy (8), it is easy to produce here the ratio of the long run variance of orders to demand. Let \( V(y) \) and \( V(D) \) denote the steady state variances of \( y_t \) and \( D_t \) respectively. With a little algebra we have

\[
V(y)/V(D) = 1 + 2(1 - \alpha) \left( \gamma^2 + \gamma_l \right) .
\] (12)

From (12) we see that \( V(y) > V(D) \), i.e., the orders of the optimal policy amplify the variance of the demands, and the following proposition follows easily from examining the first derivative of \( V(y) \) with respect to \( \alpha \). See Figure 1 for the shape of this ratio as a function of \( \alpha \).
Proposition 3.1 Under the optimal policy for system $\Gamma$, i) for $\alpha > 0$, $V(y)/V(D) > 1$ and increases and then decreases with $\alpha$, ii) for $\alpha \leq 0$, $V(y)/V(D) \leq 1$ and decreases and then increases to 1 with $\alpha$.

Interestingly the relationship between the correlation and the bullwhip effect depends on the metric used for the bullwhip effect, whether it is based on the difference or on the ratio between the variance of orders and demand. In a model with lost sales and a similar but slightly different AR(1) model compared to ours, Kahn examines, under the optimal production policy, the difference between the variance of production (orders) and sales and shows that, for $\alpha > 0$, this difference is always positive and increases with the serial correlation. Lee, Padmanabhan and Whang in a model similar to Kahn however with backlogs and arbitrary leadtime, examine the ratio of the variance of orders to that of demand. Our ratio is the same as that in Lee, Padmanabhan and Whang.

Now let $\tilde{y}$ denote the subjective forecasting based order quantity $\tilde{y}_t$, in steady state. From orders based on the perceived demand leadtime (11), a little algebra shows

$$V(\tilde{y}) = (1 + 2(1 - \alpha)(\tilde{\gamma}_l^2 + \tilde{\gamma}_l)) V(D).$$

(13)

Differentiating $V(\tilde{y})$ with respect to $\tilde{\alpha}$, we obtain

$$\frac{dV(\tilde{y})}{d\tilde{\alpha}} = 2\tilde{\gamma}_l'(1-\alpha)(2\tilde{\gamma}_l + 1) V(D),$$

where $\tilde{\gamma}_l' = d\tilde{\gamma}_l/d\tilde{\alpha}$.

The following lemma, which can be proved using simple algebra, is used to prove Proposition 3.3, which describes the effect on the bullwhip effect and the variance of orders due to over-reacting to demand changes.

Lemma 3.2 For $l$ even, $\frac{dV(\tilde{y})}{d\tilde{\alpha}} \geq 0$ if and only if $\tilde{\alpha} \geq \tilde{\alpha}_l(< 0)$ where $(2\tilde{\gamma}_l + 1)|_{\tilde{\alpha} = \tilde{\alpha}_l} = 0$. For $l$ odd, $\frac{dV(\tilde{y})}{d\tilde{\alpha}} \geq 0$ if and only if $\tilde{\alpha} \geq \tilde{\alpha}_l(0)$ where $\tilde{\gamma}_l'|_{\tilde{\alpha} = \tilde{\alpha}_l} = 0$.

Proposition 3.3 i) $V(\tilde{y})/V(D) \leq 1$ if and only if $\tilde{\alpha} \leq 0$.

ii) For $\alpha \geq 0$, the bullwhip effect persists when the manager over-estimates the correlation and this variance of the order process increases with the over-estimate of the correlation, or equivalently, with the level of the over-reaction to demand changes.

For iii) and iv) below, let $\tilde{\alpha}_l$ be such that $(2\tilde{\gamma}_l + 1)|_{\tilde{\alpha} = \tilde{\alpha}_l} = 0$ if $l$ is even and be such that $\tilde{\gamma}_l'|_{\tilde{\alpha} = \tilde{\alpha}_l} = 0$ if $l$ is odd.

iii) For $\tilde{\alpha}_l \leq \alpha < 0$ and an over-reaction (under-reaction) to demand changes, i.e., $\tilde{\alpha} \leq \alpha < 0$ ($\alpha \leq \tilde{\alpha}$), if $\tilde{\alpha}_l \leq \tilde{\alpha}$ then the variance of orders is less (greater) than that of the optimal policy.

iv) For $\alpha < \tilde{\alpha}_l < 0$ and an over-reaction to demand changes, i.e., $\tilde{\alpha} < \alpha < 0$, the variance of orders is greater than that of the optimal policy. For an under-reaction to demand changes $\alpha \leq \tilde{\alpha}$, if $\tilde{\alpha} < \tilde{\alpha}_l$ then the variance of orders is less than that of the optimal policy.
Comparing the results on the bullwhip effect for the optimal policy in Proposition 3.1 with results for the perceived demand leadtime in Proposition 3.3 i) we see that it is the estimate of the serial correlation that drives the existence of the bullwhip effect and not the serial correlation itself. If the manager assumes a negative correlation, the bullwhip effect is not introduced even if the actual serial correlation is positive. The results Proposition 3.3 ii)-iv) also show that it is for positive serial correlation or very strong negative serial correlation that an over-reaction results in a higher variance of orders than the optimal policy. For moderate negative serial correlation an over-reaction results in a lower variance of orders than the optimal policy.

**Remark 1:** It is interesting to contrast the above proposition with Proposition 3.1 for \( \alpha \geq 0 \). Proposition 3.3 states that the order volatility increases with the reaction level. Proposition 3.1 states that the order volatility does not necessarily increase with the correlation.

We note however that the (long run) variance of the order (demand) process is a measure that mixes uncertainty with some predictable variability. Therefore, in next subsection, we discuss an alternative measure of the bullwhip effect, which reflects only uncertainty.

### 3.3.2 Bullwhip Effect modulo Full Information

Gilbert (2005) uses an alternative measure for the bullwhip effect: the ratio of the standard error of orders to that of demand, and shows for a more general demand model that a myopic policy can increase this ratio, and therefore, causes this type of bullwhip effect.

Following Gilbert, (8) can be written

\[
y_t = \alpha y_{t-1} + (1 - \alpha) \mu + \sqrt{1 - \alpha^2 \sigma} \left( (1 + \gamma_l) \epsilon_{t-1} - \gamma_l \epsilon_{t-2} \right)
\]  

which shows that \( y_t \) is now an ARMA(1,1) process (Box, Jenkins and Reinsel 1994). Let’s consider the supplier’s forecast on \( y_t \). With full distributional knowledge of the demand process and the buyer’s inventory policy, at the beginning of period \( t - 1 \), before demand occurs and \( \epsilon_{t-1} \) is realized, the supplier has the least uncertainty about \( y_t \). The supplier’s forecast error is \( \sqrt{1 - \alpha^2 \sigma} (1 + \gamma_l) \epsilon_{t-1} \) while our buyer’s forecast error at the beginning of period \( t - 1 \) is \( \sqrt{1 - \alpha^2 \sigma} \epsilon_{t-1} \). The ratio of these forecast errors is \( 1 + \gamma_l \) and it measures the best case increase in uncertainty for the supplier in the following sense: if the supplier did not know the demand distribution or the buyer’s inventory policy, then the uncertainty facing the supplier would be higher. Since \( 1 + \gamma_l > 1 \), the uncertainty facing the supplier is greater than that facing the buyer. We refer to this increase in uncertainty as the bullwhip effect *modulo full information*.

**Proposition 3.4** Under the optimal policy for system \( \Gamma \), if \( \alpha > 0 \), a bullwhip effect *modulo full information*
information exists and increases with $\alpha$.

For our manager forecasting subjectively, their orders (11) can be expressed as

$$\tilde{y}_t = \alpha \tilde{y}_{t-1} + (1 - \alpha) \mu + \sqrt{1 - \alpha^2} \sigma ((1 + \tilde{\gamma}_t) \epsilon_{t-1} - \tilde{\gamma}_t \epsilon_{t-2}),$$

where similar to $y_t$, $\tilde{y}_t$ is an ARMA(1,1) process where the autoregressive coefficient is still the serial correlation $\alpha$ but the moving average parameter and variance of the noise terms are based on the misestimate $\tilde{\alpha}$. It is easy to observe here that the ratio of the standard error of orders to demand gives $1 + \tilde{\gamma}_t$ which increases with $\tilde{\alpha}$. Thus we have arrived at the following proposition.

**Proposition 3.5** In the inventory system $\Gamma$, the bullwhip effect modulo full information persists if and only if the manager misestimates the correlation with $\tilde{\alpha} > 0$ whether under-reacting or over-reacting to demand changes. For $\alpha > 0$ the bullwhip effect increases with the level of the over-reaction to demand changes.

Similarly as observed with the traditional measure of the bullwhip effect, it is the misestimate of the correlation which drives the bullwhip effect modulo full information. The bullwhip effect modulo full information will not exist if the manager assumes a negative correlation even if the actual serial correlation is positive.

**Remark 2:** Propositions 3.3 and 3.5 imply that for $\alpha > 0$ while over-reaction increases the bullwhip effect, under-reaction reduces the bullwhip effect (of both measures). While increased uncertainty in orders would increase the mismatch costs of the newsvendor type to the supplier, more variable orders, even if predictable, would also cost more for suppliers with limited capacity (or with convex production cost structure) to meet. Usually then, from the perspective of the supplier, her buyer’s under-reaction is more preferable than over-reaction. However, our manager usually cares less about the external effect than his internal efficiency. Then the next question is how does over-reaction compare to under-reaction in terms of the internal costs? Are there incentives there that should drive behavior more efficient for the chain? This question is addressed in the next subsection.

### 3.4 Internal Cost Comparison Between Over-reaction and Under-reaction

In this subsection, we compare over-reaction to under-reaction in terms of the expected costs incurred in the system. Unlike the case of the external effect, neither over-reaction nor under-reaction dominates the other in the absolute sense. Therefore, we establish more specific results, which favor underreaction to over-reaction for $\alpha \geq 0$ when the absolute deviation from the true serial correlation
reaction levels is equal. For $\alpha < 0$, numerical examples suggest that over-reaction could be favored to under-reaction for short leadtimes.

To facilitate our comparisons we parameterize $\tilde{\alpha}$ as $\tilde{\alpha}(b) = \alpha + b$ ($-1 - \alpha \leq b \leq 1 - \alpha$). With this range of $b$, we are able to model both over-estimation and under-estimation of the serial correlation. In the following let $\tilde{\alpha}_l = -1$ if $l$ is even and let $\tilde{\alpha}_l$ be such that $\tilde{\gamma}'_l|_{\tilde{\alpha} = \tilde{\alpha}_l} = 0$ if $l$ is odd.

The main result of the section for system $\Gamma$ is the following:

**Proposition 3.6** i) For $\alpha \geq 0$, the expected cost caused by an over-reaction (under-reaction) associated with perceived correlation $\tilde{\alpha}(b) = \alpha + b$ ($\tilde{\alpha}(-b) = \alpha - b$) strictly increases with $b > 0$.

ii) For $\alpha > 0$, the expected cost caused by an over-reaction associated with perceived correlation $\tilde{\alpha}(b) = \alpha + b$, (such that $\min\{\alpha, 1 - \alpha\} \geq b > 0$ and $V_l \geq \tilde{\gamma}_l^2$ hold) is higher than that caused by an under-reaction with perceived correlation $\tilde{\alpha}(-b) = \alpha - b$. An equality in costs holds only when $l = 0$ hold simultaneously with $p = h$.

iii) For $\alpha = 0$, the expected cost caused by an over-reaction associated with perceived correlation $\tilde{\alpha}(b) = b$, (such that $-\tilde{\alpha}_l \geq b > 0$ and $V_l \geq \tilde{\gamma}_l^2$ hold) is higher than that caused by an under-reaction with perceived correlation $\tilde{\alpha}(-b) = b$. An equality in costs holds only when $l = 0$.

iv) For $\tilde{\alpha}_l \leq \alpha < 0$, the expected cost caused by an over-reaction (under-reaction) associated with perceived correlation $\tilde{\alpha}(-b) = \alpha - b$ ($\tilde{\alpha}(b) = \alpha + b$) strictly increases with $b > 0$.

We outline the main ideas of the proof here, and relegate the supporting lemmas to the Appendix. The manager, as assumed, responds to his perceived leadtime demand

$$D[t, t + l] = \tilde{\gamma}_l (b) D_{t-1} + (l + 1 - \tilde{\gamma}_l (b)) \mu + \left(\sqrt{V_l - (\tilde{\gamma}_l (b))^2}\right) \sigma \epsilon,$$

with the order up-to level

$$\tilde{z}_l^* (b) = \tilde{\gamma}_l (b) D_{t-1} + (l + 1 - \tilde{\gamma}_l (b)) \mu + \left(\sqrt{V_l - (\tilde{\gamma}_l (b))^2}\right) \sigma k^*,$$

where $k^* = \Phi^{-1}\left(\frac{p}{p+k}\right)$ and $\tilde{\gamma}_l (b) = \sum_{i=1}^{l+1} (\alpha + b)^i$ with $\tilde{\gamma}_{-1} (b) = 0$. As a result, the inventory level at the end of period $t+l$ is

$$IL(t+l) = \tilde{z}_l^* (b) - D[t, t+l],$$

which can be expressed as

$$IL(t+l) = S(b)k^* - \Delta_l (b),$$

where $S(b) = \sqrt{V_l - (\tilde{\gamma}_l (b))^2}$ and $\Delta_l (b) = \sqrt{V_l - \gamma_l^2 \sigma \epsilon} - (\tilde{\gamma}_l (b) - \gamma_l) (D_{t-1} - \mu)$. Note that $\Delta_l (b)$ is normal with mean 0 and variance $V(b) = \sigma^2 \left(V_l + (\tilde{\gamma}_l (b))^2 - 2\tilde{\gamma}_l (b) \gamma_l\right)$. Therefore, $S(b)k^*$ can be viewed as the “safety stock” to cover the “random noise” $\Delta_l (b)$.

We make two observations on the consequence of the manager’s response to his perceived demand. First, compared with the optimal policy, for any $\alpha$, his response increases the variance of the inventory level process from $V(0) = \sigma^2 (V_l - \gamma_l^2)$ to $V(b) = \sigma^2 (V_l + (\tilde{\gamma}_l (b))^2 - 2\tilde{\gamma}_l (b) \gamma_l)$. We note that $V(b)$
increases with the absolute value of $b$, and that $V(b) \geq V(-b)$ for $b > 0$, where the equality holds if and only if $l = 0$ (Lemma A.2).

Second, for $\alpha > 0$ in his response, except for $b = -\alpha$, the safety stock level is misaligned with the variance of $\Delta_t(b)$: $S(b)k^* \neq \sqrt{V(b)k^*}$. Lemma A.3 shows that $S(b)$ is a decreasing function of $b$ for $b \geq -\alpha$ and that when $k^* > 0$, the over-reacting manager under-orders (i.e., the safety stock level is less than optimal when $b > 0$) and the under-reacting manager over-orders (i.e., more than optimal when $-\alpha < b < 0$). Lemma A.4 shows that the monotonic conditions on $S(b)$ and $V(b)$ imply that costs increase monotonically for increased level of over-reacting (under-reacting). Note that under-ordering is potentially more costly than over-ordering when $k^* > 0$, i.e., $p > h$. Moreover, Lemma A.5 shows that the difference $\sqrt{V(b)k^*} - S(b)k^*$ is larger than $S(-b)k^* - \sqrt{V(-b)k^*}$ for any given $b > 0$. As a result, the expected cost caused by an over-reacting manager is higher than that by an under-reacting manager when their reaction levels are the same (Lemma A.6). The ideas for the case of $p < h$ are similar.

For $\alpha = 0$, in his response the safety stock level is completely aligned with the variance of $\Delta_t(b)$: $S(b)k^* = \sqrt{V(b)k^*}$. Therefore neither the over-reacting manager nor under-reacting manager over-orders or under-orders. However because of the introduced variance, Lemma A.4 shows that the monotonic conditions on $S(b)$ and $V(b)$ imply that costs still increase monotonically for increased level of over-reacting (under-reacting). Furthermore since $V(b) \geq V(-b)$ for $b > 0$ where the equality holds if and only if $l = 0$, the expected cost of over-reacting is higher than that caused by a similar level of under-reacting. An equality in costs holds if and only if $l = 0$.

For $\alpha < 0$, in his response, except for $b = -\alpha$, the safety stock level is again misaligned with the variance of $\Delta_t(b)$: $S(b)k^* \neq \sqrt{V(b)k^*}$. Lemma A.3 shows that $S(b)$ is an increasing function of $b$ for $b \geq \alpha - \bar{\alpha}_t$. Lemma A.4 shows that the monotonic conditions on $S(b)$ and $V(b)$ imply that costs increase monotonically for increased level of over-reacting (under-reacting).

To obtain a feeling for the magnitude of the cost increases, we have conducted a numerical study. Figure 2 shows the percentage increase in costs for various levels of under-reaction ($b < 0$) and over-reaction ($b > 0$) for system $\Gamma$ with $\alpha = 0, 0.2, 0.4, 0.6$, $l = 0$, $p = 3$, $h = 1$, $(1 - \alpha)\mu = 100$ and $\sqrt{1 - \alpha^2}\sigma = 25$. Figure 3 shows the results for the case of $l = 4$. The examples show that the cost increase from over-reaction and under-reaction can be significant. Take $\alpha = 0.4$ as an example. For $l = 0$, over-reaction with $b = 0.4$ induces a 13.3% cost increase, and under-reaction with $b = -0.4$ induces a 9.1% increase. For $l = 4$, over-reaction with $b = 0.4$ induces a 30.9% cost increase, and under-reaction with $b = -0.4$ induces a 2.4% increase.

For evidence of the plausibility of our examples, Figure 4 shows a graph of the three leadtime
demand processes implied by \( b + \alpha = 0.4 \) & 0.8 for \( l = 0 \). The similarity across the three processes suggests that subjective forecasting would struggle with properly identifying any one of the graphs without a benchmark. Furthermore for \( \alpha = 0.4 \), \( b = -0.4 \) assumes that demand is i.i.d. which is a conceivable response especially for decision making on inventory. At the other extreme, naïve forecasting, used as a forecasting benchmark, e.g., Makridakis, Wheelwright & Hyndman (1998), uses the most recent demand realization as the forecast of the next period, effectively assuming for \( \alpha = 0.4 \) that \( b = 0.6 \).

Figures 5 & 6 show the percentage increase in costs for negative serial correlation \( \alpha = -0.2 \), -0.4, -0.6, \( p = 3 \), \( h = 1 \), \((1 - \alpha) \mu = 100 \) and \( \sqrt{1 - \alpha^2} \sigma = 25 \) for \( l = 0 \) and \( l = 4 \). The graphs show that for short leadtime, over-reaction is less costly than under-reaction however for \( l = 4 \), under-reaction is slightly less costly than over-reaction.

To summarize, in this subsection we established specific results, which favor under-reaction to over-reaction for \( \alpha \geq 0 \) when the absolute deviation from the true serial correlation reaction levels is equal. Since the managers tend to care more about their internal costs than their external effects on the supply chain, our results relate mitigating the bullwhip effect to decision making that would help reduce costs for the manager since, when in doubt about the appropriate interpretation of observed demand changes, being conservative is argued better than being radical. This is valid for the inventory system itself and the supply chain of which it is a part. The context of our model however best fits a product that is in the basic replenishment phase of its lifecycle due to the required stationary quality of the stochastic processes used to represent demand.

4 Reaction to Demand Changes due to Moving Average Forecasting

In this section, we consider the scenario where the moving average, a popular forecasting technique, is used to forecast the leadtime demand for system \( \Gamma \). An analysis of orders based on using the moving average for forecasting easily shows that orders react to the change in demand between the current demand and the demand at the end of the moving average’s rolling period. This reaction to demand changes is increasing in the leadtime and decreasing in the number of periods used for the moving average. Given the relationship between the moving average and our model for subjective forecasting, we expect some similarities in the dynamics for over-reaction to demand. In particular we expect that the moving average with a low number of periods and long leadtimes could lead to over-reaction to demand changes especially since Chen et. al., show the conventional bullwhip effect always exists with the moving average and positive correlation and sometimes with negative correlation. As evidence of the similarities to over-reacting to demand changes using subjective forecasting, we find that a
bullwhip effect modulo full information exists with the moving average and decreases with the number of periods used. We also show that unless the demand correlation is extremely strong and positive, (a required strength that is increasing in the leadtime,) using moving average forecasting does not help reduce the uncertainty of the demand over the lead time for the buyer but increases it. Translating these dynamics on forecast accuracy into inventory cost performance, implies that the non-stationary base stock policy based on moving average forecasts is even more costly than the stationary base stock policy based on the (long run) mean demand over the lead time. Consequently, the system would be better off both internally and externally without the use of the moving average.

4.1 Moving Average Forecasting and its Orders

We assume that the manager of system uses \((l + 1)\) times the average of the most recent \(n\) periods of realized demands \(\hat{D}_{n,t} = (l+1)\frac{\sum_{i=1}^{n} D_{t-i}}{n}\) to forecast the mean of the leadtime demand. He subsequently sets his order up-to level in period \(t\) equal to \(\hat{D}_{n,t}\) plus a safety stock \(s_t\) to cover the forecast error. Under this scenario the order process is

\[
\hat{y}_t = D_{t-1} + \hat{D}_{n,t} - \hat{D}_{n,t-1} = D_{t-1} + \frac{(l + 1)}{n} (D_{t-1} - D_{t-n-1}) + (s_t - s_{t-1}).
\]

The orders from the moving average (15) show a similar form as with subjective forecasting. The orders consist of replenishment of the demand realized, along with a reaction to the change in demand between the current demand and demand at the end of the moving average’s rolling period. The orders also reflect any changes in the safety stock factor. The reaction level to the change in demand is proportional to the leadtime and inversely proportional to the number of periods used by the moving average. Our analysis of the subjective forecasting as a cause of reacting to demand changes suggests that the moving average with a low number of periods and long leadtimes could lead to over-reaction to demand changes. We further develop this idea in the following sections by considering the use of moving average in the inventory system \(\Gamma\).

4.2 The Moving Average Forecasting and the Bullwhip Effect modulo Full Information

We examine the bullwhip effect modulo full information and make the convenient assumption that the safety factor \(s_t\) is a constant. Substituting the expression for AR(1) demand (1) in the above expression (15) for orders using the moving average, we have

\[
\hat{y}_t = \left(\frac{n + l + 1}{n}\right) \alpha D_{t-2} - \left(\frac{l + 1}{n}\right) D_{t-n-1} + \left(\frac{n + l + 1}{n}\right) \left((1 - \alpha)\mu + \sqrt{1 - \alpha^2} \sigma_{t-1}\right).
\]
Under full information, at the beginning of period \( t - 1 \), \( D_{t-2}, D_{t-n-1}, s_t \) and \( s_{t-1} \) are known and only \( \epsilon_{t-1} \) is unknown. The ratio of the standard error of orders to that of demand is \( \frac{n+l+1}{n} \) which is greater than 1 and decreasing in \( n \), and therefore, we have the following proposition:

**Proposition 4.1** In the inventory system \( \Gamma \), when the manager uses the moving average to forecast the mean of the leadtime demand and the safety stock chosen is a constant, the bullwhip effect modulo full information exists and decreases with the number of periods used in the moving average.

Proposition 4.1 can be compared with the results for the bullwhip effect modulo full information for the manager forecasting subjectively in Proposition 3.5 where a bullwhip effect modulo full information was only present if the subjectively estimated correlation was positive.

### 4.3 The Moving Average Forecasting and Its Error

The moving average forecast for the leadtime demand, \( \hat{D}_{n,t}^l \), has an error

\[
e_{n}^l = D[t, t + l] - (l + 1) \frac{\sum_{i=1}^{n} D_{t-i}}{n},
\]

that is normal with mean 0 and variance \( V(e_{n}^l) \). (Here, we suppress the dependency of the forecast error on \( t \) for notational simplicity.) That is, the moving average is an unbiased estimate of the lead time demand, and \( V(e_{n}^l) \) reflects the remaining uncertainties of the leadtime demand after using the moving average forecast. It is clear that the smaller this variance, the better the forecast.

For an AR(1) demand process with low \( \alpha \geq 0 \) or \( \alpha < 0 \), our analysis in the prior section suggests the greater is the likelihood that the moving average over-reacts to demand changes. In this case, we show that when \( \alpha \) is smaller than a threshold value, \( V(e_{n}^l) > V(D[t, t + l]) \) holds for any finite \( n \). That is, when \( \alpha \) is low, using the moving average forecast with a finite \( n \) actually amplifies, rather than reduces, the variance of the leadtime demand.

We expect something of the converse, when the serial correlation is high. In this case, we show that when \( \alpha \) is higher than the threshold, the moving average forecast with the number of periods equal to one reduces the variance of the leadtime demand. We further show a stronger result that, when \( \alpha \) is higher than the threshold, \( V(e_{1}^l) < V(e_{n}^l) \) for all \( n > 1 \), i.e., \( \hat{D}_{1,t}^l \), which uses only the most recent demand realization for its forecast, minimizes the forecasting errors among \( \hat{D}_{n,t}^l \) for all \( n \). This is because for AR(1) process, the most recent demand realization is the best indicator of future demand.

We proceed to compare \( V(D[t, t + l]) \) with \( V(e_{1}^l) \) first. Rewrite (5) as \( D[t, t + l] = \gamma_l D_{t-1} + \zeta_l(t) \), where \( \zeta_l(t) = \mu (l + 1 - \gamma_l) + \sqrt{1 - \alpha^2} \sigma \sum_{i=0}^{l} (1 + \gamma_{i-1}) \epsilon_{t+i-l} \). Clearly, \( D_{t-1} \) and \( \zeta_l(t) \) are independent.
Therefore, we have
\[
V(D[t, t + l]) = V(\zeta_l(t) + \gamma_l D_{t-1}) = V(\zeta_l(t)) + \gamma_l^2 V(D).
\] (17)
and
\[
V(e_1^l) = V \left( D[t, t + l] - \hat{D}_{1,t}^l \right)
= V (\zeta_l(t) + \gamma_l D_{t-1} - (l + 1)D_{t-1})
= V (\zeta_l(t)) + (\gamma_l^2 - 2(l + 1)\gamma_l + (l + 1)^2) V(D).
\]
Therefore,
\[
V(e_1^l) \leq V(D[t, t + l])
\iff \gamma_l^2 - 2(l + 1)\gamma_l + (l + 1)^2 \leq \gamma_l^2
\iff 2\gamma_l \geq l + 1.
\] (18)

Since \(2\gamma_l < 0\) for \(-1 < \alpha < 0\) and is a continuous, increasing function of \(\alpha \geq 0\) ranging from 0 (\(\alpha = 0\)) to \(2(l + 1)\) (\(\alpha = 1\)), there exists \(\hat{\alpha}(l)\), namely the solution of \(2\gamma_l = l + 1\) such that \(2\gamma_l \geq l + 1\) if and only if \(\alpha \geq \hat{\alpha}(l)\). It is easy to see that \(\hat{\alpha}(l)\) increases with \(l\). Thus, we have established the following proposition.

**Proposition 4.2** The moving average forecast with \(n = 1\), \(\hat{D}_{1,t}^l\), increases (reduces) the variance of the leadtime demand, i.e., \(V(e_1^l) > (<) V(D[t, t + l])\), if and only if \(\alpha < (>) \hat{\alpha}(l)\). Here \(\hat{\alpha}(l)\) is the unique solution of \(\gamma_l = (l + 1)/2\), which increases in \(l\).

The next lemma shows that \(V(e_n^l)\) is bounded below by either \(V(e_1^l)\) or \(V(D[t, t + l])\).

**Lemma 4.3** For all \(n > 1\), \(V(e_n^l) \geq V(D[t, t + l])\) holds if \(\alpha \leq \hat{\alpha}(l)\), and \(V(e_n^l) \geq V(e_1^l)\) holds otherwise.

The proof of the above lemma, which is mostly algebra, is provided in Appendix B. Combining Proposition 4.2 and Lemma 4.3 leads to the following theorem:

**Theorem 4.4** For \(\alpha < \hat{\alpha}(l)\), the moving average forecast \(\hat{D}_{n,t}^l\) (with any finite \(n\)) amplifies the variance of the leadtime demand. For \(\alpha \geq \hat{\alpha}(l)\), the forecast \(\hat{D}_{1,t}^l\) reduces the variance of the leadtime demand and minimizes the forecast error among \(\hat{D}_{n,t}^l\) for all \(n\).

In order to obtain some feeling about the threshold correlation level, we list \(\hat{\alpha}(l)\) for various \(l\) in Table 1. Note that for most consumer goods, the serial correlation of demand is usually less than 0.7,
Table 1: The Threshold $\hat{\alpha}(l)$

<table>
<thead>
<tr>
<th>$l$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\alpha}(l)$</td>
<td>0.5</td>
<td>0.62</td>
<td>0.69</td>
<td>0.74</td>
<td>0.78</td>
<td>0.80</td>
</tr>
</tbody>
</table>

(Erkip, Hausman and Nahmias 1990; Lee, So and Tang). Therefore, this table suggests that if an AR(1) process provides a good fit for the demand for these goods and if the leadtime is greater than 1, one should not use the moving average with any finite $n$. Instead, one would be better off to use the long run average $(l + 1)\mu$ to estimate the leadtime demand.

### 4.4 The Base Stock Policy using Moving Average Forecasts

At this point we can examine the expected cost of the base stock policy whose order upto level in period $t$ is set at $z_{n,t} = \hat{D}_{n,t} + s_t$, where $s_t$ is the safety stock. Assuming costless returns allowed, the expected cost attributed to the decision in period $t$ is

$$G_t(z_{n,t}) = hE(z_{n,t} - D[t, t + l])^+ + pE(z_{n,t} - D[t, t + l])^-$$

$$= hE(s_t - \epsilon_n^l)^+ + pE(s_t - \epsilon_n^l)^-.$$

The above is the cost of a newsvendor whose demand is normal with mean 0 and variance $V(e_n^l)$ and who sets the order quantity at $s_t$. Clearly, this cost depends not only on $V(e_n^l)$, but also on $s_t$.

We consider two cases of how the safety stock is set. In the first case, $s_t$ is chosen to minimize the cost of the base stock policy. That is, $s_t$ is set at $s_t^l = k^*\sqrt{V(e_n^l)}$, where $k^* = \Phi^{-1}(\frac{p}{\pi + p})$. For this case, it is intuitively clear that the larger the variance $V(e_n^l)$, the higher the cost (see Lemma A.7 for proof). Therefore, the same superiority results established in Theorem 4.4 carry over when the criterion is changed from the variance of the forecast error to the long run average cost of the base stock policy.

However, if the inventory manager is using the moving average forecasting because she does not know that the underlying demand process is AR(1), it is unlikely that she would be able to find or use this optimal safety stock level. Therefore, we consider the second case where a constant $s_t$ is adopted regardless of the choice of $n$, which does not necessarily minimize the expected cost. This is a more plausible scenario that would more likely occur in practice. We are also able to show that the results established for the first case hold true for the second case.

For convenience, in the sequel, the base stock policy using the $n$-period moving average with safety stock $s_n^l$ is referred to as $\overline{MA}(n)$ policy; and the same base stock policy with the safety stock level set at a constant $K$ is referred to as $\overline{MA}(n, K)$ policy. Since the long run average leadtime demand can be viewed as the moving average with $n = \infty$ (times $l + 1$), the constant base stock policies with
safety stock set at \( k^* \sqrt{V[D[t, t+l]]} \) and \( K \) are referred to as \( \overline{MA}(\infty) \) and \( \overline{MA}(\infty, K) \), respectively. Then we have the following two propositions:

**Proposition 4.5** Within the \( \overline{MA}(n) \) class and for the long run average cost criterion, \( \overline{MA}(\infty) \) is optimal if \( \alpha \leq \hat{\alpha}(l) \), else \( \overline{MA}(1) \) is optimal.

**Proposition 4.6** For all \( K \), within the \( \overline{MA}(n, K) \) class and for the long run average cost criterion, \( \overline{MA}(\infty, K) \) is optimal if \( \alpha \leq \hat{\alpha}(l) \), else \( \overline{MA}(1, K) \) is optimal.

Finally, some insight is shed by comparing the order quantities of \( \overline{MA}(1) \) and \( \overline{MA}(1, K) \) with those of the optimal policy for the system \( \Gamma \). For both policies, the order quantities are the same:

\[
y_{t,1} = D_{t-1} + (l+1)(D_{t-1} - D_{t-2})
\]

Here, the reaction level \((l+1)\) is always larger than \( \gamma_l \), the optimal reaction level, see (8). Therefore, using \( \overline{MA}(1) \) although optimal within the class of \( \overline{MA}(n) \) policies leads to an over-reaction to demand changes when \( \alpha > 0 \) and when \( \alpha < 0 \), a greater under-reaction to demand changes versus not using the moving average.

5 **Concluding Remarks**

By virtue of a sequential development from their predecessors in classic inventory theory, most of the existing models of supply chain research have retained the assumption of completely rational and capable decision makers. In this paper, we deviated from this tradition and conducted a formal exploration of the effect of systematic errors that may be committed by managers by examining the effects of one type of error: the over-reaction to demand changes and common contributor of decision making biases: demand forecasting. We show that our representative manifestations of forecasting - managers’ subjective response to demand signals and the use of simple quantitative forecasting techniques - share similar consequences: both result in an increase in internal costs and in the uncertainty and volatility of the system’s replenishment orders.

Our approach suggests that, as a cause of the bullwhip effect, responding to demand signals can be broadened beyond considering only optimal responses as proposed in Lee, Padmanabhan and Whang where the bullwhip effect is related to the best possible reaction to demand changes for the manager. The expectation of our approach in this paper was that broadening the range of behavioral responses to contexts, as we propose by our focus on errors, could, contribute to our understanding of the cause and mitigation of the bullwhip effect. The expectation is given confirming evidence in the results of the paper. In particular, a complementary perspective provided on the bullwhip effect, is that not only can it result from a demand process with a positive serial correlation but from a manager who is behaving
as if he is facing positive serial correlation, when in fact demand could be otherwise. Furthermore we
provide argument and thus incentive for mitigating the bullwhip effect by relating such mitigation to
decision making that would help reduce costs for the manager as well. In particular we show that an
asymmetry in costs exist which favors under-reaction to demand changes. Discovering such incentives
for mitigating the bullwhip effect is made possible by broadening the range of responses to demand
signals as we do.

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The proof for Proposition 3.6 uses Lemmas A.2-A.6. Lemmas A.1 and A.7 are used for Propositions 4.5 and 4.6.

Let $\eta_v$ be a normal random variable with mean 0 and variance $v^2$, $G(x) = h(x^+) + p(x^-)$, and define $g(r|v) = EG(r - \eta_v)$.

**Lemma A.1** For all $r$, $g(r|v)$ strictly increases in $v$.

**Proof.** Using the identity $g(r|v) = v g(r/v|1)$, it is easy to show that $\frac{\partial g(r|v)}{\partial v} = -(h+p) \int_{-\infty}^{r/v} x d\Phi(x)$ which is strictly positive because $\int_{-\infty}^{r/v} x d\Phi(x) < \int_{-\infty}^{x} d\Phi(x) = 0$. \hfill $\blacksquare$

**Lemma A.2** Let $\bar{\alpha}_l = -1$ if $l$ is even and if $l$ is odd, be such that $\bar{\gamma}_l(b) |_b = \bar{\alpha}_l - \alpha = 0$.

1. For $\alpha > \bar{\alpha}_l$, $V(b)$ and $V(-b)$ strictly increase in $|\bar{\alpha}_l - \alpha| \geq b \geq 0$.

2. For $\alpha \geq 0$ and $\min\{\alpha - \bar{\alpha}_l, 1 - \alpha\} \geq b > 0$, then $V(b) \geq V(-b)$. The inequality becomes an equality for $l = 0$.

**Proof.** Recall that $V(b) = \sigma^2 \left( V_l + (\bar{\gamma}_l(b))^2 - 2\bar{\gamma}_l \bar{\gamma}_l(b) \right)$. We have then that $V'(b) = \sigma^2 \bar{\gamma}_l'(b) (2\bar{\gamma}_l(b) - 2\gamma_l)$. Since $\bar{\gamma}_l(b)$ strictly increases in $b$ and $\bar{\gamma}_l(b) \geq \gamma_l$ if and only if $b \geq 0$, we have $V'(b) > 0$ and $V'(-b) > 0$ if and only if $b > 0$ and thus we have 1). For $l = 0$, $V(b) - V(-b) = 0$. For $l > 0$, $\alpha \geq 0$ and $\min\{\alpha - \bar{\alpha}_l, 1 - \alpha\} \geq b > 0$,

\[
V(b) - V(-b) = \sigma^2 \left( (\bar{\gamma}_l(b))^2 - (\bar{\gamma}_l(-b))^2 - 2\gamma_l (\bar{\gamma}_l(b) - \bar{\gamma}_l(-b)) \right)
= \sigma^2 (\bar{\gamma}_l(b) + \bar{\gamma}_l(-b) - 2\gamma_l) (\bar{\gamma}_l(b) - \bar{\gamma}_l(-b))
\geq 0
\]

since $\bar{\gamma}_l(0) + \bar{\gamma}_l(0) - 2\gamma_l = 0$ and $\bar{\gamma}_l'(b) - \bar{\gamma}_l'(-b) > 0$ imply that $\bar{\gamma}_l(b) + \bar{\gamma}_l(-b) - 2\gamma_l > 0$ and since $\bar{\gamma}_l(b) - \bar{\gamma}_l(-b) > 0$. Thus we have 2). \hfill $\blacksquare$

The proofs of Lemmas A.3 and A.5 require the following lemma.

**Lemma A.A.1.** Let $W(b) = (S(b))^2$. Let $W'(b)$ ($W''(b)$) refer to the first (second) derivative of $W(b)$ w.r.t. $b$.

1. For $-\alpha \leq b \leq 1 - \alpha$, s.t. $V_l - (\bar{\gamma}_l(b))^2 \geq 0$, i) $W'(b) \leq 0$ and $W''(b) \leq 0$.

2. Let $\bar{\alpha}_l = -1$ if $l$ is even and if $l$ is odd be such that $\bar{\gamma}_l'(b) |_{b = \bar{\alpha}_l - \alpha} = 0$. For $\bar{\alpha}_l - \alpha \leq b \leq -\alpha$, s.t. $V_l - (\bar{\gamma}_l(b))^2 \geq 0$, $W'(b) \geq 0$.
3. For \( \alpha > 0 \) and \( \alpha \geq b \geq 0 \), \( W(-b) \geq V(-b) \).

4. For \( \alpha < 0 \) and \( -\alpha \geq b \geq 0 \), \( W(b) \leq V(b) \).

**Proof.** Recall that \( S(b) = \left( \sqrt{V_l((\gamma_l(b))^2)} \right) \sigma \) which gives that \( W(b) = \left( V_l((\gamma_l(b))^2) \right)^2 \) and that \( V_l = \left( l + 1 + 2 \sum_{i=0}^l \gamma_{l-i} \right) \). Differentiating we get \( W'(b) = -2\tilde{\gamma}_l(b)\tilde{\gamma}_l(b)\sigma^2 \leq 0 \), and that \( W''(b) = (-2\tilde{\gamma}_l(b)\tilde{\gamma}_l(b) - 2(\gamma_l(b))^2)\sigma^2 \leq 0 \), for \(-\alpha \leq b \leq 1 - \alpha \) which prove 1). For \( \tilde{\alpha}_l - \alpha \leq b \leq -\alpha \), s.t. \( V_l - (\gamma_l(b))^2 \geq 0 \), \( W'(b) = -2\tilde{\gamma}_l(b)\tilde{\gamma}_l(b)\sigma^2 \geq 0 \) which proves 2). To see 3), recall that 
\[ V(b) = \sigma^2 \left( V_1 + (\gamma_l(b))^2 - 2\tilde{\gamma}_l(b)\gamma_l \right). \] 
This gives \( W(-b) - V(-b) = 2\tilde{\gamma}_l(-b)\gamma_l - 2(\gamma_l(b))^2 \geq 0 \), since \( \tilde{\gamma}_l(-b) \leq \gamma_l \) which proves 3). To see 4), we have \( W(b) - V(b) = 2\tilde{\gamma}_l(b)\gamma_l - 2(\tilde{\gamma}_l(b))^2 \leq 0 \), since \( \tilde{\gamma}_l(b) \geq \gamma_l \) which proves 4). \( \blacksquare \)

**Lemma A.3** Let \( S'(b) \) refer to the first derivative of \( S(b) \) w.r.t. \( b \).

1. For all \(-\alpha \leq b \leq 1 - \alpha \), \( S'(b) \leq 0 \).

2. Let \( \bar{\alpha}_l = -1 \) if \( l \) is even and if \( l \) is odd be such that \( \tilde{\gamma}_l(b) |_{b=\bar{\alpha}_l-\alpha} = 0 \). For all \(-\alpha \leq b \leq 1 - \alpha \), \( S'(b) \geq 0 \).

3. For \( \alpha > 0 \) and \( \alpha \geq b \geq 0 \), \( S(-b) \geq \sqrt{V(-b)} \).

4. For \( \alpha < 0 \) and \(-\alpha \geq b \geq 0 \), \( S(b) \geq \sqrt{V(b)} \).

5. For \( \alpha > 0 \) and \( b \geq 0 \), \( S(b) \leq \sqrt{V(0)} < \sqrt{V(b)} \).

**Proof.** From Lemma AA.1 (which proves results for \((S(b))^2\)) we have directly 1), 2) 3) and 4). Since from Lemma A.2, \( V(b) \) is strictly increasing in \( b \geq 0 \), 5) also follows from Lemma AA.1 since for \( \alpha > 0 \) and \( b \geq 0 \), we have \((S(b))^2 \leq (S(0))^2 = V(0) < V(b) \). \( \blacksquare \)

**Lemma A.4** For all \( k^* \),

1. if \( r(x) \leq v(x) \) and \( r(x) \) is a decreasing function and \( v(x) \) is an (strictly) increasing function then \( g(r(x)k^*v(x)) \) (strictly) increases in \( x \).

2. if \( r(x) \geq v(x) \) and \( r(x) \) is an increasing function and \( v(x) \) is an (strictly) increasing function then \( g(r(x)k^*v(x)) \) (strictly) increases in \( x \).

**Proof.** To establish claim 1) and 2), assume w.l.o.g. \( x_2 > x_1 \). We have necessarily \( g(r(x_2)k^*v(x_2)) \geq g(r(x_2)k^*v(x_1)) \) \( \geq g(r(x_1)k^*v(x_1)) \) where the first equality follows from Lemma A.1 and is strict if \( v \) is strictly increasing. The second inequality follows since we have \( v(x_1) \geq r(x_1) \geq r(x_2) \), if \( r(x) \leq v(x) \) and \( r(x) \) is a decreasing function and \( v(x_1) \leq r(x_1) \leq r(x_2) \), if \( r(x) \geq v(x) \) and \( r(x) \) is an increasing function. In both cases, \( r(x_1)k^* \) is closer to the optimal solution \( v(x)k^* \) than \( r(x_2)k^* \) and thus \( g(r(x_2)k^*v(x_1)) \geq g(r(x_1)k^*v(x_1)) \). \( \blacksquare \)

**Lemma A.5** For \( \alpha > 0 \) and \( \min\{\alpha, 1 - \alpha\} \geq b > 0 \),

\[ |S(b) - \sqrt{V(b)}| > |S(-b) - \sqrt{V(-b)}|. \]
Proof. For $\alpha > 0$ and $\min\{\alpha, 1 - \alpha\} \geq b > 0$, from Lemma A.3, we have, $S(b) \leq \sqrt{V(0)} < \sqrt{V(-b)} \leq S(-b)$ and $\sqrt{V(0)} < \sqrt{V(b)}$. Thus we have $|S(b) - \sqrt{V(b)}| > \sqrt{V(0)} - S(b)$ and $S(-b) - \sqrt{V(0)} > |S(-b) - \sqrt{V(-b)}|$. From Lemma AA.1 we have further that $S_t(b)$ is an increasingly decreasing function and thus we have $\sqrt{V(0)} - S(b) \geq S(-b) - \sqrt{V(0)}$ (since $S(0) = \sqrt{V(0)}$ which gives 1). □

Lemma A.6 Fix $x > 0$ and $\hat{v} \geq v$. For all $w$ in the range of $0 \leq w \leq x$,

$$g(\hat{v}k^* - x|\hat{v}) \geq g(vk^* + w|v) \quad \text{if } p \geq h,$$

and

$$g(\hat{v}k^* + x|\hat{v}) \geq g(vk^* - w|v) \quad \text{if } p \leq h.$$

The above inequalities become equalities if and only if $h = p$, $w = x$ and $\hat{v} = v$ all hold simultaneously.

Proof. Let $F \sim N(0, \hat{v}^2)$. Since $\frac{\partial g(\hat{v}k^* + q|\hat{v})}{\partial q} = (h + p)F(\hat{v}k^* + q) - p$, which is positive if and only if $q$ is positive, this means that

$$\left| \frac{\partial g(\hat{v}k^* + q|\hat{v})}{\partial q} \right| \geq \left| \frac{\partial g(\hat{v}k^* - q|\hat{v})}{\partial q} \right|$$

$$\iff (h + p)F(\hat{v}k^* + q) - p \geq p - (h + p)F(\hat{v}k^* - q)$$

$$\iff F(\hat{v}k^* + q) - F(\hat{v}k^*) \geq F(\hat{v}k^*) - F(\hat{v}k^* - q)$$

$$\iff e^{-(\hat{v}k^* + x)^2/2\hat{v}^2} \geq e^{-(\hat{v}k^* - x)^2/2\hat{v}^2}, \forall x > 0$$

$$\iff k^* \leq 0$$

$$\iff h \geq p,$$

where (19) follows since $F(\hat{v}k^*) = \left(\frac{p}{p+n}\right)^n$, (20) follows since $F'(x) = e^{-x^2/2\hat{v}^2}/\sqrt{2\pi \hat{v}^2}$ and (21) follows from simple algebra. The above result implies that for $0 \leq w \leq x$ and $p \geq h$, $g(\hat{v}k^* - x|\hat{v}) \geq g(\hat{v}k^* + w|\hat{v})$. Given $\hat{v} \geq v$, from Lemma A.4 we have $g(\hat{v}k^* + w|\hat{v}) \geq g(vk^* + w|v)$ and thus $g(\hat{v}k^* - x|\hat{v}) \geq g(vk^* + w|v)$. The required result can be similarly shown for $h \leq p$. □

Lemma A.7 $g(vk^*|v)$ strictly increases in $v$.


B Proof of Lemma 4.3

The proof of Lemma 4.3 requires the following lemma:

Lemma B.1 i) For $1 \geq \alpha \geq 0$, $\sum_{j=0}^{n-1} \alpha^j \leq \frac{1}{n} \left(n + 2 \sum_{j=1}^{n-1} (n - j)\alpha^j\right)$.

ii) For $-1 \leq \alpha < 0$, $\sum_{j=0}^{n-1} \alpha^j \geq 0$. 

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Proof. Assume $1 \geq \alpha \geq 0$. We proceed by examining the terms in the RHS.

\[ \text{RHS} = \frac{1}{n} \left( n + 2(n-1)\alpha + 2 \sum_{j=2}^{n-1} (n-j)\alpha^j \right) \]

\[ = \frac{1}{n} \left( n + n\alpha + (n-2)\alpha + 2 \sum_{j=2}^{n-1} (n-j)\alpha^j \right) \]

\[ \geq \frac{1}{n} \left( n + n\alpha + 3(n-2)\alpha^2 + 2 \sum_{j=3}^{n-1} (n-j)\alpha^j \right) \]

\[ \geq \frac{1}{n} \left( n \left( \sum_{j=0}^{k-1} \alpha^j \right) + (k+1)(n-k)\alpha^k + 2 \sum_{j=k+1}^{n-1} (n-j)\alpha^j \right) \]

\[ \geq \sum_{j=0}^{n-1} \alpha^j. \]

Now assume $-1 \leq \alpha < 0$. It is easy to show that $\sum_{j=1}^{n-1} \alpha^j \geq \alpha$ for $n > 1$ since each term in the series is less in magnitude but of a different sign than the previous term. Thus $\sum_{j=0}^{n-1} \alpha^j \geq 1 + \alpha \geq 0$.

Lemma 4.3 For all $n > 1$, $V(e_n^l) \geq V(D[t, t+l])$ holds if $\alpha \leq \hat{\alpha}(l)$, and $V(e_n^l) \geq V(e_1^l)$ holds otherwise.

Proof. We first show that $2\gamma_l \leq (l+1)$ implies $V \left( D[t, t+l] - \hat{D}_{n,t}^l \right) \geq V(D[t, t+l])$ for all positive integer $n$.

Recall $V(D[t, t+l]) = V(\zeta_l(t)) + \gamma_l^2 V(D)$. Noting that $\zeta_l(t)$ and $\hat{D}_{n,t}^l$ are independent we have,

\[ V \left( D[t, t+l] - \hat{D}_{n,t}^l \right) = V \left( \zeta_l(t) + \gamma_l D_{t-1} - \hat{D}_{n,t}^l \right) \]

\[ = V(\zeta_l(t)) + \gamma_l^2 V(D) + V(\hat{D}_{n,t}^l) - 2\gamma_l Cov(D_{t-1}, \hat{D}_{n,t}^l). \]

Thus, it suffices to show

\[ V(\hat{D}_{n,t}^l) - 2\gamma_l Cov(D_{t-1}, \hat{D}_{n,t}^l) \geq 0. \]

Now

\[ V(\hat{D}_{n,t}^l) = V \left( \frac{l+1}{n} \sum_{i=1}^{n} D_{t-i} \right) \]

\[ = \frac{(l+1)^2}{n^2} \left( n + 2 \sum_{j=1}^{n-1} (n-j)\alpha^j \right) V(D), \]
and
\[ 2\gamma_l \text{Cov}(D_{t-1}, \hat{D}_{n,t}^l) = 2\gamma_l \text{Cov} \left( D_{t-1}, \frac{l+1}{n} \sum_{i=1}^{n} D_{t-i} \right) = 2\gamma_l(l+1) \left( \sum_{j=0}^{n-1} \alpha^j \right) V(D). \] \tag{25}

When \(1 \geq \alpha \geq 0, 2\gamma_l \leq (l+1)\) implies
\[ 2\gamma_l \text{Cov}(D_{t-1}, \hat{D}_{n,t}^l) \leq (l+1) \left( \sum_{j=0}^{n-1} \alpha^j \right) V(D) \leq V(\hat{D}_{n,t}^l). \]

The last inequality is due to Lemma B.1(i).

When \(-1 \leq \alpha < 0, 2\gamma_l < 0 \leq (l+1)\) implies \(2\gamma_l \text{Cov}(D_{t-1}, \hat{D}_{n,t}^l) \leq 0 \leq V(\hat{D}_{n,t}^l).\) The first inequality is due to Lemma B.1(ii). (23) is proven.

Next we show that for all positive integer \(n, 2\gamma_l > (l+1)\) implies
\[ V \left( D[t,t \pm l] - \hat{D}_{n,t}^l \right) \geq V \left( D[t, t \pm l] - \hat{D}_{1,t}^l \right) \] \tag{26}

Note that \(2\gamma_l > (l+1)\) implies that \(\alpha > 0\). It follows from (22) that (26) is equivalent to
\[ V(\hat{D}_{n,t}^l) - 2\text{Cov}(\gamma_l D_{t-1}, \hat{D}_{n,t}^l) \geq V(\hat{D}_{1,t}^l) - 2\text{Cov}(\gamma_l D_{t-1}, \hat{D}_{1,t}^l). \]

It follows from (24) and (25) that we have
\[ V(\hat{D}_{n,t}^l) - 2\text{Cov}(\gamma_l D_{t-1}, \hat{D}_{n,t}^l) = \left( \frac{(l+1)^2}{n^2} \left( n + 2 \sum_{j=1}^{n-1} (n-j)\alpha^j \right) - \frac{2\gamma_l(l+1)}{n} \sum_{j=0}^{n-1} \alpha^j \right) V(D) \]
\[ \geq \frac{(l+1)^2 - 2\gamma_l(l+1)}{n} \sum_{j=0}^{n-1} \alpha^j V(D) \]
\[ \geq (l+1)^2 - 2\gamma_l(l+1) V(D) \]
\[ = V(\hat{D}_{1,t}^l) - 2\text{Cov}(\gamma_l D_{t-1}, \hat{D}_{1,t}^l), \]

where the first inequality follows from Lemma B.1, and the second follows from \(2\gamma_l > (l+1)\) and since \((1/n) \sum_{j=0}^{n-1} \alpha^j < 1\). Thus we have that \(2\gamma_l > (l+1)\) implies (26). This completes the proof.
Figure 1: Order Variance Amplification as a function of $\alpha$

Fig 2: Percentage increase in costs of Over-reaction and Under-reaction ($l=0$)
- Positive auto-correlation

Fig 3: Percentage increase in costs of Over-reaction and Under-reaction ($l=4$)
- Positive Serial Correlation

Figure 4: Graph of demand processes implied by $b+\alpha=0, 0.4$ & $0.8$ given $\alpha=0.4$
**Fig 5:** Percentage increase in costs of Over-reaction and Under-reaction (l=0) - Negative Auto-correlation

**Fig 6:** Percentage increase in costs of Over-reaction and Under-reaction (l=4) - Negative Auto-correlation

**Figure 7:** Inventory Costs of MA(n) and the Optimal Policy (l=0)

**Figure 8:** Inventory Costs of MA(n) and the Optimal Policy (l=0)
Figure 9: Inventory Costs of MA(n) and the Optimal Policy
(\(l=5\))

\(\alpha = 0.2\)
\(\alpha = 0.4\)
\(\alpha = 0.6\)
\(\alpha = 0.8\)
\(\alpha = 0.9\)