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Assortment rotation – the retailing practice of changing the assortment of products offered to customers – has recently been used as a competitive advantage for both brick-and-mortar and online retailers. We focus on product categories where consumers may purchase multiple products during a season and investigate a new reason why frequent assortment rotations can be valuable to a retailer. Namely, by distributing its seasonal catalog of products over multiple assortments rotated throughout the season – as opposed to selling all products in a single, fixed assortment – the retailer effectively conceals a portion of its full product catalog from consumers, injecting uncertainty into the consumer’s relative product valuations. Rationally-acting consumers may respond to this structural difference by purchasing more products, thereby generating additional sales for the retailer. We refer to this phenomenon as the *value of concealment* and show that the retailer enjoys a positive and significant value of concealment under quite general conditions. However, we show that when consumers are forward-looking, the value of concealment is context-dependent; we present insights and discuss intuition regarding which product categories likely lead to a positive vs. negative value of concealment. **Keywords:** assortment optimization, retailing, imperfect information

1. Introduction

Assortment rotation – the retailing practice of changing the assortment of products offered to customers throughout a selling season – has recently been used by both brick-and-mortar and online retailers as a strategy for gaining competitive advantage. A notable category of retailers who have employed this strategy successfully are “fast fashion retailers” such as Zara and H&M, who have differentiated themselves from other retailers by rotating their assortment multiple times throughout the fashion industry standard 6-month selling season; these two companies were the two largest global fashion retailers in terms of revenue earned in 2016 (Olanubi 2017). Interestingly, the entire industry of online flash sales (e.g., Rue La La and Groupon) has been created using the idea of frequent assortment rotation as a cornerstone of its business strategy: It is not uncommon for such companies to rotate their assortments on a daily basis. Finally, there has been a recent trend of large apparel retailers purchasing flash sales companies and thus diversifying their assortment rotation frequency across different channels (Forte 2016).

From a normative perspective, whether a retailer should adopt frequent assortment rotation as a strategy clearly depends on the trade-off between a host of factors. For certain factors, their impact on the retailer's profitability are intuitive and well-understood. For example, on the positive side, brick-and-mortar retailers who have limited shelf space benefit from frequent assortment rotations because it allows them to offer more products throughout a given selling season, increasing the likelihood that a customer finds products that she likes. Furthermore, frequent assortment rotations mitigate the challenge of predicting trends and product demand; if retailers have a short enough production cycle, they are able to adapt assortments rapidly based on the latest trends (Fisher and Raman 1996). Retailers can also time new product introductions with consumer budget constraints; that way, consumers are able to spend their monthly budget on new products as opposed to products introduced in previous months that they were unable to afford at the time, assuming that product attractiveness decays over time (Caro et al. 2014, Caro and Martínez-de-Albéniz 2012). On the other hand, assortment refreshes may entail switching costs, and the higher these costs are, the less appealing frequent refreshes become. In addition, if the retailer does not sufficiently engage the consumer (e.g., through inadequate marketing channels) frequent assortment refreshes may lead a consumer to miss out on purchasing items that she would have purchased, had it been offered throughout the season.

For other factors, the direction of their impact is *a priori* less obvious and merit detailed study. For example, Bernstein and Martínez-de-Albéniz (2017) investigate the interaction of product rotation costs and retailer capacity constraints on the frequency of assortment refreshes in the setting where each consumer may purchase up to a single product over the course of a selling season. They find interesting insights (reviewed in §1.1) about how assortments should be designed.

This study adopts a similar line of inquiry, and focuses on a different factor that has not received much research attention to date. Namely, we consider product categories within which consumers may make *multiple purchases* over the course of a selling season. As seen from consumer surveys, examples of such product categories include many apparel subcategories, accessories, and toys. For example, Pentecost and Andrews (2010) find that 66% of women purchase fashion products at least once per month, and Cilley (2016) finds that 42% of women purchase clothing at least once per month. In such settings with multiple purchases, the impact of frequent assortment rotation on the retailer is *a priori* unclear, particularly when products are priced differently. Both outcomes seem plausible: the retailer might see higher revenues from either (a) displaying all products in the catalog in a single assortment held fixed throughout a selling season or (b) distributing its products over multiple assortments rotated throughout the season. In the latter case, products to be sold in future assortments are “concealed” from the consumer, injecting uncertainty into the consumer's relative product valuations that may affect her purchase decisions. Thus throughout this paper,

we will say that the retailer has a positive *value of concealment* if its expected revenue from the latter (sequential assortment) exceeds that of the former (fixed assortment), and a negative value of concealment otherwise.

The main objective of this research is to gain an understanding around the drivers that influence the value of concealment for a retailer when the consumer may consider purchasing multiple products in the same category. To do this, we analyze two extreme assortment rotation strategies that the retailer could choose:

1. *Fixed assortment strategy*: offer all products concurrently throughout the entire selling season.
2. *Sequential assortment strategy*: offer each product sequentially over the course of the selling season, only offering one product at a time to consumers.

We study these extremes partly for the sake of analytical tractability, and partly because the clear distinction between these two strategies is helpful in generating sharper managerial insights from our analysis. The sequential assortment strategy is akin to the fast fashion and flash sales models where the retailer frequently rotates assortments, whereas the fixed assortment strategy is akin to the more traditional retail model of offering all products concurrently.

To achieve our objective, we first combine two aspects of consumer utility theory to characterize a consumer's utility for purchasing products throughout the selling season. The first, deterministic, component of our utility model is known to the customer and retailer *a priori*, and we assume that this component obeys the law of diminishing marginal utility (see, e.g., Marshall (1890), Ch.3). The second, random, component of our utility model represents the heterogeneity in valuations across different products (see, e.g., McFadden 1980). It is unknown to both the retailer and consumer *a priori* and thus reflects the consumer's uncertainty about future product options; the consumer realizes this random component, and thus her valuation of the product, only after the retailer offers the product in an assortment. We note that our utility model is a generalization of the one recently used in Fox et al. (2018), which we further discuss in Section 1.1.

We then consider a retailer who chooses to follow either the fixed or sequential assortment strategy in order to maximize its revenue over the selling season. Furthermore, the retailer must select the assortment composition, as well as the sequence in which to offer each product under the sequential assortment strategy. We show that the value of concealment is generally positive (i.e., it is advantageous for the retailer to offer products sequentially), and although it is difficult to fully characterize the optimal sequence in which to offer products, we show that a heuristic product sequencing policy that offers products in decreasing order of prices performs well in numerical tests and is guaranteed to provide more revenue than the fixed assortment strategy. Furthermore, we show that when consumers are forward looking, the value of concealment can be either positive or negative, and therefore the retailer's strategy is context-dependent. We present insights and

discuss intuition regarding which product categories likely lead to positive vs. negative values of concealment.

Our work contributes to the literature and to practice in two primary dimensions. First, our work extends the scarce literature analyzing assortment rotation strategies with multiple consumer purchases. Second, our work identifies a new reason why some retailers may want to rotate assortments throughout the selling season: to capture the value of concealment.

1.1. Literature Review

There are two papers that are most closely related to ours. The first is by Fox et al. (2018), who develop a model where a consumer first chooses a set of differentiated products to purchase, and subsequently decides which product to *consume* in each period given uncertain future consumption preferences. Despite the semantic differences between purchasing/consuming, our utility model is nearly identical to the “generalized model” of Fox et al. (2018, §4), where consumers have a “no purchase” outside option. In fact, our utility model generalizes their model in two respects: first and most importantly, the deterministic component of our utility model exhibits diminishing marginal utility, whereas Fox et al. (2018) essentially assume that this component is linear in the number of products purchased; it is worth noting that earlier work in this stream of literature models diminishing returns in the deterministic component of utility, as we do, but does not include uncertainty in future valuations (see, e.g., Dube (2004)). Second, we do not impose distributional assumptions on the form of the random utility shock, whereas their model is based on Gumbel-distributed utility shocks. Beyond differences in the utility model, Fox et al. (2018) consider a different decision environment than we do. In their model, consumers are fully strategic and make consumption choices from an inventory of available products (which diminishes as items are consumed) in each time period. Whereas Fox et al. (2018) are focused primarily on developing optimal shopping and consumption policies, this paper has a broader objective to not only develop policies characterizing optimal consumer decisions, but also to consider how these optimal decisions impact a retailer’s optimal assortment strategy.

The second is Bernstein and Martínez-de-Albéniz (2017) who also study the retailer’s optimal assortment rotation strategy and the value of concealment. To our knowledge, only our paper and Bernstein and Martínez-de-Albéniz (2017) consider assortment rotation strategies when a consumer’s purchase decisions in each period affect her purchase decisions in subsequent periods. The main difference with our paper is that they allow for each consumer to only purchase a single product within the selling season, which allows them to model their problem as an optimal stopping problem. Because we allow each consumer to purchase multiple products throughout the selling season, we must use a different modeling technique since our problem cannot be cast as an optimal

stopping problem. Interestingly, they find that in the absence of capacity constraints, it is optimal for the retailer to offer all products in a single period; thus in their model, capacity constraints are one underlying cause of why retailers may want to rotate assortments. For the special case of our model when consumers are willing to purchase at most one product throughout the selling season, we show that the fixed assortment strategy outperforms the sequential assortment strategy, corroborating the results of Bernstein and Martínez-de-Albéniz (2017); however, the underlying cause of our result stems from the consumer's low/zero value of purchasing multiple products.

Several other recent papers have proposed and studied various reasons why a retailer may rotate assortments throughout the selling season. The key modeling difference between our work and the following is that ours incorporates a dynamic model of consumer purchases - where each consumer may make multiple purchases over the season - whereas the following do not. Caro et al. (2014) study the optimal timing of assortment changes when product attractiveness decays after the product is introduced. Davis et al. (2015) study which product, if any, to add to the assortment in each time period to maximize revenue throughout the selling season; they assume that at most one product can be added at a time, and once a product is added, it cannot be removed from the assortment. Bernstein et al. (2015) and Golrezaei et al. (2014) explore how a retailer can best offer dynamic, customized assortments given a customer's type and remaining inventory levels. Several papers have studied how a retailer can learn consumer demand by dynamically changing assortments (Agrawal et al. (2019), Caro and Gallien (2007), Farias and Madan (2011), Rusmevichientong et al. (2010), Sauré and Zeevi (2013), Ulu et al. (2012)). Bernstein et al. (2019) bridges both of these literatures by showing how a retailer can offer dynamic assortments to jointly learn heterogeneous consumer demand and offer customized assortments. There is also a vast literature on new product introduction timing due to technological advances, which can influence a firm's assortment rotation strategy (see, e.g., Krankel et al. 2006, Ramachandran and Krishnan 2008).

The above papers study dynamic assortment decisions, where the retailer chooses to offer a changing assortment of products over time. In contrast, there is also a vast literature on *static* assortment optimization, where the retailer selects a single assortment of products to offer all customers in order to maximize total revenue. Advancements in this stream of work focus on first defining a choice model that specifies the likelihood of a customer choosing each product in the assortment; then, optimization algorithms are developed - typically specific to the chosen choice model - to select the subset of available products that the retailer should offer to maximize its revenue. Examples of work that address static assortment optimization under various choice models include: Talluri and van Ryzin (2004) (multinomial logit model), Rusmevichientong et al. (2014) (mixture of multinomial logits model), Davis et al. (2014) and Li et al. (2015) (nested multinomial

logit model), Rusmevichientong and Topaloglu (2012) (robust multinomial logit model), Wang (2018) (multinomial logit model with reference prices), Alptekinoglu and Semple (2016) (exponential choice model), Feldman and Topaloglu (2017) and Désir et al. (2019) (Markov chain choice model), and Bertsimas and Misis (2019) and Paul et al. (2018) (ranking-based choice model). The two key differences between this work on static assortment optimization and our paper is that (i) the choice models in this stream of work do not (scalably) incorporate the situation where consumers may purchase more than one product in the assortment, and (ii) this stream of work does not consider the dynamics of a consumer's purchase behavior over time. For these reasons, we chose to model consumer choice more similarly to that of the dynamic utility model in Fox et al. (2018), which has required very different techniques to solve for the optimal assortment rotation strategy.

Our work also relates to a stream of research on the effect of assortment size or variety on consumer demand and sales; however, unlike the other work referenced above, this stream of work does not consider assortment rotation strategies and rather focuses on the optimal single-period assortment size (see, e.g., Boatwright and Nunes 2001, Borle et al. 2005, Broniarczyk et al. 1998, Gourville and Soman 2005, Kuksov and Villas-Boas 2010).

Although allowing for multiple purchases in a category has been quite understudied in the academic literature, one notable exception is for dynamic purchases of consumer packaged goods (CPG); in this stream of literature, nearly all papers study identical products offered over time so there is no uncertainty in future product valuations (or the valuations can be learned by experiencing goods, e.g., Erdem and Keane (1996)), although some papers do model uncertainty in future prices (see, e.g., Erdem et al. (2003), Hendel and Nevo (2006), Seiler (2013), and Sun et al. (2003) for initial work in this area). Within this stream of work, our utility model has some similarities to two recent utility models proposed in Seiler (2013) and Ruiz et al. (2019). In Seiler (2013), the author considers a consumer who visits a grocery store on a periodic basis and during each shopping trip, decides whether or not to buy more laundry detergent (or other CPG). The consumer may choose to stockpile detergent if the price is right. Her utility is assumed to be a linearly decreasing function of the amount of detergent she has at home; our model has a similar, but slightly more general, feature where the consumer's utility is a general decreasing function of the number of products the consumer has previously purchased in the category. In addition, our model shares the common feature that the consumer experiences uncertainty in product valuations. However, the model in Seiler (2013) incorporates search costs, whereas ours does not. In Ruiz et al. (2019), the authors consider a customer who sequentially adds items to her grocery basket. In their base model, at each step, the consumer chooses the item that myopically maximizes her utility, conditional on the other items already in her basket; in an extension, they also consider consumers who

have a one-step lookahead. Their model considers latent attributes of items and complementarities between items, which we do not incorporate. Conversely, their model requires Gumbel-distributed error terms in their utility specification, whereas we do not require this assumption. Conceptually, these differences in our modeling approaches arise because our studies have markedly different objectives. Seiler (2013) and Ruiz et al. (2019) each focus on estimating a dynamic, structural utility model with data, whereas we aim to study the retailer’s assortment rotation strategy.

2. Model

During a given selling season, we suppose that a retailer (it) has a potential catalog of N unique products, within a certain product category, for sale to a market of homogeneous consumers, whose size is normalized to 1 without loss of generality. Before the season begins, the retailer may choose either: (i) a fixed assortment strategy, where the assortment is unchanged throughout the entire selling season, or (ii) a sequential assortment strategy where it offers each product sequentially over the course of the selling season; in the latter, we assume that the retailer divides the season into disjoint time periods, and each product is offered in exactly one period and each period has exactly one product. We note that the retailer also has secondary decisions associated with each strategy. Namely, it has to decide the optimal composition of products (i.e., which subset of $\{1, \dots, N\}$) to make available. If it chooses the sequential assortment strategy, it additionally has to decide the product sequence of available products. We allow the products to be priced differently, and denote their prices as p_1, \dots, p_N . We assume that these prices have been pre-determined by the retailer (e.g., based on margin targets or competitors’ pricing) and thus do not include them in the present set of retailer decisions; this assumption is implicitly made in other related work (see, e.g., Bernstein and Martínez-de-Albéniz (2017), Caro et al. (2014), and Davis et al. (2015)).

We now turn to the consumer’s utility, and do so by first describing a classical random utility model (McFadden 1980) that forms the basis of our model. Let X_1, \dots, X_N each represent scalar-valued random variables that are mutually independent, but not necessarily identically distributed. We will refer to X_i as a *valuation shock* for product i . If the consumer purchases some subset of products $S \subseteq \{0, \dots, N\}$, in the classical model, her total utility is given by

$$u(S) := \sum_{j \in S} (X_j - p_j). \quad (1)$$

Fox et al. (2018) use this utility model directly (further assuming Gumbel-distributed shocks) to study consumers’ optimal strategic consumption behavior from an inventory of different products, as we described in Section 1.1. Other studies start from this basic framework and add modifications to the model to capture essential features relevant to the problem that they are interested in. For example, in both the “independent assortment search” and “overlapping assortment search” models

by Cachon et al. (2005), a consumer’s total utility over two decision stages can be interpreted as having the form of (1) with an additional disutility term (representing a fixed search cost) that is potentially incurred in the second stage. Similarly, Wang and Sahin (2017) also consider a two-stage decision problem, with (1) appearing in the second stage of the consumer’s decision problem, and add the feature that a consumer incurs disutility in the first stage by paying a per-product search cost to form a consideration set of N products. Wang and Wang (2017) use a single stage model and modify the non-stochastic components of each product’s utility (represented by $\mathbb{E}[X_i] - p_i$ in our notation) to capture network effects. In the latter three papers, consumers are restricted to purchasing a single item (i.e., $|S| = 1$), whereas in Fox et al. (2018) and our paper, we may have $|S| > 1$.

In our consumer model, we adopt a similar approach, modifying the basic random utility framework to capture a feature of interest when consumers can make multiple purchases, namely, diminishing marginal utility from purchasing multiple items. This is a classical modeling element that traces back to Marshall (1890), and more recently has been analyzed by Baucells and Sarin (2007) and Caro and Martínez-de-Albéniz (2012) using the notion of “satiation”, where multiple similar purchases over time yield diminishing marginal utilities. To this end, we modify (1) with an additive term, assuming that the consumer’s utility from purchasing products in set S is given by

$$u(S) := \sum_{j \in S} (X_j - p_j) + V(|S|), \quad (2)$$

where $V : \mathbb{N}_0 \rightarrow \mathfrak{R}_+$ is a given increasing, concave function. In our model, as in the full satiation model of Baucells and Sarin (2007), the consumer’s marginal utility diminishes in the *quantity* of products purchased. Our model adds to Baucells and Sarin (2007) by further incorporating uncertainty in each product’s utility.

It turns out to be more convenient to equivalently express the consumer’s valuation in marginal terms, and we do so as follows. Define the function $v : \mathbb{N}_0 \rightarrow \mathfrak{R}_+$ as $v(0) := V(0)$, and $v(j) := V(j+1) - V(j)$ for all $j > 0$. It follows from the assumptions on V that v is nonnegative and decreasing. Then, we may write the consumer’s utility from purchasing products in set S as

$$u(S) := \sum_{j \in S} (X_j - p_j) + \sum_{j=0}^{|S|-1} v(j). \quad (3)$$

We now describe the sequence of events in our model. If the retailer chooses the fixed assortment strategy, the consumer (she) visits the retailer once, upon which she observes the realization of all the valuation shocks of all available products, and chooses which product(s), if any, to purchase from the subset of products that are available. If the retailer chooses the sequential assortment strategy, she visits the retailer in each period, views the product available in that period, observes

the realization of the valuation shock for that product, and then decides whether or not to purchase it. Note that in either the fixed or sequential assortment, the consumer may purchase multiple products. This feature is one of our paper's key points of differentiation from the literature.

We emphasize that when the consumer faces the sequential assortment, her purchase policy has to be *non-anticipative*. From (3), we can infer that the utility gained from purchasing a product in a given period depends on the number of previous items purchased. Namely, if she enters period i having already purchased some $k = 0, \dots, i - 1$ products, then, her utility gained from purchasing product i would be $X_i - p_i + v(k)$. One might reasonably posit an alternative utility model where this utility gain is a function of the *utility* of previous purchases, instead of the *quantity* of previous purchases. Although this alternative is plausible, we chose our model because (i) it is analytically tractable, (ii) it more closely resembles the classical notion of diminishing marginal utility, i.e., marginal utility decreases in the *quantity* of items, rather than their net utility, and (iii) it is used by other authors, e.g., Baucells and Sarin (2007) and Caro and Martínez-de-Albéniz (2012). We are unaware of any existing utility models where the utility gained from the purchase of an additional product is a function of the net utility of previous purchases, and creating a new utility model to account for such consumer behavior would be beyond the scope of this paper.

In our model, we further assume that:

- (a) The retailer has sufficient inventory to satisfy the maximum demand for each product.
- (b) The consumer knows when assortment rotations occur and visits the retailer at negligible cost every time a new assortment is displayed.
- (c) The consumer cannot return or exchange any purchased products.
- (d) The valuation shocks, X_1, \dots, X_N are mutually independent.

Our model is purposefully stylized in order to focus on the impact of the retailer's assortment rotation strategy on revenue, as opposed to pricing, sourcing, or inventory decisions. Assumption (a) is quite common in the literature (see, e.g., Bernstein and Martínez-de-Albéniz 2017). Practically, there is anecdotal evidence that assumption (b) is representative for many retailers who frequently change their assortment. In the fast fashion industry, it has been documented that customers visit Zara much more frequently than customers visit their more traditional retail competitors - an average of 17 visits per customer per year compared to just three (Kumar and Linguri 2006). As another example, many flash sales retailers send emails to their customers every time a new assortment is displayed on their site, and customers can simply click on the link in the email to visit the site and view the new assortment. Assumption (c) is satisfied either via the return policy or naturally when consumers start using the product as soon as it is purchased, i.e., the consumer wears her new pair of sandals the day after she buys them. The final assumption is a technical assumption that is required for the analysis.

We first study the consumer’s optimal purchase policy that maximizes her utility under both the fixed and sequential assortment strategies in Section 3, and then study the retailer’s optimal assortment rotation strategy and the value of concealment in Section 4. In Section 5, we extend our analysis to forward-looking consumers. We supplement our theoretical analysis with numerical studies in Section 6.

Notation. First, we denote the cdf of X_i as F_i , and define $\bar{F}_i := 1 - F_i$. We will generally allow F_i to be non-identical, but to avoid technical minutiae we will further assume that X_1, \dots, X_N are continuous, integrable random variables. We will omit explicitly presenting measure-theoretic details: The random valuation shocks X_1, \dots, X_N should be interpreted as being implicitly defined a suitably-constructed probability space. Second, we adopt the convention that all otherwise unqualified references to monotonicity (e.g., increasing/decreasing) refer to them in the weak sense, and all (in)equalities involving random variables are meant to hold with probability 1. All proofs can be found in Appendix A.

3. Consumer’s Optimal Policy

We will use π with superscripts to denote the consumer’s purchase policy, where $\pi_i = 1$ is interpreted as “purchase product i ” and $\pi_i = 0$ is interpreted as “do not purchase product i ”. In the following subsections, we will derive the optimal policies that maximize the consumer’s utility under each of the retailer’s assortment rotation strategies.

3.1. Under a Fixed Assortment

Under the fixed assortment strategy, the retailer offers the same subset of products throughout the entire selling season. Consider an arbitrary subset containing $M \leq N$ products, and assume that products are re-indexed such that these M products are now products $1, \dots, M$. The consumer, upon visiting the retailer, realizes her valuation for all M products. Thus, given a realization of X_1, \dots, X_M , the consumer’s choice can be represented by the following optimization problem:

$$\max_{z_i \in \{0,1\}, y = \sum_{i=1}^M z_i} \left\{ \sum_{i=1}^M z_i (X_i - p_i) + \sum_{k=0}^{y-1} v(k) \right\}.$$

The *price-adjusted realized valuations*, $X_i - p_i, i = 1, \dots, M$ play an important role in our analysis. In particular, several results are naturally stated through a permutation of the indices $\{1, \dots, M\}$ such that these price-adjusted valuations are decreasing. It is convenient to develop notation that captures this.

DEFINITION 1. Let $\sigma = (\sigma(1), \dots, \sigma(M))$ represent a permutation of $\{1, \dots, M\}$ such that the *price-adjusted valuations* are decreasing under σ , i.e., $X_{\sigma(1)} - p_{\sigma(1)} \geq X_{\sigma(2)} - p_{\sigma(2)} \geq \dots \geq X_{\sigma(M)} - p_{\sigma(M)}$.

We note that σ depends on the realized values of X_i (and is therefore a random variable), and further depends on the subset of products chosen by the retailer. Using this definition, Lemma 1 characterizes an optimal consumer purchase policy, π^f , when the retailer uses a fixed assortment. Intuitively, it implies that the consumer maximizes her total utility by purchasing products in decreasing order of their price-adjusted valuations, as long as each incremental purchase gives her positive utility.

LEMMA 1. *Define the set $S^f := \{j : v(j-1) + X_{\sigma(j)} - p_{\sigma(j)} \geq 0\}$ and let $D^f := |S^f|$ represent its cardinality. When the retailer follows the fixed assortment strategy, the consumer maximizes her utility by policy $\pi^f = (\pi_1^f, \dots, \pi_M^f)$, where $\pi_i^f := \mathbb{1}_{\{\sigma(i) \leq D^f\}}$, $i = 1, \dots, M$.*

Using Lemma 1, we may write an expression for the retailer's (random) revenue when it follows the fixed assortment strategy, which is

$$R^f := \sum_{i=1}^M p_{\sigma(i)} \mathbb{1}_{\{v(i-1) + X_{\sigma(i)} - p_{\sigma(i)} \geq 0\}}. \quad (4)$$

3.2. Under a Sequential Assortment

Now suppose that the retailer adopts the sequential assortment strategy and consider an arbitrary subset of $M \leq N$ products to be sold over the selling season (this subset may be different than the subset in the fixed assortment). Recall that in this setting, each assortment contains only a single product, and the retailer presents the M assortments/products sequentially over M periods. Suppose the retailer has chosen to sell these M products in some given arbitrary order, and assume that the M products are also indexed according to that order. We emphasize that the consumer realizes her valuation for each product only when the assortment/product is revealed and that she can only use the realized valuations at each point in time to make her purchase decisions.

In period $i = 1, \dots, M$, assuming that the consumer has already purchased $k \in \{0, \dots, i-1\}$ products, the consumer solves the maximization problem $\max \{v(k) + X_i - p_i, 0\}$. Her total utility is therefore maximized by the threshold policy, π^s , where

$$\pi_i^s(k, X_i) := \mathbb{1}_{\{v(k) + X_i - p_i \geq 0\}} \quad i = 1, \dots, M. \quad (5)$$

We now present a useful expression for the total revenue of products purchased by the consumer over the entire selling season, which we denote as R^s . Define the stochastic process $\{W_i\}_{i=0}^M$ as follows: Let $W_0 := 0$, and recursively define $W_i := W_{i-1} + \mathbb{1}_{\{v(W_{i-1}) + X_i - p_i \geq 0\}}$, $i = 1, \dots, M$. W_i can be interpreted as the number of products purchased by the consumer through period i . The total revenue earned by the retailer over the selling season is

$$R^s := \sum_{i=1}^M p_i (W_i - W_{i-1}) = \sum_{i=1}^M p_i \mathbb{1}_{\{v(W_{i-1}) + X_i - p_i \geq 0\}}. \quad (6)$$

REMARK 1. From (6), we observe that if $M < N$, adding product $M + 1$ will only increase R^s . Hence, it is optimal for the retailer to include all N products when choosing the sequential assortment strategy. As we will soon show, this is not necessarily the case for the fixed assortment strategy.

4. Retailer's Optimal Assortment Strategy

We now analyze whether the fixed or sequential assortment strategy generates higher expected revenue for the retailer. To this end, we define the retailer's *value of concealment* (VoC) as $\text{VoC} = \mathbb{E}[R_*^s] - \mathbb{E}[R_*^f]$, where R_*^s represents the retailer's revenue when it uses an optimized sequential assortment strategy (i.e., with an optimal sequence of products), and where R_*^f represents the retailer's revenue when it uses an optimized fixed assortment strategy (i.e., with an assortment composition chosen to maximize its expected revenue, potentially with fewer products). In other words, when the VoC is positive (negative), it is optimal for the retailer to choose the sequential (fixed) assortment strategy.

4.1. Optimal Assortment Composition

When the retailer uses a fixed assortment strategy, it must decide on the subset of products to offer in the assortment. As we show in the following example, it is not always optimal for the retailer to offer all products.

EXAMPLE 1. Consider the case where $N = 2$, and label the two products A and B , with $p_A = \$75$ and $p_B = \$50$, and let $X_A = X_B = 0$ with probability 1. Consider the case where $v(0) = \$100$ and $v(1) = \$25$. If both products A and B are offered under the fixed assortment strategy, the customer will purchase only product B since $X_B - p_B > X_A - p_A$, $v(0) + X_B - p_B > 0$, and $v(1) + X_A - p_A < 0$; the retailer earns revenue of $p_B = \$50$. On the other hand, if the retailer only offers product A , the consumer will purchase product A since $v(0) + X_A - p_A > 0$, and the retailer will earn $p_A = \$75$. Thus, in this example, it is optimal for the retailer to offer only a subset of its products (product A) under the fixed assortment strategy. \square

The example above raises a natural question: what is the optimal product composition for the retailer in a fixed assortment? Unfortunately, the stochastic and combinatorial nature of this problem prevented us from obtaining a simple, closed-form answer to this question. Nevertheless, using ideas from stochastic programming, we developed an approach to solve this problem numerically through a technique called Sample Average Approximation (SAA), which comes with large sample approximation guarantees (see, e.g., Kleywegt et al. (2002)). We defer the discussion of this numerical approach to §6.

4.2. Optimal Product Sequence

When the retailer uses a sequential assortment strategy, it must choose the sequence in which to offer the N products. We begin this analysis with some preliminaries. For $n = 0, \dots, N$, and $k = 0, \dots, N - n$, let $R_k(n)$ be a random variable representing the revenue earned by the retailer under the sequential assortment strategy when the customer has already purchased k items and has n products left to view. Formally, define the stochastic process $\{Y_i\}_{i=0}^n$ as $Y_0 := k$, and $Y_i := Y_{i-1} + \mathbb{1}_{\{X_{N-n+i} > p_{N-n+i} - v(Y_{i-1})\}}$, and further define

$$R_k(n) := \sum_{i=1}^n p_{N-n+i} (Y_i - Y_{i-1}) = \sum_{i=1}^n p_{N-n+i} \mathbb{1}_{\{v(Y_{i-1}) + X_{N-n+i} - p_{N-n+i} > 0\}}. \quad (7)$$

(In this notation, $R_0(N) = R^s$). The following lemma bounds the difference $R_k(n) - R_{k+1}(n)$.

LEMMA 2. *Let $n = 0, \dots, N$ and $k = 0, \dots, N - n - 1$ be given. Then, with probability 1,*

$$R_{k+1}(n) \leq R_k(n) \leq \rho_n^* + R_{k+1}(n), \quad (8)$$

where $\rho_0^* := 0$ and $\rho_n^* := \max\{p_{N-n+1}, p_{N-n+2}, \dots, p_N\}$ if $n > 1$.

Intuitively, this result relates the revenue earned from the remaining products purchased in the last n periods for two different hypothetical customers (call them A and B), where A has purchased k products and B has purchased $k + 1$ products prior to the n th period from the end of the season. Because the base valuation $v(k)$ exhibits diminishing marginal utility and thus decreases in k , we naturally have $R_{k+1}(n) \leq R_k(n)$, reflecting that, *ceteris paribus*, A is more willing to make purchases than B . However, the second bound of Lemma 2 shows that this increased willingness to purchase of customer A is bounded. In fact, what it means is that A purchases at most one more product than B , and therefore contributes at most ρ_n^* (i.e., the highest price of the remaining n products) more to the retailer. We further define $\Delta_k : \mathbb{Z} \rightarrow \Re$ for any $k = 0, \dots, N - 1$, as the remaining expected revenue loss for the retailer if the customer entered the period with $k + 1$ versus k products purchased:

$$\Delta_k(n) := \begin{cases} \mathbb{E}[R_{k+1}(n)] - \mathbb{E}[R_k(n)] & \text{if } n = 1, 2, \dots, N - k - 1 \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Building on Lemma 2 and Equation (9), the following lemma characterizes necessary and sufficient conditions under which interchanging the order of two products would hurt the retailer's total expected revenue, thus implying the optimal sequence of products.

LEMMA 3. For a given period $j = 1, \dots, N - 1$, consider the pair of products X_j, X_{j+1} that are assigned to the j^{th} and $(j + 1)^{\text{th}}$ position. Suppose that the customer arrives at period j having already purchased $k = 0, \dots, j - 1$ products. Fixing j and k , for $i = j, j + 1$, define

$$\beta_i := \begin{cases} \frac{\mathbb{P}(X_i > p_i - v(k+1))}{\mathbb{P}(X_i > p_i - v(k))} & \text{if } \mathbb{P}(X_i > p_i - v(k)) > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

Then,

- (a) If $\mathbb{P}(X_j > p_j - v(k)) = 0$ or $\mathbb{P}(X_{j+1} > p_{j+1} - v(k)) = 0$, interchanging products j and $j + 1$ does not change the retailer's expected revenue.
- (b) If $\mathbb{P}(X_j > p_j - v(k)) > 0$ and $\mathbb{P}(X_{j+1} > p_{j+1} - v(k)) > 0$, interchanging products j and $j + 1$ decreases the retailer's expected revenue if, and only if,

$$p_j(1 - \beta_j) - p_{j+1}(1 - \beta_{j+1}) + (\beta_{j+1} - \beta_j) \Delta_{k+1}(N - j - 1) \geq 0. \quad (11)$$

Two special cases are worth noting: The first is when products have different random valuation shock distributions, but identical prices. In this case, our characterization of the optimal sequencing will turn out to rely on a type of stochastic ordering called the *hazard rate ordering*. Although this ordering is most commonly defined for nonnegative random variables, an equivalent general definition of this ordering for generic real-valued signed random variables is given by Shaked and Shanthikumar (2007)[p16, equation (1.B.4)]. We reproduce this below for completeness.

DEFINITION 2. Let U, V be real-valued random variables. We say that U is smaller than V in the *hazard rate order*, denoting this as $U \leq_{hr} V$, if $\frac{\mathbb{P}(U > x + \delta)}{\mathbb{P}(U > x)} \leq \frac{\mathbb{P}(V > x + \delta)}{\mathbb{P}(V > x)}$ for every $\delta \geq 0$ and $x \in \mathfrak{R}$.

The hazard rate order implies the usual stochastic order. Thus, the intuitive interpretation of $X_j \leq_{hr} X_{j+1}$ is that, *ceteris paribus*, product j is (stochastically) less appealing to the consumer than product $j + 1$. The following corollary implies that when prices are equal, it is optimal to order products from least to most appealing.

COROLLARY 1 (Optimal Sequencing: Equal Prices, Unequal Valuation Distributions).
If all products have the same prices, i.e., for some $p \geq 0$, we have $p_i = p$ for every $i = 1, \dots, N$, then any ordering such that $X_1 \leq_{hr} X_2 \leq_{hr} \dots \leq_{hr} X_N$ is optimal.

The second special case is when products have different prices, but identically-distributed valuation shocks. Here, subject to a mild technical condition on that common distribution, it is optimal to order products decreasing in price. The condition is that this distribution has the *increasing failure rate* (IFR) property, namely, that the function $\mathbb{P}(X > x)$ is log-concave in x (see, e.g., Shaked and Shanthikumar 2007, p1). Many common random variables have IFR distributions (e.g., uniform, exponential, normal; see Banciu and Mirchandani (2013) for a longer list).

COROLLARY 2 (Optimal Sequencing: Unequal Prices, Equal Valuation Distributions).

Let X be a random variable with an IFR distribution. If $X_i \stackrel{d}{=} X, i = 1, \dots, N$, then any sequence of products where $p_1 \geq p_2 \geq \dots \geq p_N$, is optimal.

Interestingly, both Corollaries 1 and 2 lead to similar intuition: the retailer's optimal sequencing of products in the sequential assortment strategy is to offer products from least to most appealing. Customers become more selective as they purchase more products, and will therefore only buy additional products if they are very appealing.

4.3. Value of Concealment

Building upon our optimal product sequencing results from §4.2, we are ready to state the central result of this paper about the retailer's value of concealment.

THEOREM 1. *Suppose that an optimized fixed assortment contains $M \leq N$ products. Let R be a random variable representing the retailer's revenue when it follows a sequential assortment strategy for the same M products sequenced in decreasing order of prices, i.e., $p_1 \geq p_2 \geq \dots \geq p_M$.*

(a) *Then, $R \geq R_*^f$ with probability 1.*

(b) *The retailer's value of concealment is positive, that is, $\mathbb{E}[R_*^s] \geq \mathbb{E}[R_*^f]$.*

The combinatorial nature of the problem prevented us from finding a simple, closed-form characterization of an optimal assortment composition (in the case of the fixed assortment strategy), or an optimal product sequence (in the case of the sequential assortment strategy). However, despite not having these characterizations, we are still able to show that the value of concealment is positive (part (b)). That said, part (b) alone would be hard to take advantage of in practice since the optimal sequence of products is typically difficult to explicitly characterize and would require the retailer to perfectly characterize the valuation distributions $\{F_i, i = 1, \dots, N\}$. Therefore, part (a) is particularly important because it shows that a sequential assortment that decreases in prices is a reasonable heuristic, at least from the perspective of generating revenue that exceeds that of the fixed assortment strategy. In §6 we conduct a numerical study to investigate the relative value of this heuristic.

In addition, we note that in the context of our model, there is no useful information that the retailer gleans from offering products sequentially. The retailer in our model cannot make any dynamic assortment adjustments based on consumer decisions or purchases, which is the most natural mechanism that one expects the retailer to profit from adopting a sequential assortment. Therefore, what this implies is that the additional value that the retailer enjoys from product concealment is *structural* and fundamental, as opposed to *informational*.

We further develop the key contributors of a retailer's *strictly* positive value of concealment in the following two theorems.

THEOREM 2 (Conditions for Zero VoC). *For any N , the retailer's VoC is zero if*

- (a) *For some $v \in \mathfrak{R}$, $v(k) = v$ for all $k = 0, \dots, N - 1$, or*
- (b) *For some $x \in \mathfrak{R}, p \in \mathfrak{R}_+$, $X_i = x$ with probability 1 and $p_i = p$ for all $i = 1, \dots, N$.*

THEOREM 3 (Symmetric Asymptotic Analysis). *Assume that $X_i \stackrel{d}{=} X$ and $p_i = p$ for all i . As $N \rightarrow \infty$, for any $t \in \{f, s\}$,*

- (a) *Suppose that $\lim_{N \rightarrow \infty} \mathbb{P}(X \geq p - v(N - 1)) = \delta > 0$. Then $R_*^t / N \xrightarrow{a.s.} p\delta$.*
- (b) *Define $N_0 := \sup\{N \geq 1 : \mathbb{P}(X \geq p - v(N - 1)) > 0\}$. Suppose that $N_0 < \infty$. Then $R_*^t \xrightarrow{a.s.} pN_0$.*

Together, these results reveal that a non-zero value of concealment is driven by the *combination* of three key factors: diminishing marginal utility (Theorem 2(a)), uncertainty in future product valuations or price heterogeneity (Theorem 2(b)), and a limited number of products in the category (Theorem 3). We note that Theorem 3(b) may be interpreted as incorporating a fixed consumer budget that limits the maximum number of purchases (i.e., N_0) that a given consumer is willing to purchase over the season.

5. Forward-Looking Consumers

In this section, we extend our model and analysis to consider forward-looking consumers. Such consumers seek to maximize their utility over the course of the selling season; thus under the sequential assortment strategy, they consider the expected utility they would gain from future purchases when making their current purchase decision. In other words, their purchase decisions are derived from solving a dynamic program. In practice, customers are likely unable to form complex purchasing strategies over many future periods, and therefore may only be able to consider a few future periods of decisions when making their current purchase decision. Moreover, it is likely that only a fraction of consumers are forward-looking, and it may be difficult for retailers to estimate this fraction. We consider the impact of these pragmatic considerations in our numerical study (§6). In this section, we aim to first develop directional insights for the retailer and thus will analyze the setting of where all the retailer's customers are fully forward-looking.

In order to avoid having to make strong assumptions on the customer's knowledge about the retailer's product sequencing decisions, we focus on the special case where each product provides the same expected utility, specifically, $p_i = p$ and $X_i \sim_{iid} F \forall i = 1, \dots, N$; in addition to tractability reasons, these are reasonable assumptions in horizontally differentiated product categories. In this case, the ex-ante homogeneity of the products imply that the product sequence does not matter for either the retailer or the consumer. Specifically, customers need only be forward-looking with respect to the realizations of future products' random valuation shocks. Even in the absence of price and random valuation shock distribution heterogeneity, we show that the value of concealment may

be positive or negative; we present insights and discuss intuition regarding which product categories likely lead to positive vs. negative values of concealment. In Section 6, we present numerical studies for the more general case of heterogeneous prices and valuation shocks.

5.1. Consumer's Optimal Policy

Since the consumer faces no uncertainty under the fixed assortment strategy, her optimal purchase policy is the same as described in Section 3.1. Under the sequential assortment strategy, however, her optimal purchase policy differs in that she now forms expectations over future product valuations. When referring to forward-looking consumers, we will modify our original notation with a “tilde” when necessary.

Forward-looking consumers maximize their utility over the course of the selling season, and thus their optimal policy may be represented as an optimal solution to the following Bellman equation: For $i = 1, \dots, N$ and $k = 0, \dots, i - 1$, the forward-looking consumer's expected utility-to-go at the start of period i , i.e., before her valuation of product i is realized, is

$$\begin{aligned} U_i(k) &= \mathbb{E}[\max\{v(k) + X_i - p + U_{i+1}(k+1), U_{i+1}(k)\}] \\ U_{N+1}(k) &= 0, \end{aligned} \tag{12}$$

where the expectation is over the consumer's random valuation of product i , namely, X_i . The first term in the maximization, $v(k) + X_i - p + U_{i+1}(k+1)$, represents the consumer's (random) utility for purchasing product i plus the expected future utility that she would earn given that she purchases product i . The second term in the maximization, $U_{i+1}(k)$, represents the consumer's expected future utility that she would earn given that she does not purchase product i .

As before, a threshold policy $\tilde{\pi}^s$ is optimal, where in this case

$$\tilde{\pi}_i^s(k, X_i) := \mathbb{1}_{\{X_i \geq s_i(k)\}} \quad i = 1, \dots, N. \tag{13}$$

The thresholds $s_i(k)$ do not admit simple expressions, but can be implicitly defined as:

$$s_i(k) := p - v(k) - U_{i+1}(k+1) + U_{i+1}(k). \tag{14}$$

Next, we define the stochastic process $\{\tilde{W}_i\}_{i=0}^N$ as follows: Let $\tilde{W}_0 := 0$, and recursively define $\tilde{W}_i := \tilde{W}_{i-1} + \mathbb{1}_{\{X_i \geq s_i(\tilde{W}_{i-1})\}}$, $\forall i = 1, \dots, N$. \tilde{W}_i can be interpreted as the number of products purchased by the forward-looking consumer through period i . Similar to (6), the total revenue earned by the retailer over the selling season is

$$\tilde{R}^s := p\tilde{W}_N = p \sum_{i=1}^N \mathbb{1}_{\{X_i \geq s_i(\tilde{W}_{i-1})\}}. \tag{15}$$

The next result collects several key properties of the thresholds in (14), and characterizes how they relate to other parameters of our model.

PROPOSITION 1. *When the retailer follows the sequential assortment strategy, the thresholds $s_i(k)$ that determine the forward-looking consumer's optimal threshold policy satisfy the following properties:*

- (a) *The thresholds increase in k for every fixed i ; that is, $s_i(k) \leq s_i(k+1)$.*
- (b) *The thresholds decrease in i for every fixed k ; that is, $s_i(k) \geq s_{i+1}(k)$.*
- (c) *The thresholds increase as the valuation distribution F increases stochastically.*
- (d) *The thresholds are bounded from above; specifically, $s_i(k) \leq p - v(k + N - i)$.*

In Proposition 1, part (a) implies that the forward-looking consumer gets increasingly selective as she purchases more products – her $(k+1)$ th purchase has to meet a higher bar of attractiveness to her than her k th purchase, given that she has already viewed i products. Intuitively, since $v(k)$ is a decreasing function of k , she experiences diminishing returns with each product she purchases and therefore is more selective. Part (b) implies that the forward-looking consumer gets less selective the closer she is to the end of the selling season. This is because she knows she'll be less likely to find a better product in a future assortment simply because there will be fewer products available and opportunities to purchase. For part (c), if the valuation distribution increases stochastically, products become generally more attractive to the consumer; her expectation of future product valuations is more optimistic (higher), and she therefore becomes more selective in the current period about making a purchase. Finally, part (d) is mostly a technical property, but it reveals a simple sufficient condition for the forward-looking consumer's optimal purchasing behavior: It is optimal to purchase product i if $X_i \geq p - v(k + N - i)$.

5.2. Retailer's Optimal Assortment Strategy

Under the sequential assortment strategy, it is clear from Lemma 3 that any sequence of products leads to the same expected revenue. Furthermore, without a priori differentiated products, the optimal assortment composition under the fixed assortment strategy is to offer all N products. Therefore the retailer only has one decision to make in order to maximize its revenue over the selling season: Offer the N products under the fixed or sequential assortment strategy. It turns out that this is generally a difficult problem, primarily because the consumer's optimal purchasing policy is defined by solving the dynamic program in (12).

5.2.1. Special Case: $N = 2$. For the special case of just two products and when the retailer follows the sequential assortment strategy, the following equations describe the expected utility-to-go functions for the forward-looking consumer:

$$U_1(0) = \mathbb{E}[\max\{v(0) + X_1 - p + U_2(1), U_2(0)\}], \quad (16)$$

$$U_2(0) = \mathbb{E}[\max\{v(0) + X_2 - p, 0\}], \text{ and} \quad (17)$$

$$U_2(1) = \mathbb{E}[\max\{v(1) + X_2 - p, 0\}]. \quad (18)$$

THEOREM 4 (VoC: Forward-Looking Consumers, $N = 2$). *For $N = 2$, the retailer's value of concealment is positive if and only if*

$$F(a)(F(b) - F(a)) \leq \bar{F}(c)(\bar{F}(b) - \bar{F}(c)), \quad (19)$$

where $a := p - v(0)$, $c := p - v(1)$, and $b := a + \int_a^c \bar{F}(u) du$. Thus if condition (19) holds, the retailer maximizes its revenue by following the sequential assortment strategy. Otherwise, it maximizes its revenue by following the fixed assortment strategy.

Note that when $N = 2$, equations (16)–(18) illustrate that the consumer must consider three purchasing thresholds: $p - v(0) - U_2(1) + U_2(0)$, $p - v(0)$, and $p - v(1)$. Furthermore, we have $b = a + \int_a^c \bar{F}(u) du = p - v(0) - U_2(1) + U_2(0)$, where the second equality follows from (14) and (29) in the proof of Proposition 1 with $i = 2$ and $k = 0$. Therefore, Theorem 4 shows us that whether the retailer should choose the sequential or fixed assortment strategy depends on the probability density that falls between each of these three thresholds.

The result above shows that for these forward-looking consumers, there are regions of the parameter space in which the VoC is positive. This result reveals that the assumption of myopia in customers is not a necessary condition for the VoC to be positive, that is, it is possible for retailers to have a higher expected revenue from a sequential assortment (compared to a fixed assortment) even if consumers are forward-looking. However, the setting of forward-looking consumers is more nuanced as the sign of the VoC is now parameter-dependent.

To gain additional insights, we first present analytical results when $X_i \sim \text{Uniform}(\alpha, \beta)$ before interpreting condition (19) for the general valuation distribution F . We will consider the interesting set of parameters where $\alpha \leq p - v(0) \leq p - v(1) \leq \beta$. Note that the restriction $\alpha \leq p - v(0)$ implies that for some realizations of X_1 and X_2 , the consumer will choose not to purchase either product $i = 1, 2$. The restriction $\beta \geq p - v(1)$ implies that for some realizations of X_1 and X_2 , the consumer will purchase both products. The following result provides necessary and sufficient conditions under which the retailer maximizes its revenue by following the sequential assortment strategy.

COROLLARY 3. *For forward-looking consumers and $N = 2$, $X_i \sim \text{Uniform}(\alpha, \beta)$, and $\alpha \leq p - v(0) \leq p - v(1) \leq \beta$, the retailer's value of concealment is positive if and only if*

$$v(0) + v(1) \geq 2p - \alpha - \beta. \quad (20)$$

Thus if condition (20) holds, the retailer maximizes its revenue by following the sequential assortment strategy. Otherwise, it maximizes its revenue by following the fixed assortment strategy.

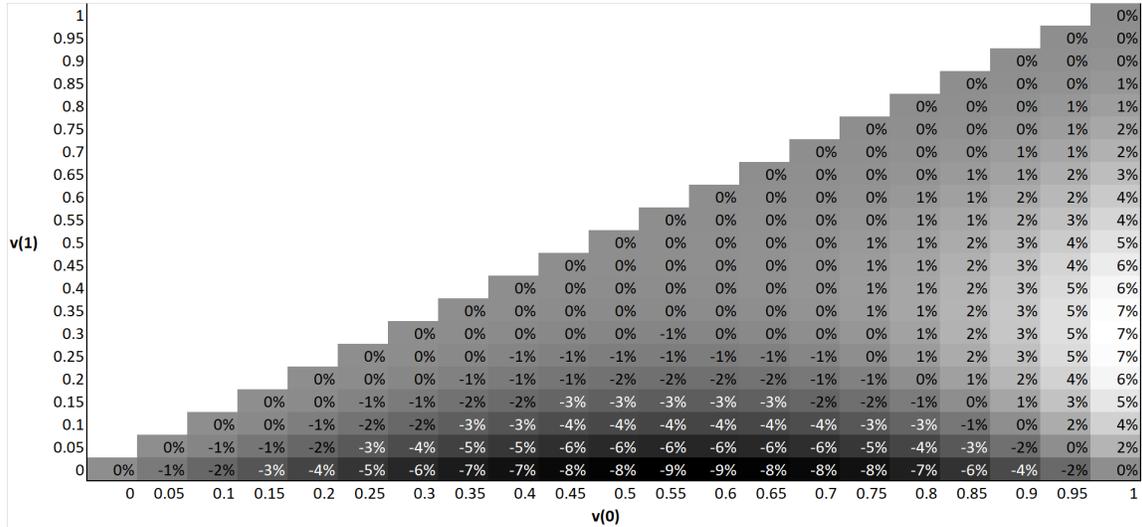


Figure 1 Value of concealment as a percentage of expected revenue from the fixed assortment strategy: $p = 1$ and $X_i \sim \text{Uniform}(0, 1); i = 1, 2$

Corollary 3 follows from Theorem 4 after algebraic manipulation of (19). Corollary 3 tells us that as long as the total base valuation ($v(0) + v(1)$) that the forward-looking consumer would obtain from purchasing both products is large enough, the retailer maximizes its expected revenue by following the sequential assortment strategy. By recognizing that $\mathbb{E}[X_i] = \frac{\alpha + \beta}{2}$ and rearranging terms in condition (20), condition (20) can be re-written as

$$(v(0) + \mathbb{E}[X_1] - p) + (v(1) + \mathbb{E}[X_2] - p) \geq 0. \quad (21)$$

In other words, as long as the consumer's expected total valuation from purchasing *both* products is non-negative, the retailer maximizes its revenue by following the sequential assortment strategy; note that this is less restrictive than requiring her expected valuation from purchasing *each* product to be non-negative.

Figure 1 shows a heat map of the value of concealment, expressed as a percentage of the retailer's expected revenue from following the fixed assortment strategy, for the case where $p = 1, X_i \sim \text{Uniform}(0, 1); i = 1, 2$ and for different values of $v(0) \in [0, 1], v(1) \in [0, v(0)]$. In other words, Figure 1 illustrates the percent increase in demand when the retailer follows the sequential assortment strategy compared to the fixed assortment strategy.

Figure 1 illustrates that for some values of $v(0)$ and $v(1)$, the value of concealment can be substantial in both directions. To help the retailer understand when it should follow each assortment rotation strategy, we will focus on two extremes to help gain some intuition. First, we look at the case when $v(0) = 1$, at the right-most portion of Figure 1, which is generally where the retailer has the greatest value of concealment. In this case, the consumer will certainly purchase at least one

product regardless of which assortment strategy the retailer follows. The increase in expected sales comes from the situation where, after viewing the first product and realizing her valuation, the consumer believes that she will not like the second product as much as the first; in this case she may decide to purchase the first product. Subsequently, she realizes her valuation for the second product which she ends up liking even more than the first, so much so that she may also purchase the second product. Had the retailer offered both products in a fixed assortment, the consumer may have only purchased her favorite product.

Next, we look at the case when $v(1) = 0$, at the bottom-most portion of Figure 1, generally where the retailer has the smallest (and negative) value of concealment. In this case, the consumer will purchase at most one product regardless of which assortment strategy the retailer follows. The decrease in expected demand comes from the situation where, after viewing the first product and realizing her valuation, the consumer believes that she will like the second product more than the first; in this case she decides not to purchase the first product. Subsequently, she realizes her valuation for the second product which she ends up not liking as much as the first, so much so that she may also decide not to purchase the second product. Had the retailer offered both products in a fixed assortment, the consumer may have purchased her favorite product.

Having discussed the interpretation of Theorem 4 for the special case of a uniform valuation distribution, we note that the same intuition holds for normally-distributed valuations (see Appendix B). We now turn toward interpreting condition (19) of Theorem 4 for a general valuation distribution F , and will see that the broad ideas remain the same in certain parametric regimes. As $F(a)$ decreases, the probability of buying at least 1 product increases. Indeed, in the limit, when $F(a) = 0$, this means that $\mathbb{P}(X + v(0) \geq p) = 1$ which means that the consumer purchases at least 1 product with probability 1. All else being equal, for small $F(a)$, as $F(a)$ decreases, the LHS of (19) also decreases, and the retailer tends to prefer a sequential assortment. In the limit when $F(a) = 0$, the LHS of (19) is zero and thus, in this limiting case, a sequential assortment is always preferred. Conversely, as $\bar{F}(c)$ decreases, the probability of purchasing both products also decreases. In the limit as $\bar{F}(c) = 0$, we have $\mathbb{P}(X + v(1) \geq p) = 0$, meaning that the consumer purchases both products with probability 0. All else being equal, for small $\bar{F}(c)$, as $\bar{F}(c)$ decreases, the RHS of (19) also decreases, and the retailer tends to prefer a fixed assortment. In the limit when $\bar{F}(c) = 0$, the RHS of (19) is zero and thus, in this limiting case, the retailer always prefers the fixed assortment.

In sum, Theorem 4 suggests that if consumers are predisposed towards purchasing at least 1 product, the retailer tends to favor a sequential assortment; conversely, if consumers are predisposed toward purchasing at most 1 product, the retailer tends to favor a fixed assortment. In the sequel, we investigate this result for general N .

5.2.2. General N : Next we investigate whether this intuition holds more generally beyond the case of $N = 2$ products and uniform valuation shocks. For general N , we believe that it will be very difficult – if not impossible – to develop simple conditions that are both necessary and sufficient to determine whether the value of concealment is positive or negative. In Appendix B.3, we show that even for $N = 3$ with uniform valuation shocks, while a sharp characterization is possible, it is also very complicated. Therefore, we focus on developing sufficient conditions for the retailer’s optimal strategy and continue to build our intuition.

THEOREM 5 (VoC: Forward-Looking Consumers, Sufficient Conditions). *For any $N \geq 2$,*

- (a) *If $X_i \geq p - v(N - 2)$ with probability 1, then the value of concealment is positive.*
- (b) *If $X_i \leq p - v(1)$ with probability 1, then the value of concealment is negative.*

Theorem 5 shows us that directionally, the intuition that was developed for the case where $N = 2$ continues to hold more generally. Specifically, for product categories where consumers tend to purchase multiple products, the retailer will maximize revenue by offering products sequentially; this is suggested by part (a). Conversely, for product categories where consumers tend to purchase just a few products, the retailer will maximize revenue by offering products in a fixed assortment; this is suggested by part (b). Even though the sufficient conditions for the theorem can be stringent, one can apply a continuity argument to arrive at less stringent sufficient conditions.

Fast fashion and flash sales retailers are typified by their frequent assortment rotations, which corresponds to the sequential assortment strategy in this paper. We note that for most product categories associated with such retailers (e.g., fashionable apparel), consumers often purchase multiple products within a category. Our results above suggest that for such categories, retailers may be able to realize a positive value of concealment by offering products in such categories sequentially. On the other hand, retailers occasionally use the same strategy to sell products in categories where it is expected that a consumer will purchase at most one product (e.g., engagement rings). Our results suggest that product concealment has a negative value in those cases, and that the retailer could generate more revenue by offering all products in a fixed assortment. This intuition also gives guidance to retailers who are choosing to offer some product categories via the sequential assortment strategy and others via the fixed assortment strategy.

6. Numerical Study

In this section we discuss two numerical simulation studies. These studies had several overarching goals: (a) to investigate various forms of forward-looking consumers where we lift key assumptions used in our theoretical analysis, (b) to develop a better understanding of the magnitude of the VoC, and (c) to understand the robustness of the results with respect to the model parameters. In both studies, we assumed a catalog of $N = 10$ products.

6.1. Study 1: VoC with Limited Lookahead

In this study, we evaluated the VoC in a setting of unequal prices and valuation shocks when customers were assumed to solve a rolling-horizon problem with a lookahead of $L \in \{0, \dots, N\}$ periods. We note that the case of $L = 0$ corresponds to the case where customers are not forward-looking at all, making purchases using our basic model in §3. Conversely, the case of $L = N$ corresponds to the case where customers are fully forward-looking, as described in §5. Intermediate values of L represent cases where customers have a limited lookahead. Their purchasing behavior can be derived from solving a modified version of (12).

In this study, valuation shocks were assumed to be independently normally distributed but non-identical, having means and variances that were themselves randomly-drawn parameters. We generated 1,000 such problem instances. In each instance, we assumed that the components of the baseline valuation vector were exponentially decaying, and drew the decay parameter randomly from a standard lognormal distribution. Product prices were drawn from another independent standard lognormal distribution. The means of the valuation shocks were drawn from a lognormal distribution centered around the product prices to introduce positive correlation between prices and valuations as one might expect in practice, whereas the standard deviations of the shocks were drawn from another independent standard lognormal distribution.

To evaluate the retailer's revenue from the fixed assortment, we had to find an optimal assortment composition for each problem instance. We did so numerically using a Sample Average Approximation (SAA) (see, e.g., Kleywegt et al. (2002)). What is particularly appealing about the SAA approach is that the problem can be formulated as a mixed-integer linear program (MILP), which can be solved for moderate problem sizes using modern optimization packages. The MILP formulation is provided in Appendix B.1.

For each problem instance, we generated 1,000 SAA samples of valuation shocks using the distribution parameters corresponding to that instance, and solved the MILP to determine the approximately-optimal assortment composition for the retailer's fixed assortment strategy. We then used this optimal composition to implement a product ordering heuristic for the sequential assortment that was based on Theorem 1, which ordered the included products (from the fixed assortment) from highest to lowest prices, followed by the excluded products (from the fixed assortment) from highest to lowest prices. Here, because product prices and valuation shocks are unequal, we assume that forward-looking customers are strategic both with respect to the realizations of future products' random valuation shocks, and the sequence in which products are presented. To be specific, we assume for tractability that forward-looking customers fully anticipate the sequence of products that would be offered by the retailer.

For each problem instance, we generated 10,000 independent sets of demand shocks and estimated the retailer's expected revenues under the (approximately) optimal fixed assortment strategy and the heuristically-ordered sequential assortment strategy (the latter was repeated for each distinct value of the customers' lookahead period L). Figures 2 and 3 tabulate our results over the 1,000 problem instances.

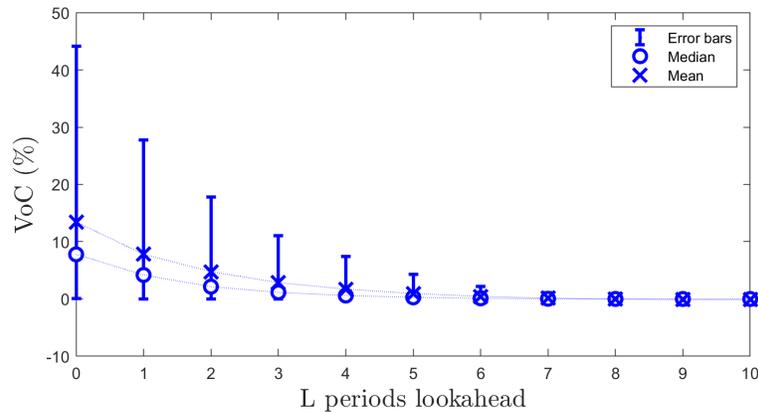


Figure 2 Relationship of mean/median VoC and lookahead period L . Error bars depict 5% and 95% quantiles of the VoC for each L .

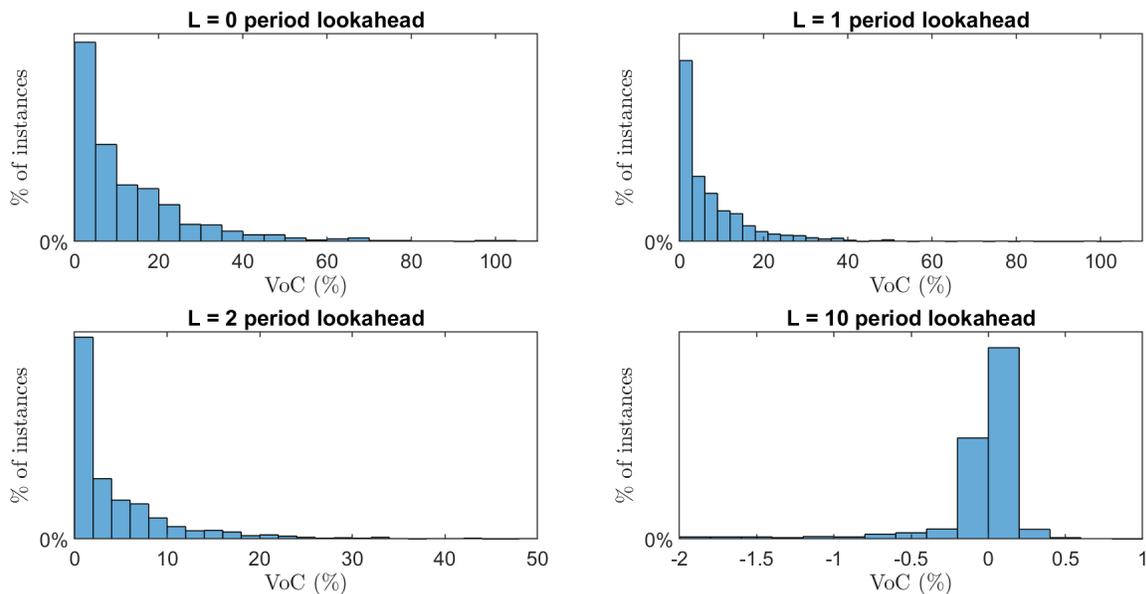


Figure 3 Histogram of VoC for selected numbers of lookahead periods L , expressed as a percentage of the expected revenue from the fixed assortment.

By solving the MILP to obtain the approximately optimal fixed assortment composition, the vast majority of instances (around 92%), had an optimal composition of either 9 or 10 products.

In other words, the fixed assortment seldom excluded more than 1 product.

The comparative results depicted in Figures 2 and 3 reveal that in the basic case ($L = 0$), the VoC was always nonnegative (as we expect, from Theorem 1) with mean of 13.5% and median of 7.8%. Variation in the VoC was high, for approximately 38% of instances, the value of concealment was less than 5%. In the top 5% of instances, the value of concealment exceeded 44%.

As the number of lookahead periods increased, both the median and variation in VoC decreased. For moderately low values of lookahead ($L = 1$ and $L = 2$), the VoC remained largely nonnegative, and in the top 5% of instances, exceeded 27% and 17% respectively. At the far end, with fully forward-looking consumers ($L = 10$), the mean VoC was -0.1% and the median VoC was 0.0%, with substantially reduced variation in the VoC.

This numerical study suggests that the VoC can be substantial; in other words, there can be a large gain by switching from a fixed assortment to a sequential assortment, and ordering products using the heuristic outlined in Theorem 1. The potential gains are especially large when L is small. This is the case that we think is more likely to hold in many practical settings, as it is unlikely that individuals would be able to form the complex purchasing strategies that a large L would entail. Conversely, it is possible for the VoC to be negative for customers who are very forward-looking, but the downside is considerably smaller in magnitude.

6.2. Study 2: Unknown Consumer Types

In reality, a retailer may not know whether its customers are forward-looking or not (or a combination of both), which may affect whether the retailer can earn a positive value of concealment. How then should a retailer choose its assortment strategy in this setting of unknown consumer types? To model this case, we assume that only some fraction $\gamma \in [0, 1]$ of the retailer's consumers make purchase decisions as described in Section 3, and the remaining proportion are fully forward-looking and make purchase decisions as described in Section 5.1; given our results from Study 1, using *fully* forward-looking consumers likely leads to the most conservative estimate of the value of concealment. Note that in practice, retailers may have trouble accurately estimating γ . The retailer's expected revenue under the sequential assortment strategy is then the corresponding weighted combination of its expected revenue from each consumer type, and therefore, its value of concealment is also the corresponding weighted combination of its value of concealment from each consumer type.

This numerical study aims to develop intuition for the impact of γ and other parameters on the VoC. Unlike the previous study, here we considered equal prices and i.i.d. valuation shocks in order to more cleanly assess how the various model parameters affected the VoC. Because products are ex-ante homogeneous, consumers need only be forward-looking with respect to product

valuations. We assumed that $p = 1$, and the baseline valuation vector declined exponentially, so that $v(k) = e^{-\lambda k}$, where the decay rate, λ , is a parameter that we varied. In addition, we assumed that the valuation shocks were independent normal $N(\mu, \sigma^2)$ random variables, where μ, σ were also parameters that were varied. We note that, all else being equal, higher values of μ and lower values of λ represent parameters where consumers tend to purchase more products and vice-versa. Finally, we also repeated this study under the assumption that the baseline valuation vector was linearly decreasing but lower bounded by zero, namely $v(k) = (1 - \frac{\lambda k}{N})_+$. Figure 4 illustrates the results of this study.

There are three key observations that follow from the results, which are common across both the exponential and linearly decreasing functional forms of the baseline valuation vector. First, even though the relationship between the value of concealment and μ was non-monotonic, we observed a general pattern in which there was a “cutoff” value of μ below which the VoC was negative, and above which it was positive. In other words, the *sign* of the VoC was weakly increasing in μ . Similarly, as λ increased, the sign of the VoC was weakly decreasing in all but only 1.33% (exponentially decaying v) and 0.67% (linearly decreasing v) of the combinations of μ and σ that we investigated. Together, these observations support the broad intuition that was established in special cases by Theorem 5 - that the VoC tends to be positive in scenarios where one anticipates consumers purchasing more products on average, and negative otherwise.

Second, as the variance in random valuation shocks (i.e., σ) increased, the sign of the VoC also weakly increased, and this pattern held for all combinations of λ and μ . Intuitively, this observation means that as consumers become more uncertain about their product valuations, retailers can capitalize on this uncertainty by concealing future products to induce consumers to purchase more.

Third, although we expected the value of concealment to increase as the composition of consumers becomes less forward-looking (i.e., as γ increases), this numerical study adds a more nuanced sense of the magnitude of this effect. We observed that when uncertainty was high ($\sigma = 2.0$), the value of concealment was positive for all μ in our range as long as 75% of customers or fewer were forward-looking.

The broad overall recommendation that we derive from these results is that retailers should be more inclined to use sequential assortment strategies for a product category the larger the number of purchases (per consumer) it anticipates, the more uncertain the product valuations are in that category, and the lower its estimated proportion of forward-looking consumers that it serves. Overall, it appears that several categories of apparel tend to fit these criteria. For example, categories with low-cost products tend to be purchased in greater quantities, categories containing fashion-forward products are likely to have more uncertain product valuations, and categories that are advertised as having limited availability might tend to induce less forward-looking consumer

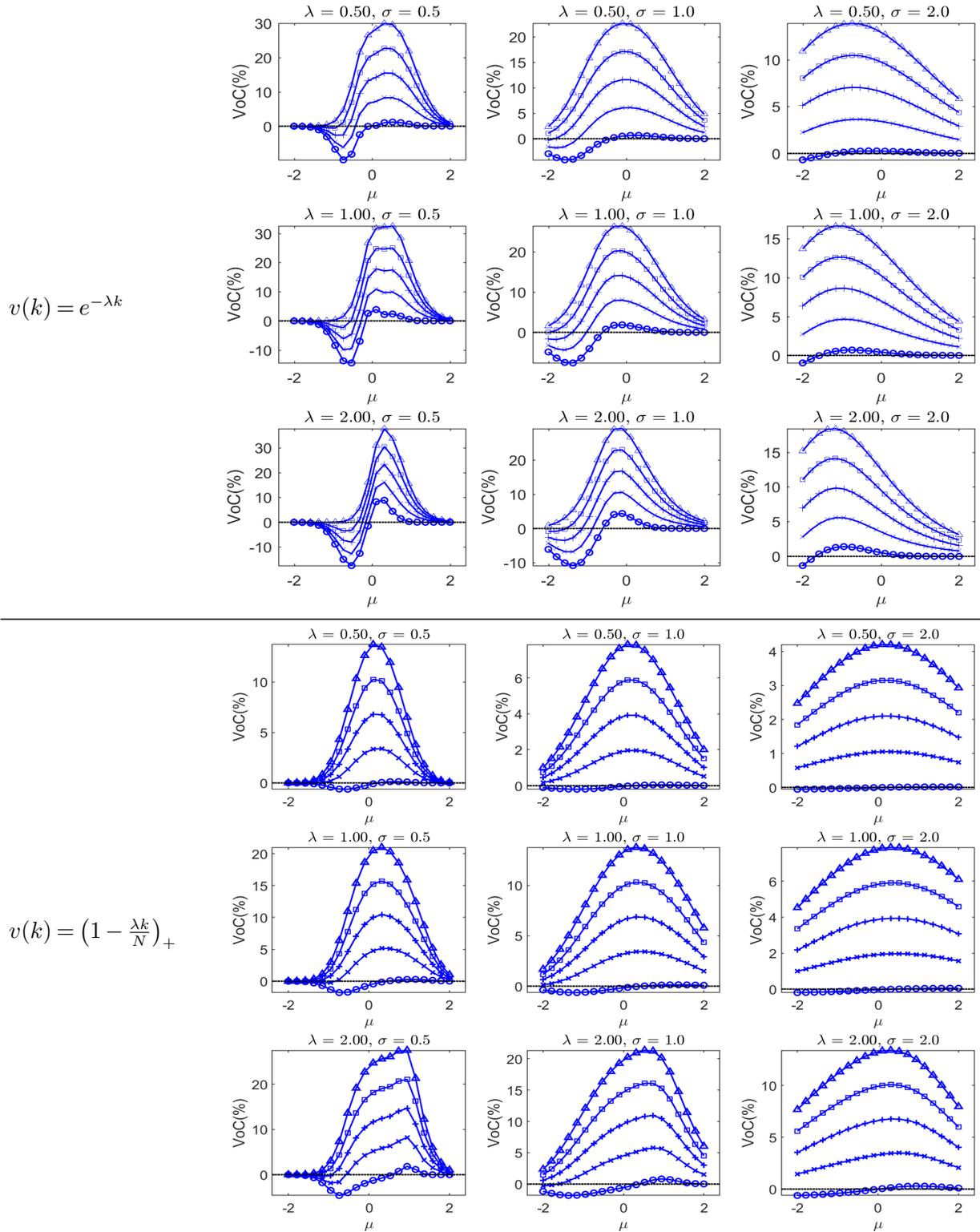


Figure 4 Value of concealment (VoC), expressed as a percentage of the expected revenue from consumers in the fixed assortment, against mean valuation shock, μ . Dashed horizontal line shows where $V_{oC} = 0$.

Legend: \circ $\gamma = 0.00$ \times $\gamma = 0.25$ $+$ $\gamma = 0.50$ \square $\gamma = 0.75$ \triangle $\gamma = 1.00$

purchase behavior. At least anecdotally, it seems that the major categories of products that are currently being sold with frequent assortment rotations tend to have one or more of these features outlined above, suggesting that retailers possess some intuitive sense of the value of concealment. The contribution of our work in this regard is that it provides a conceptual framing of its drivers.

7. Conclusion

As more and more retailers are choosing to frequently rotate their assortments, it becomes increasingly important to understand the implications of such frequent assortment rotation on the consumer's purchase decisions. Our work identifies and studies a new reason for retailers to have assortment rotations – they may be able to obtain a positive value of concealment and thereby generate additional revenue. In this paper, we developed a model of a consumer's purchase decisions throughout a selling season when she considers purchasing multiple products in the same category, and applied this model to gain an understanding around when this value of concealment is positive vs. negative.

We found that the retailer can benefit from a positive value of concealment by following the sequential assortment strategy and offering products in decreasing order of prices. For two special cases, we were able to fully characterize the optimal sequence of products, allowing the retailer to further increase its value of concealment. First, when all products share the same price, we showed that it was optimal for the retailer to offer products such that their valuation shock distributions are in increasing hazard rate order. Second, when all products have identical valuation shock distributions, we showed that it was optimal for the retailer to offer products in decreasing order of prices. In both of these special cases, the retailer's optimal sequencing of products is to offer products from least to most appealing, since customers become more selective as they purchase more products. Simulations extending beyond the two special cases and using the decreasing price heuristic suggest that the value of concealment may be substantial: increasing the retailer's expected revenue by 13.5% averaged across all problem instances.

We further identified the key drivers of a positive value of concealment, namely the combination of (i) diminishing marginal utility, (ii) uncertainty in future product valuations or price heterogeneity, and (iii) a limited number of products in the category. Interestingly, many fast fashion and flash sales retailers – who employ a sequential assortment strategy – sell products that are likely characterized by these key drivers. For example, consumers shopping for fashion apparel likely face diminishing marginal utility of each additional fashion item and have high uncertainty in their valuation of future fashion items; furthermore, prices vary greatly within this category and each season there are limited products for sale. As another example, some restaurants are employing a type of sequential assortment strategy: some sushi restaurants serve unique sushi rolls

on a conveyor belt that passes in front of customers. Again, consumers likely face diminishing marginal utility of eating one more sushi roll and are uncertain about the valuation of new sushi rolls to come, and the total number of sushi rolls is limited over the course of the meal. For such characterizations of products and consumers, retailers can reap substantial benefits from following the sequential assortment strategy due to the value of concealment.

For forward-looking consumers, we identify conditions under which the value of concealment is positive vs. negative. Our results and intuition provide guidance to retail executives regarding which product categories should be managed following the sequential vs. fixed assortment strategies. Specifically, for product categories where consumers are likely to buy many products within the category throughout the selling season, the uncertainty in future product valuations that the sequential assortment strategy introduces can cause the forward-looking consumer to purchase more products than if she had been offered them in a fixed assortment; in this case, the retailer has a positive value of concealment and maximizes its revenue by following the sequential assortment strategy. Again corroborating our intuition, for many of the product categories that fast fashion and flash sales retailers sell – such as fashionable apparel, accessories, and children’s toys – consumers are likely to buy many products within the category. In contrast, when consumers consider buying at most one or very few products within the category throughout the season, the uncertainty in future product valuations that the sequential assortment strategy introduces can cause the forward-looking consumer not to purchase a product that she would have attained positive utility from; in this case, the retailer’s value of concealment is negative, and it instead maximizes its revenue by following the fixed assortment strategy. Examples of such product categories include automobiles, home appliances, and even some apparel and accessories categories (e.g. engagement rings).

Finally, we conducted two in-depth numerical studies to investigate various forms of forward-looking consumers and to develop a better understanding of the magnitude of the value of concealment. Both studies suggest that the value of concealment can be substantial, i.e. there can be a significant increase in expected revenue by switching from a fixed assortment to a sequential assortment, and ordering products using our proposed heuristic. The potential gains are especially large when customers are limited in their forward-looking behavior, product valuations are more uncertain, and for categories where retailers expect customers to purchase multiple products.

In practice, a retailer’s optimal assortment rotation strategy should be determined by weighing a combination of factors. As noted in the Introduction, factors such as constraints on shelf space and consumer budgets, or consumers’ desire for “newer” products tend to favor more frequent refreshes, whereas high visitation costs from consumers or a high switching cost for each refresh tend to favor infrequent refreshes. Furthermore, operational decisions such as a generous return policy

may dampen the value of concealment by allowing a consumer facing the sequential assortment strategy to effectively reverse purchase decisions after realizing her valuation of future products. However, before weighing the pros and cons of frequent assortment rotations, it is imperative for retailers to understand whether the balance scale is even level to begin with. This study shows that in the context of categories where consumers make multiple purchases, the scale is not level, and, in particular, tends to tilt in favor of frequent assortment refreshes when some fraction of the consumer population is not forward-looking.

In conducting and sharing this research, we have realized that there is very little academic work that studies operations and marketing decisions in the context of consumers making multiple purchases within category and over a selling season (with the exception of CPG), yet this context is prevalent in practice. We see numerous directions of future work in this area. In addition to applying common problems (e.g. pricing) to this area, perhaps one of the most important streams of future work will be to propose and study new utility models that are suitable to multiple consumer purchases. We chose to use a utility model that has been well-supported and used in previous literature, but there may be others that are reasonable, too. Namely, one could conceptualize a utility model where the utility earned from a previous purchase would affect the utility earned in subsequent purchases (recall that in our model, only the quantity of previous purchases - not their utility - affects the utility earned in subsequent purchases). We are unaware of existing utility models that capture this feature. There are some non-classical behavioral models based on attaining a prescribed target consumption level that can incorporate this feature (Brown et al. 2012, Chen et al. 2015). However, these models have well-recognized limitations, including challenges associated with how to set targets and issues of dynamic consistency (Chen et al. 2015, §4). Therefore, it could be fruitful for future research to either focus on developing multi-purchase decision models based on the classical framework of expected utility maximization, or to develop model variants that circumvent some of the limitations associated with non-classical behavioral models. The development of such decision models could open the horizon to future interesting research and lead to new insights on various aspects of practical retail operations.

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Appendix A: Proofs

Proof of Lemma 1. Fix X_1, \dots, X_N . Suppose that if the consumer acts optimally, she purchases ℓ products, where $\ell \in \{0, \dots, N\}$. Given that she purchases ℓ products, it is obvious that her utility-maximizing choice of ℓ products are the ℓ products with the highest price-adjusted valuations, that is $\sigma(1), \dots, \sigma(\ell)$. Therefore, the consumer's problem can be written as $\max_{\ell=0, \dots, N} \left\{ \sum_{i=1}^{\ell} (v(i-1) + X_{\sigma(i)} - p_{\sigma(i)}) \right\}$. Define $L := \{1, \dots, \ell\}$, and consider

$$\begin{aligned} & \sum_{i=1}^{D^1} [v(i-1) + X_{\sigma(i)} - p_{\sigma(i)}] - \sum_{i=1}^{\ell} [v(i-1) + X_{\sigma(i)} - p_{\sigma(i)}] \\ &= \sum_{i \in S^f \setminus L} \underbrace{[v(i-1) + X_{\sigma(i)} - p_{\sigma(i)}]}_{\geq 0} - \sum_{i \in L \setminus S^f} \underbrace{[v(i-1) + X_{\sigma(i)} - p_{\sigma(i)}]}_{\leq 0} \geq 0, \end{aligned}$$

implying that her utility is maximized by only purchasing products i that have $\sigma(i) \leq D^1$. Q.E.D.

Proof of Lemma 2. We prove both bounds by induction on n . When $n = 0$, because $R_k(0) = 0$ for each k , (8) holds trivially noting that $\rho_0^* = 0$ by construction. Now suppose that (8) holds for some n and every $k = 0, \dots, N - n - 1$. We aim to show that (8) holds when there are $n + 1$ periods remaining (this is the period when the product with random valuation X_{N-n} is observed). Fix any $k = 0, \dots, N - n - 2$, and define the following events:

$$\begin{aligned} A &:= \{X_{N-n} > p_{N-n} - v(k+1)\}, \\ B &:= \{p_{N-n} - v(k) < X_{N-n} \leq p_{N-n} - v(k+1)\}, \\ C &:= \{X_{N-n} \leq p_{N-n} - v(k)\}. \end{aligned}$$

Note that $\{A, B, C\}$ form a partition of the probability space. To prove the lower bound,

$$\begin{aligned} & R_{k+1}(n+1) - R_k(n+1) \\ &= \mathbb{1}_A \left(\underbrace{R_{k+2}(n) - R_{k+1}(n)}_{\leq 0} \right) + \mathbb{1}_B \left(\underbrace{R_{k+1}(n) - R_{k+1}(n) - p_{N-n}}_{=0} \right) + \mathbb{1}_C \left(\underbrace{R_{k+1}(n) - R_k(n)}_{\leq 0} \right) \\ &\leq 0, \end{aligned}$$

where we have applied the induction hypothesis to establish that the first and third terms are bounded from above by 0. To prove the upper bound,

$$\begin{aligned} & \rho_{n+1}^* + R_{k+1}(n+1) - R_k(n+1) \\ &= \rho_{n+1}^* + \mathbb{1}_A \left(\underbrace{R_{k+2}(n) - R_{k+1}(n)}_{\geq -\rho_n^*} \right) + \mathbb{1}_B \left(\underbrace{R_{k+1}(n) - R_{k+1}(n) - p_{N-n}}_{=0} \right) + \mathbb{1}_C \left(\underbrace{R_{k+1}(n) - R_k(n)}_{\geq -\rho_n^*} \right) \\ &\geq 0, \end{aligned}$$

where we have applied the induction hypothesis to establish that the first and third terms are bounded from below by $-\rho_n^*$ with probability 1. The final inequality follows because we may write $\rho_{n+1}^* = \max\{p_{N-n}, \rho_n^*\}$, which implies the result. Q.E.D.

Proof of Lemma 3. The expected revenues of the original and interchanged sequences are the same from periods 1 through $j - 1$, and therefore, it suffices to compare the expected residual revenue of each sequence from periods j through N . For notational convenience, throughout the proof, define $n := N - j - 1$. Also,

denote expected revenue for periods j through N as $\mathbb{E}[R_k^{orig}(n+2)]$ and $\mathbb{E}[R_k^{swap}(n+2)]$ for the original and interchanged product sequences.

For notational brevity, for $i = j, j+1$, define

$$\alpha_i := \mathbb{P}(X_i > p_i - v(k)) \quad \text{and} \quad \tilde{\alpha}_i := \mathbb{P}(X_i > p_i - v(k+1))$$

By the assumption that v is decreasing, we have $0 \leq \tilde{\alpha}_i \leq \alpha_i \leq 1$. Note that by construction, we have $\beta_i \alpha_i = \tilde{\alpha}_i$, even if $\alpha_i = 0$.

For the original sequence, expanding the expression for the retailer's expected revenue by considering cases, we have

$$\begin{aligned} \mathbb{E}[R_k^{orig}(n+2)] &:= \alpha_j p_j + \alpha_j (\tilde{\alpha}_{j+1} p_{j+1} + \tilde{\alpha}_{j+1} \mathbb{E}[R_{k+2}(n)] + (1 - \tilde{\alpha}_{j+1}) \mathbb{E}[R_{k+1}(n)]) \\ &\quad + (1 - \alpha_j) (\alpha_{j+1} p_{j+1} + \alpha_{j+1} \mathbb{E}[R_{k+1}(n)] + (1 - \alpha_{j+1}) \mathbb{E}[R_k(n)]) \end{aligned}$$

The first term is the retailer's expected revenue from the j^{th} item; the second term is the expected revenue from the $(j+1)^{th}$ item onwards conditional on the customer purchasing item j ; the third and final term is the corresponding expected revenue conditional on the customer not purchasing item j .

In the interchanged sequence, we have a similar expression

$$\begin{aligned} \mathbb{E}[R_k^{swap}(n+2)] &:= \alpha_{j+1} p_{j+1} + \alpha_{j+1} (\tilde{\alpha}_j p_j + \tilde{\alpha}_j \mathbb{E}[R_{k+2}(n)] + (1 - \tilde{\alpha}_j) \mathbb{E}[R_{k+1}(n)]) \\ &\quad + (1 - \alpha_{j+1}) (\alpha_j p_j + \alpha_j \mathbb{E}[R_{k+1}(n)] + (1 - \alpha_j) \mathbb{E}[R_k(n)]) \end{aligned}$$

We note that interchanging j and $j+1$ have no effect on the distributions of $R_k(n)$, $R_{k+1}(n)$ or $R_{k+2}(n)$. Therefore, the difference $\mathbb{E}[R_k^{orig}(n+2)] - \mathbb{E}[R_k^{swap}(n+2)]$ can be obtained by taking the difference and eliminating common terms, to get

$$\begin{aligned} &\mathbb{E}[R_k^{orig}(n+2)] - \mathbb{E}[R_k^{swap}(n+2)] \\ &= \alpha_j \tilde{\alpha}_{j+1} (p_{j+1} + \mathbb{E}[R_{k+2}(n)] - \mathbb{E}[R_{k+1}(n)]) - \tilde{\alpha}_j \alpha_{j+1} (p_j + \mathbb{E}[R_{k+2}(n)] - \mathbb{E}[R_{k+1}(n)]) \\ &\quad - \alpha_j \alpha_{j+1} (p_{j+1} - p_j) \\ &= \alpha_j \alpha_{j+1} \beta_{j+1} (p_{j+1} + \Delta_{k+1}(n)) - \alpha_j \alpha_{j+1} \beta_j (p_j + \Delta_{k+1}(n)) + \alpha_j \alpha_{j+1} (p_j - p_{j+1}) \\ &= \alpha_j \alpha_{j+1} [(1 - \beta_j) p_j - \beta_j \Delta_{k+1}(n) - (1 - \beta_{j+1}) p_{j+1} + \beta_{j+1} \Delta_{k+1}(n)], \end{aligned}$$

where the second equality is because $\tilde{\alpha}_i = \beta_i \alpha_i, \forall i = j, j+1$. Parts (a) and (b) follow directly from the expression above. Q.E.D.

Proof of Corollary 1. The proof will proceed by first showing that such a sequence of products satisfies condition (11) in Lemma 3(b). Fix an arbitrary period j , and assume that for some $k \leq j-1$, the consumer had purchased k products before that period. We assume WLOG that $\mathbb{P}(X_i > p - v(k)) > 0, i = j, j+1$; otherwise, there is nothing to prove.

By assumption, we have $X_j \leq_{hr} X_{j+1}$, which, by (10), implies that $\beta_j \leq \beta_{j+1}$. Further,

$$\Delta_{k+1}(N - j - 1) + p = \mathbb{E}[R_{k+2}(N - j - 1)] - \mathbb{E}[R_{k+1}(N - j - 1)] + p \geq 0, \quad (22)$$

where the equality is by definition (9) and the inequality is by Lemma 2, using the fact that all prices are equal to p . Thus, considering the LHS of (11),

$$\begin{aligned}
 & p_j(1 - \beta_j) - p_{j+1}(1 - \beta_{j+1}) + (\beta_{j+1} - \beta_j)\Delta_{k+1}(N - j - 1) \\
 &= p(\beta_{j+1} - \beta_j) + (\beta_{j+1} - \beta_j)\Delta_{k+1}(N - j - 1) \quad [p_i = p, \forall i] \\
 &= (\beta_{j+1} - \beta_j)(\Delta_{k+1}(N - j - 1) + p) \\
 &\geq 0,
 \end{aligned}$$

where the final inequality is by $\beta_j \leq \beta_{j+1}$ and (22).

Given that such a sequence of products satisfies condition (11) in Lemma 3(b), we can conclude that interchanging any two consecutive products leads to a suboptimal sequence. It remains to be shown that interchanging any two non-consecutive products also leads to a suboptimal sequence. Consider products in the original sequence j and $j' > j + 1$. We will show that by swapping j and j' , condition (11) in Lemma 3(b) no longer holds for period j (now occupied by product j'), implying that it is a suboptimal sequence.

The LHS of (11) for period j after the swap becomes

$$\begin{aligned}
 & p(1 - \beta_{j'}) - p(1 - \beta_{j+1}) + (\beta_{j+1} - \beta_{j'})\Delta_{k+1}(N - j - 1) \\
 &= p(\beta_{j+1} - \beta_{j'}) + (\beta_{j+1} - \beta_{j'})\Delta_{k+1}(N - j - 1) \\
 &= (\beta_{j+1} - \beta_{j'}) (\Delta_{k+1}(N - j - 1) + p) \\
 &\leq 0,
 \end{aligned}$$

where the inequality follows from $\beta_{j+1} \leq \beta_{j'}$ (transitive property of hazard rate ordering) and $\Delta_{k+1}(N - j - 1) + p \geq 0$ from (22). Q.E.D.

Proof of Corollary 2. We prove this by induction on N . It suffices to prove the inductive step. For some $N \geq 1$, inductively assume that the statement is true for $N - 1$. Specifically, we inductively assume that products $2, 3, \dots, N$ are ordered such that $p_2 \geq p_3 \geq \dots \geq p_N$, and this is an optimal order for those sequence of products, regardless of whether the consumer has purchased one or zero products by period 2. Now, the upper bound of Lemma 2 implies that

$$\Delta_1(N - 2) + p_2 \geq 0. \quad (23)$$

Now suppose that $p_1 \geq p_2$. We aim to show that the current product ordering for all N products is optimal. Because X_i are identically distributed according to an IFR distribution, we note that Shaked and Shanthikumar (2007, Theorem 1.A.30) implies $\frac{\mathbb{P}(X_1 > p_1 - v(1))}{\mathbb{P}(X_1 > p_1 - v(0))} \leq \frac{\mathbb{P}(X_2 > p_2 - v(1))}{\mathbb{P}(X_2 > p_2 - v(0))}$. That is, $\beta_1 \leq \beta_2$.

Consider the LHS of (11) for $j = 1$, which, after rearranging, is $(p_1 - p_2)(1 - \beta_1) + (\beta_2 - \beta_1)(\Delta_1(N - 2) + p_2) \geq 0$, where the inequality holds as a consequence of (23), $\beta_1 \leq \beta_2$, and the assumption that $p_1 \geq p_2$.

Given that such a sequence of products satisfies condition (11) in Lemma 3(b), we can conclude that interchanging any two consecutive products leads to a suboptimal sequence. It remains to be shown that interchanging any two non-consecutive products also leads to a suboptimal sequence. Consider products in the original sequence j and $j' > j + 1$. We will show that by swapping j and j' , condition (11) in Lemma 3(b) no longer holds for period j (now occupied by product j'), implying that it is a suboptimal sequence.

The LHS of (11) for period j after the swap becomes

$$\begin{aligned} & p_{j'}(1 - \beta_{j'}) - p_{j+1}(1 - \beta_{j+1}) + (\beta_{j+1} - \beta_{j'})\Delta_{k+1}(N - j - 1) \\ & \leq p_{j+1}(\beta_{j+1} - \beta_{j'}) + (\beta_{j+1} - \beta_{j'})\Delta_{k+1}(N - j - 1) \\ & = (\beta_{j+1} - \beta_{j'}) (\Delta_{k+1}(N - j - 1) + p_{j+1}) \\ & \leq 0, \end{aligned}$$

where the first inequality is because $p_{j'} \leq p_{j+1}$ and the final inequality follows from $\beta_{j+1} \leq \beta_{j'}$ (property of IFR distribution; Shaked and Shanthikumar (2007, Theorem 1.A.30)) and $\Delta_{k+1}(N - j - 1) + p_{j+1} \geq 0$ from the upper bound in Lemma 2. Q.E.D.

Before proving Theorem 1, we first state an intermediate result that is used in the proof:

LEMMA 4. *Suppose that $p_1 \geq \max\{p_2, \dots, p_N\}$. Let R represent the retailer's revenue under the original sequence of product prices and valuations $(p_1, X_1), (p_2, X_2), \dots, (p_N, X_N)$ and \hat{R} the revenue that interchanges both the prices and realized valuations of products 1 and 2 in the event that $X_1 - p_1 \leq X_2 - p_2$. Then, $R \geq \hat{R}$.*

Proof of Lemma 4. Define the following events

$$\begin{aligned} A_{11} & := \{X_2 - p_2 > v(0), X_1 - p_1 > v(1)\}, \quad A_{10} := \{X_2 - p_2 > v(0), X_1 - p_1 \leq v(1)\}, \\ A_{01} & := \{X_2 - p_2 \leq v(0), X_1 - p_1 > v(0)\}, \quad A_{00} := \{X_2 - p_2 \leq v(0), X_1 - p_1 \leq v(0)\}, \\ B & := \{X_2 - p_2 \geq X_1 - p_1\} \end{aligned}$$

Note that $A_{11}, A_{10}, A_{01}, A_{00}$ form a partition of the probability space. By construction,

$$R - \hat{R} = (R - \hat{R}) \mathbb{1}_B. \quad (24)$$

Furthermore, by inspection, we have

$$\mathbb{P}(A_{01} \cap B) = 0, \quad (R - \hat{R}) \mathbb{1}_{A_{11} \cap B} = 0, \quad \text{and} \quad (R - \hat{R}) \mathbb{1}_{A_{00} \cap B} = 0 \quad (25)$$

Therefore, from (24) and (25), we have $R - \hat{R} = (R - \hat{R}) \mathbb{1}_{A_{10} \cap B}$.

Recall that $R_k(n)$ denotes the revenue from the purchases in the last n remaining periods with k products already purchased. For the interchanged sequence, we have $\hat{R} \mathbb{1}_{A_{10} \cap B} = (p_2 + R_1(N - 2)) \mathbb{1}_{A_{10} \cap B}$. Define the events $C := \{X_1 - p_1 > v(0)\}$ and $D := \{X_2 - p_2 > v(1)\}$. For the original sequence, by considering cases, we get:

$$\begin{aligned} R \mathbb{1}_{A_{10} \cap B} & = (p_1 + R_1(N - 2)) \mathbb{1}_{A_{10} \cap B \cap C \cap D^c} + (p_1 + p_2 + R_2(N - 2)) \mathbb{1}_{A_{10} \cap B \cap C \cap D} \\ & \quad + (p_2 + R_1(N - 2)) \mathbb{1}_{A_{10} \cap B \cap C^c} \end{aligned}$$

Therefore, putting these together, we have

$$\begin{aligned} R - \hat{R} & = (p_1 + R_1(N - 2)) \mathbb{1}_{A_{10} \cap B \cap C \cap D^c} + (p_1 + p_2 + R_2(N - 2)) \mathbb{1}_{A_{10} \cap B \cap C \cap D} \\ & \quad - (p_2 + R_1(N - 2)) \mathbb{1}_{A_{10} \cap B \cap C} \\ & = (p_1 + R_1(N - 2)) \mathbb{1}_{A_{10} \cap B \cap C \cap D^c} + (p_1 + R_2(N - 2) - R_1(N - 2)) \mathbb{1}_{A_{10} \cap B \cap C \cap D} \\ & \quad - (p_2 + R_1(N - 2)) \mathbb{1}_{A_{10} \cap B \cap C \cap D^c} \\ & = (p_1 - p_2) \mathbb{1}_{A_{10} \cap B \cap C \cap D^c} + (p_1 + R_2(N - 2) - R_1(N - 2)) \mathbb{1}_{A_{10} \cap B \cap C \cap D} \\ & \geq 0, \end{aligned}$$

where the final inequality is by Lemma 2, and the assumption that $p_1 \geq \max\{p_2, \dots, p_N\}$. Q.E.D.

Proof of Theorem 1. To prove (a), let X_1, \dots, X_M represent the realized valuation shocks from the M products. With a slight abuse of notation, let $\sigma(\cdot)$ be a permutation of $\{1, \dots, M\}$ so that

$$X_{\sigma(1)} - p_{\sigma(1)} \geq X_{\sigma(2)} - p_{\sigma(2)} \geq \dots \geq X_{\sigma(M)} - p_{\sigma(M)}.$$

It follows from an immediate generalization of Lemma 1 that consumers, when presented with a fixed assortment, will purchase products in the order specified by σ . Therefore, part (a) will hold if there exists a set of successive pairwise interchanges of products (both prices and valuations), starting from the original sequence of products (under which the retailer earns revenue R), that satisfies two conditions: (i) the set of successive interchanges terminates with the permutation $(\sigma(1), \dots, \sigma(M))$, and (ii) the retailer's revenue decreases (with probability 1) on each interchange. If these hold, then at the end of this chain of interchanges, the retailer's revenue would coincide with R^f , and we have shown $R \geq R^f$.

We will show that such a sequence of interchanges exists by induction on the last m products, for $m = 2, \dots, M$. First consider the case that $m = 2$. Then, the only interchange that is potentially necessary to satisfy (i) is to swap the two products, in which case, (ii) follows from Lemma 4 (stated immediately preceding this proof).

Now, inductively assume that for some $m - 1$, there exists a set of successive interchanges made such that these last $m - 1$ products are sorted according to the permutation σ , and each interchange decreases the retailer's revenue. Assume therefore that this set of interchanges has been performed. Notice, however, that these interchanges (to the last $m - 1$ products) do not change the position of the m th product from the end, which, by assumption, also has the highest price of all the last m products. For brevity, let us call this the "critical product". Since the last $m - 1$ products have already been sorted, it suffices to additionally perform a finite number (at most m) additional pairwise interchanges to ensure that all m products are sorted by (i). Each interchange involves this critical product, which has the highest price of the last m products. Hence, each of these interchanges satisfies the conditions of Lemma 4, which implies that the retailer's revenue decreases with each interchange, thus satisfying (ii). Therefore, by induction, the proof of (a) is complete.

To prove part (b), we observe that $\mathbb{E}[R_*^s] \geq \mathbb{E}[R] \geq \mathbb{E}[R_*^f]$, where the first inequality is because R_*^s represents the expected revenue from an optimized sequential assortment with N products, which, by definition, exceeds the expected revenue from a suboptimal sequential assortment with $M \leq N$ products. The second inequality is a consequence of part (a). Q.E.D.

Proof of Theorem 2. For part (a), suppose that $v(k) = v, k = 0, \dots, N - 1$. We first show that if the retailer chooses the fixed assortment, it is optimal to also make all N products available. Suppose WLOG that the retailer chooses the first $M < N$ products to be sold in the fixed assortment. By (4), the retailer's revenue from including these M products is $R^f(M) := \sum_{i=1}^M p_{\sigma(i)} \mathbb{1}_{\{v + X_{\sigma(i)} - p_{\sigma(i)} \geq 0\}} = \sum_{i=1}^M p_i \mathbb{1}_{\{v + X_i - p_i \geq 0\}}$. From this expression, we observe that the retailer's incremental revenue from including product $M + 1$ would be $p_{M+1} \mathbb{1}_{\{v + X_{M+1} - p_{M+1} \geq 0\}} \geq 0$, and thus it would do so optimally do so, until all N products are included. For the sequential assortment, let $\hat{\sigma}$ be an arbitrary permutation of $\{1, \dots, N\}$ representing the retailer's product sequence. By (6) and Remark 1, and using $v(k) = v$,

$$R^s = \sum_{i=1}^N p_{\hat{\sigma}(i)} \mathbb{1}_{\{v + X_{\hat{\sigma}(i)} - p_{\hat{\sigma}(i)} \geq 0\}} = \sum_{i=1}^N p_i \mathbb{1}_{\{v + X_i - p_i \geq 0\}} = R^f(N).$$

For part (b), suppose that $X_i = x$ with probability 1 and $p_i = p$ for all $i = 1, \dots, N$. As before, we first show that if the retailer chooses the fixed assortment, it is optimal to also make all N products available. Suppose WLOG that the retailer chooses the first $M < N$ products to be sold in the fixed assortment. By (4), the retailer's revenue from including these M products can be expressed as $R^f = \sum_{i=1}^M p \mathbb{1}_{\{v(i-1)+x-p \geq 0\}}$. From this expression, it is clear that the retailer gains nonnegative revenue from including product $M+1$ and so on until all N products are included. Next, recall the definition of D^f from Lemma 1. By construction, $v(i-1) + x - p \geq 0$ for all $i \leq D^f$ and $v(D^f) + x - p < 0$. Recall the definition of $\{W_i, i = 0, \dots, N\}$ via (6). We claim that $W_i = \min\{i, D^f\}$. This holds by definition for $i = 0$. Suppose it holds for some $i-1$, then, for $i \leq D^f$, we have $W_i = W_{i-1} + \mathbb{1}_{\{v(W_{i-1})+x-p \geq 0\}} = i-1 + \mathbb{1}_{\{v(i-1)+x-p \geq 0\}} = i$; for $i > D^f$, we have $W_i = W_{i-1} + \mathbb{1}_{\{v(W_{i-1})+x-p \geq 0\}} = D^f + \mathbb{1}_{\{v(D^f)+x-p \geq 0\}} = D^f$, establishing the claim for all i . To conclude the proof, we note that in this case, (6) implies that we have $R^s = pW_N = p \min\{N, D^f\} = pD^f = R^f$. Q.E.D.

Proof of Theorem 3. For this proof, denote $v(\infty) := \lim_{k \rightarrow \infty} v(k)$, and note that $v(\infty)$ is finite because v is assumed to be decreasing and bounded from below. Furthermore, note that although we do not denote this explicitly, R^t for each $t \in \{f, s\}$ all depend on N . In addition, under the assumption of equal prices and a common valuation distribution of every product, it is easy to see that (i) it is optimal in the fixed assortment to make all products available, and (ii) any product sequence is optimal in the sequential assortment.

First, we prove part (a) for $t = s$. Construct the random variable B_N as $B_N := \sum_{i=1}^N \mathbb{1}_{\{v(\infty)+X_i-p \geq 0\}}$, where we have used the assumption that $p_i = p$ for all i . Furthermore, recalling the definition of the process $\{W_i\}_{i=0}^N$ from the construction (6), for an arbitrary K , construct the random variable B_N^K as $B_N^K := \sum_{i=1}^N \mathbb{1}_{\{v(\min\{W_{i-1}, K\})+X_i-p \geq 0\}}$. Since $v(\min\{W_{i-1}, K\}) \geq v(W_{i-1}) \geq v(\infty)$, we have $B_N \leq W_N \leq B_N^K$, which, by (6), under the assumption that $p_i = p$, implies that

$$pB_N \leq R^s \leq pB_N^K. \quad (26)$$

Define the (integer-valued) random variable L_N^K as $L_N^K := \sup\{i \in \{1, \dots, N+1\} : W_{i-1} < K\}$, so that we can decompose B_N^K as

$$B_N^K = \sum_{i=1}^{L_N^K} \mathbb{1}_{\{v(W_{i-1})+X_i-p \geq 0\}} + \sum_{i=L_N^K+1}^N \mathbb{1}_{\{v(K)+X_i-p \geq 0\}}.$$

Since we have assumed that $X_i \stackrel{d}{=} X$, by the Strong Law of Large Numbers, we have $B_N/N \xrightarrow{a.s.} \delta$. From the lower bound of (26), we necessarily have $\lim_{i \rightarrow \infty} W_i = \infty$ with probability 1, which implies that $\lim_{N \rightarrow \infty} L_N^K$ is bounded almost surely. Hence,

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{B_N^K}{N} &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^{L_N^K} \mathbb{1}_{\{v(W_{i-1})+X_i-p \geq 0\}} + \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=L_N^K+1}^N \mathbb{1}_{\{v(K)+X_i-p \geq 0\}} \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=L_N^K+1}^N \mathbb{1}_{\{v(K)+X_i-p \geq 0\}} \quad [\lim_{N \rightarrow \infty} L_N^K \text{ is a.s. bounded}] \\ &\leq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{1}_{\{v(K)+X_{i+1}-p \geq 0\}} \\ &= \mathbb{P}(v(K) + X - p \geq 0) \end{aligned}$$

where the final equality is again by the assumption that X_i have a common distribution and by the Strong Law of Large Numbers. Therefore, we have shown that with probability 1,

$$p\delta \leq \lim_{N \rightarrow \infty} R^s/N \leq p\mathbb{P}(v(k) + X - p \geq 0).$$

We complete the proof by sending K to $+\infty$ on the RHS.

To prove (a) for $t = f$, we note that B_N can be equivalently written as $B_N = \sum_{i=1}^N \mathbb{1}_{\{v(\infty) + X_{\sigma(i)} - p \geq 0\}}$, and we therefore have $pB_N \leq R^f \leq R^s$, where the upper bound follows from Theorem 1. Hence, the result follows from the case of $t = s$ by a sandwich argument.

To prove (b), consider \hat{v} that is constructed as $\hat{v}(k) := \max\{v(k), v(N_0 - 1)\}$ for all k , and for each $t \in \{f, s\}$ let \hat{R}^t represent the corresponding revenue when \hat{v} is used instead of v . Since $\hat{v} \geq v$, we necessarily have $R^t \leq \hat{R}^t$, and in particular, $\mathbb{1}_{\{\hat{R}^t \leq pN_0\}} R^t = \mathbb{1}_{\{\hat{R}^t \leq pN_0\}} \hat{R}^t$. Therefore, noting that $R^t \leq pN_0$ almost surely, we may write

$$R^t = \min\{\hat{R}^t, pN_0\}. \quad (27)$$

Notice that by construction of \hat{v} , we necessarily have $\lim_{N \rightarrow \infty} \mathbb{P}(X \geq p - \hat{v}(N - 1)) = \lim_{N \rightarrow \infty} \mathbb{P}(X \geq p - v(N_0 - 1)) > 0$, where the final inequality is by construction of N_0 . This implies that we can apply part (a) for \hat{R}^t , which in turn implies that with probability 1, $\lim_{N \rightarrow \infty} \hat{R}^t = +\infty$. Therefore, using this with (27) implies that with probability 1, $\lim_{N \rightarrow \infty} R^t = pN_0$. Q.E.D.

Proof of Proposition 1. Throughout this proof, we will use the more compact notation $s_i^k := s_i(k)$. We first prove a recursive expression for s_i^k . Note that we can rewrite the Bellman equation (12) as

$$U_i(k) = U_{i+1}(k) + \mathbb{E}\left[(X_i - s_i^k)_+\right] = U_{i+1}(k) + \int_{s_i^k}^{\infty} \bar{F}(u) du, \quad (28)$$

where $(z)_+ := \max\{z, 0\}$, and the final equality is a result of the identity $\mathbb{E}[(X - x)_+] \equiv \int_x^{\infty} \bar{F}(u) du$. Now consider

$$\begin{aligned} s_{i-1}^k - s_i^k &= U_i(k) - U_i(k+1) - U_{i+1}(k) + U_{i+1}(k+1) \text{ [By (14)]} \\ &= \int_{s_i^k}^{\infty} \bar{F}(u) du - \int_{s_i^{k+1}}^{\infty} \bar{F}(u) du \quad \text{[By (28)]} \\ &= \int_{s_i^k}^{s_i^{k+1}} \bar{F}(u) du \end{aligned}$$

Therefore,

$$s_{i-1}^k = s_i^k + \int_{s_i^k}^{s_i^{k+1}} \bar{F}(u) du. \quad (29)$$

We prove part (a) by backward induction on i . For $i = N$, we have $s_N^k = s_N(k) = p - v(k)$, which is increasing in k . Inductively suppose that for some i , s_i^k is increasing in k for every k . Then, from (29), we have

$$\begin{aligned} s_{i-1}^{k+1} - s_{i-1}^k &= (s_i^{k+1} - s_i^k) + \int_{s_i^{k+1}}^{s_i^{k+2}} \bar{F}(u) du - \int_{s_i^k}^{s_i^{k+1}} \bar{F}(u) du \text{ [By (29)]} \\ &= \int_{s_i^{k+1}}^{s_i^{k+2}} \bar{F}(u) du + \int_{s_i^k}^{s_i^{k+1}} (1 - \bar{F}(u)) du \\ &= \int_{s_i^{k+1}}^{s_i^{k+2}} \bar{F}(u) du + \int_{s_i^k}^{s_i^{k+1}} F(u) du \\ &\geq 0. \end{aligned} \quad \text{[Induction hypothesis]}$$

Part (b) follows from (a) and expression (29).

For part (c), consider some valuation distribution G such that $\bar{G} \geq \bar{F}$; that is, G is stochastically larger than F . Denote $s_i^k(G)$ and $s_i^k(F)$ as the thresholds under valuation distributions G and F , respectively. Our goal is to show that for every i, k ,

$$s_i^k(G) \geq s_i^k(F). \quad (30)$$

We prove (30) by induction on i . For $i = N$, $s_i^k(G) = s_i^k(F) = p - v(k)$, and (30) holds trivially. Inductively assume that this holds for some i and for every k . We have

$$\begin{aligned} & s_{i-1}^k(G) - s_{i-1}^k(F) \\ &= s_i^k(G) - s_i^k(F) + \int_{s_i^k(G)}^{s_i^{k+1}(G)} \bar{G}(u) du - \int_{s_i^k(F)}^{s_i^{k+1}(F)} \bar{F}(u) du && \text{[By (29)]} \\ &\geq s_i^k(G) - s_i^k(F) + \int_{s_i^k(G)}^{s_i^{k+1}(F)} \bar{G}(u) du - \int_{s_i^k(F)}^{s_i^{k+1}(F)} \bar{F}(u) du && \text{[Induction Hypothesis, } s_i^{k+1}(G) \geq s_i^{k+1}(F)\text{]} \\ &= s_i^k(G) - s_i^k(F) - \int_{s_i^k(F)}^{s_i^k(G)} \bar{G}(u) du + \int_{s_i^k(F)}^{s_i^{k+1}(F)} [\bar{G}(u) - \bar{F}(u)] du \\ &\geq s_i^k(G) - s_i^k(F) - \int_{s_i^k(F)}^{s_i^k(G)} \bar{G}(u) du && \text{[By part (a) and } \bar{G} \geq \bar{F}\text{]} \\ &= \int_{s_i^k(F)}^{s_i^k(G)} G(u) du \\ &\geq 0 && \text{[Induction hypothesis, } s_i^k(G) \geq s_i^k(F)\text{]} \end{aligned}$$

Finally, we prove (d) by induction on i . This holds trivially when $i = N$. Suppose that it holds for some i .

Then we have

$$\begin{aligned} s_{i-1}^k &= s_i^k + \int_{s_i^k}^{s_i^{k+1}} \bar{F}(u) du && \text{[By (29)]} \\ &\leq s_i^k + (s_i^{k+1} - s_i^k) \bar{F}(s_i^k) && \text{[By part (a), } \bar{F} \text{ decreasing]} \\ &= s_i^k F(s_i^k) + s_i^{k+1} \bar{F}(s_i^k) \\ &\leq (p - v(k + N - i)) F(s_i^k) + (p - v(k + 1 + N - i)) \bar{F}(s_i^k) && \text{[Induction hypothesis]} \\ &\leq (p - v(k + N - i + 1)) F(s_i^k) + (p - v(k + N - i + 1)) \bar{F}(s_i^k) && \text{[} v(\cdot) \text{ decreasing]} \\ &= p - v(k + N - (i - 1)). \end{aligned}$$

Q.E.D.

Proof of Theorem 4 Let $D^f(D^s)$ represent the number of products the retailer sells when it follows the fixed (sequential) assortment strategy. Note that $R_*^f = pD^f$ and $\tilde{R}_*^s = pD^s$. Note that $\mathbb{E}[D^f] = \mathbb{P}(D^f \geq 1) + \mathbb{P}(D^f \geq 2)$, and similarly for D^s . We have

$$\mathbb{P}(D^f \geq 1) = \mathbb{P}(\max\{X_1, X_2\} > p - v(0)) = 1 - F^2(p - v(0)) = 1 - F^2(a)$$

$$\mathbb{P}(D^f \geq 2) = \mathbb{P}(\min\{X_1, X_2\} > p - v(1)) = \bar{F}^2(p - v(1)) = \bar{F}^2(c).$$

For D^s , we first note that $b = a + \int_a^c \bar{F}(u) du = p - v(0) - U_2(1) + U_2(0)$; we have

$$\mathbb{P}(D^s \geq 1) = \mathbb{P}(X_1 > b) + \mathbb{P}(X_1 \leq b, X_2 > a) = \bar{F}(b) + F(b)\bar{F}(a)$$

$$\mathbb{P}(D^s \geq 2) = \mathbb{P}(X_1 > b, X_2 > c) = \bar{F}(b)\bar{F}(c).$$

Hence, $\mathbb{E}[D^s] \geq \mathbb{E}[D^f]$ and therefore $\mathbb{E}[\tilde{R}_*^s] \geq \mathbb{E}[R_*^f]$ if and only if

$$\begin{aligned} \bar{F}(b) + F(b)\bar{F}(a) + \bar{F}(b)\bar{F}(c) &\geq 1 - F^2(a) + \bar{F}^2(c) \\ \iff \bar{F}(c)(\bar{F}(b) - \bar{F}(c)) &\geq F(b)(1 - \bar{F}(a)) - F^2(a) \\ \iff \bar{F}(c)(\bar{F}(b) - \bar{F}(c)) &\geq F(a)(F(b) - F(a)). \end{aligned}$$

Q.E.D.

Proof of Theorem 5. Throughout this proof, for notational brevity, we write $s_i^k := s_i(k)$. Again, let $D^f(D^s)$ represent the number of products the retailer sells when it follows the fixed (sequential) assortment strategy. Let $D_k(n)$ represent the number of products purchased under the sequential assortment strategy by the forward-looking consumer in the last n assortments of the season, if she has already purchased k products. Note that $D^s := D_0(N)$. By an identical argument to the proof of Lemma 2, one can show that in this setting of forward-looking consumers, we have $D_{k+1}(n) \leq D_k(n) \leq 1 + D_{k+1}(n)$ with probability one for $n = 0, \dots, N$ and $k = 0, \dots, N - n - 1$. Finally, note that if the consumer has n assortments left to view, then in the dynamic programming formulation (12), she is in period $N - n + 1$.

To prove (a), we begin with a definition and a preliminary result. For $n = 0, \dots, N$ and $k = 0, \dots, N - n - 1$, define

$$\delta_n^k := 1 + \mathbb{E}[D_{k+1}(n)] - \mathbb{E}[D_k(n)]. \quad (31)$$

Note that Lemma 2 implies that $\delta_n^k \geq 0$ for each n, k .

We can rearrange the definition of δ_n^k , and by conditioning on whether a purchase was made or not in period $N - n$, we get

$$\mathbb{E}[D_k(n+1)] = \mathbb{E}[D_k(n)] + \bar{F}(s_{N-n}^k)\delta_n^k. \quad (32)$$

We claim that

$$\delta_{n+1}^k \geq \bar{F}(s_{N-n}^{k+1})\delta_n^{k+1}. \quad (33)$$

To see this, consider

$$\begin{aligned} \delta_{n+1}^k &= 1 + \mathbb{E}[D_{k+1}(n+1)] - \mathbb{E}[D_k(n+1)] && \text{[By (31)]} \\ &= 1 + \mathbb{E}[D_{k+1}(n)] + \bar{F}(s_{N-n}^{k+1})\delta_n^{k+1} - \mathbb{E}[D_k(n)] - \bar{F}(s_{N-n}^k)\delta_n^k && \text{[By (32)]} \\ &= \delta_n^k + \bar{F}(s_{N-n}^{k+1})\delta_n^{k+1} - \bar{F}(s_{N-n}^k)\delta_n^k && \text{[By (31)]} \\ &= \bar{F}(s_{N-n}^{k+1})\delta_n^{k+1} + F(s_{N-n}^k)\delta_n^k \\ &\geq \bar{F}(s_{N-n}^{k+1})\delta_n^{k+1}, && [\delta_n^k \geq 0 \text{ by Lemma 2}] \end{aligned}$$

thus establishing (33).

Recalling that $D^s = D_0(N)$, we apply (32) for the case $k = 0, n = N - 1$ to get

$$\mathbb{E}[D_0(N)] = \mathbb{E}[D_0(N-1)] + \bar{F}(s_1^0)\delta_{N-1}^0.$$

Next, we claim that under the assumption that $X_i \geq p - v(N - 2)$ a.s., we necessarily have $D_0(N - 1) = N - 1$ with probability 1. This is because we can equivalently interpret $D_0(N - 1)$ as the demand from a consumer who only views $N - 1$ assortments. Hence, we can use representation (15) for $D_0(N - 1)$, which yields

$$D_0(N - 1) = \sum_{i=1}^{N-1} \mathbb{1}_{\{X_i \geq s_i(\widetilde{W}_{i-1})\}},$$

where $\widetilde{W}_j \leq j$ for each $j = 0, \dots, N - 1$. Consider

$$\begin{aligned} s_i(\widetilde{W}_{i-1}) &\leq p - v(\widetilde{W}_{i-1} + N - 1 - i) \quad [\text{Proposition 1(d) for } N - 1] \\ &\leq p - v(N - 2) \quad [\widetilde{W}_{i-1} \leq i - 1, v(\cdot) \text{ decreasing}] \end{aligned}$$

Therefore, $D_0(N - 1) = N - 1$ with probability 1.

For the fixed assortment, apply (4) and the condition that $X_i \geq p - v(N - 2)$ with probability 1 to get

$$\mathbb{E}[D^f] = \mathbb{E}\left[\sum_{i=1}^N \mathbb{1}_{\{X_{\sigma(i)} \geq p - v(i-1)\}}\right] = (N - 1) + \mathbb{P}(X_{\sigma(N)} \geq p - v(N - 1)) = (N - 1) + \overline{F}^N(p - v(N - 1)).$$

Comparing the expressions for $\mathbb{E}[D^s]$ and $\mathbb{E}[D^f]$, and noting that from Proposition 1(d), we have $s_1(0) \leq p - v(N - 1)$, and since \overline{F} is decreasing, $\overline{F}(p - v(N - 1)) \leq \overline{F}(s_1(0))$. Therefore, to complete the proof, it suffices to show that

$$\delta_{N-1}^0 \geq \overline{F}^{N-1}(p - v(N - 1)). \quad (34)$$

We will do this by proving that for every $n = 0, \dots, N - 1$ and $k = N - 1 - n$, we have

$$\delta_n^k \geq \overline{F}^n(p - v(N - 1)), \quad (35)$$

of which (34) is a special case.

We prove (35) by induction on n . The case of $n = 0$ holds trivially. Suppose that (35) holds for some n , and we aim to show it for $n + 1$. Then,

$$\begin{aligned} \delta_{n+1}^{k-1} &\geq \overline{F}(s_{N-n}^k) \delta_n^k && [\text{By (33)}] \\ &\geq \overline{F}(s_{N-n}^k) \overline{F}^n(p - v(N - 1)) && [\text{Induction hypothesis}] \\ &\geq \overline{F}(p - v(k + n)) \overline{F}^n(p - v(N - 1)) && [\text{Proposition 1(d) and } \overline{F} \text{ decreasing}] \\ &= \overline{F}^{n+1}(p - v(N - 1)) && [k = N - 1 - n \text{ by assumption}] \end{aligned}$$

Therefore, (34) holds, and hence, $\mathbb{E}[D^s] \geq \mathbb{E}[D^f]$ and $\mathbb{E}[\widetilde{R}_*^s] \geq \mathbb{E}[R_*^f]$.

To prove (b), we first note that in this setting,

$$\mathbb{E}[D^f] = 1 - F^N(p - v(0)).$$

Furthermore, for any $i \geq 1, k \geq 1$ we have $s_i^k \geq s_N^k = p - v(k) \geq p - v(1)$, where the first inequality is by Proposition 1(b), and the second because $v(\cdot)$ is decreasing, which implies that $\overline{F}(s_i^k) \leq \overline{F}(p - v(1))$. Because we have assumed that $X_i \leq p - v(1)$ a.s., this implies that for $k \geq 1$, we necessarily have $\overline{F}(s_i^k) = 0$ for any $i \geq 1$. In particular, using (32), this implies that for $k \geq 1$, and any $n = 0, \dots, N - 1$, we have

$$\mathbb{E}[D_k(n + 1)] = \mathbb{E}[D_k(n)] = \dots = \mathbb{E}[D_k(0)] = 0.$$

Now we use the definition of δ_n^k from (31) to expand (32) for the case of $k = 0$ to get

$$\mathbb{E}[D_0(n+1)] = \bar{F}(s_{N-n}^0) + F(s_{N-n}^0)\mathbb{E}[D_0(n)].$$

Defining $Q_n := 1 - \mathbb{E}[D_0(n)]$, the above may be written as $Q_{n+1} = F(s_{N-n}^0)Q_n$, for $n = 0, \dots, N-1$. Since $Q_0 := 1$, we have

$$Q_N = \prod_{n=0}^{N-1} F(s_{N-n}^0) \geq \prod_{n=0}^{N-1} F(p - v(0)) = F^N(p - v(0)),$$

where the inequality is by Proposition 1(b) and because F is increasing. Therefore, we conclude that

$$\mathbb{E}[D^s] = \mathbb{E}[D_0(N)] = 1 - Q_N \leq 1 - F^N(p - v(0)) = \mathbb{E}[D^f],$$

and therefore $\mathbb{E}[\tilde{R}_*^s] \leq \mathbb{E}[R_*^f]$. Q.E.D.

Appendix B: Supplementary Material

B.1. MILP formulation of SAA for optimal assortment composition

Let N_S represent the total number of i.i.d. SAA samples drawn, and let $(X_1^{(n)}, X_2^{(n)}, \dots, X_N^{(n)})$ represent the realized values of demand shocks on the n th random draw, $n = 1, \dots, N_S$. We use $\sigma_n(\cdot)$ to denote an ordering of $X_i^{(n)}$ such that $X_{\sigma_n(1)}^{(n)} - p_{\sigma_n(1)} \geq X_{\sigma_n(2)}^{(n)} - p_{\sigma_n(2)} \geq \dots \geq X_{\sigma_n(N)}^{(n)} - p_{\sigma_n(N)}$. For each $i = 1, \dots, N$, $k = 0, \dots, N - 1$, and $n = 1, \dots, N_S$, we define $U_{i,k,n} := X_{\sigma_n(i)}^{(n)} - p_{\sigma_n(i)} + v(k)$ and further define $B_{i,k,n}$ as a binary indicator that $U_{i,k,n}$ is positive, i.e., $B_{i,k,n} = 1$ if $U_{i,k,n} > 0$, and $B_{i,k,n} = 0$ otherwise. We note that for a given draw of SAA samples, both $U_{i,k,n}$ and $B_{i,k,n}$ are deterministic functions of the sample and can be pre-computed before the MILP is constructed and solved.

The MILP formulation uses two key decision variables: $z_i = 1$ is interpreted to mean that product i is excluded from the assortment and $y_{i,k,n}$ is an assignment variable that means that product i is assigned to position k for the random draw n . The key idea in the MILP is that for a given choice of z_1, \dots, z_N , there is a unique value of $y_{i,k,n}$ that satisfies the constraints, and this value corresponds to choosing products in decreasing order of $X_i^{(n)} - p_i$ if they are included in the assortment, which matches the utility-maximizing decisions of the consumer in our model when she is faced with a fixed assortment. In the MILP below, for brevity of presentation, when the ranges of indices are omitted, they are to be interpreted as being over their full range, i.e., $\forall i$ means $\forall i = 1, \dots, N$, $\forall k$ means $k = 0, \dots, N - 1$, and $\forall n$ means $n = 1, \dots, N_S$.

$$\max_{y,z,w} \quad \frac{1}{N_S} \sum_{n=1}^{N_S} \sum_{k=0}^{N-1} \sum_{i=1}^N y_{i,k,n} p_i \quad (36a)$$

$$\text{s.t.} \quad y_{i,k,n} \leq B_{i,k,n} \quad \forall i; \forall k; \forall n \quad (36b)$$

$$\sum_{k=0}^{N-1} y_{i,k,n} \leq 1 - z_i \quad \forall i; \forall n \quad (36c)$$

$$\sum_{i=1}^N y_{i,k,n} \leq 1 \quad \forall k; \forall n \quad (36d)$$

$$w_{i,k,n} = 1 - \sum_{j=i}^N y_{\sigma_n(j),k,n} \quad \forall i; \forall k; \forall n \quad (36e)$$

$$w_{i,k-1,n} \leq w_{i,k,n} \quad \forall i; \forall k = 1, \dots, N - 1; \forall n \quad (36f)$$

$$\sum_{i=1}^N w_{i,k,n} \leq k + \sum_{j=1}^{k+1} z_{\sigma_n(j)} + N \left(1 - \sum_{i=1}^N y_{i,k,n} \right) \quad \forall k; \forall n \quad (36g)$$

$$y_{i,k,n}, z_i \in \{0, 1\} \quad \forall i; \forall k; \forall n \quad (36h)$$

The objective (36a) aims to maximize the retailer's average revenue (taken over the SAA samples). Constraints (36b), (36c), and (36d) represent assignment constraints: (36b) states that only assignments with nonnegative utility for the consumer can be chosen; (36c) states that at most one product can be assigned to the k th position if that product is included in the assortment; (36d) states that each position can be filled by at most one product. Constraints (36f) and (36g) enforce the purchase of products in descending

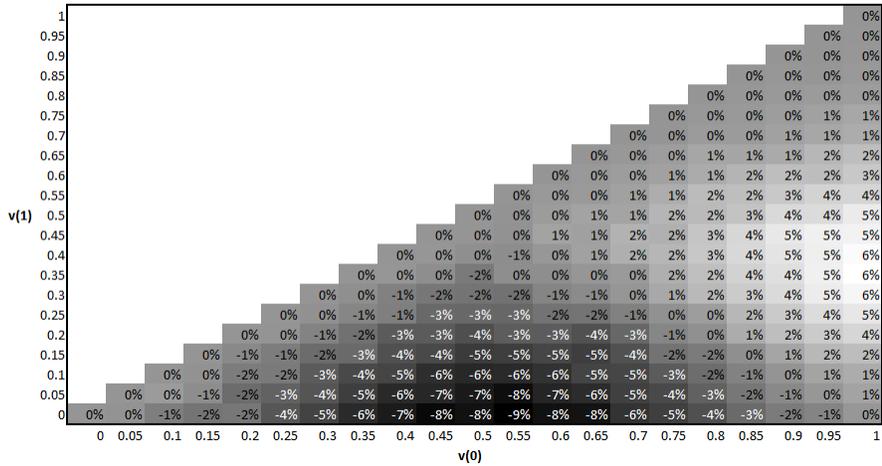


Figure 5 Value of concealment as a percentage of expected revenue from the fixed assortment strategy:

$$p = 1 \text{ and } X \sim Normal(\mu = 0.5, \sigma^2 = \frac{1}{12})$$

order of $X_i^{(n)} - p_i$ in order to match the decisions of a utility-maximizing consumer. Namely, (36f) states that the product in position $k - 1$ has to have a higher value of $X_i^{(n)} - p_i$ compared to the product in position k . Finally, (36g) is a “big-M” style constraint that ensures that if we have two products i and i' , such that $X_i^{(n)} - p_i > X_{i'}^{(n)} - p_{i'}$, and i' is assigned to a position, then product i can only be “skipped” from being assigned if, and only if, product i is permanently excluded from the assortment (i.e., from every SAA sample).

B.2. Numerical Results for forward-looking consumers when $N = 2$,

$$X_i \sim Normal(\mu = 0.5, \sigma^2 = \frac{1}{12})$$

Here we present numerical results when $X_i \sim Normal(\mu = 0.5, \sigma^2 = \frac{1}{12})$; note that the mean and variance of this distribution are identical to the mean and variance of the *Uniform*(0, 1) distribution used in Figure 1. Figure 5 shows a heat map of the value of concealment, expressed as a percentage of the retailer’s expected revenue from following the fixed assortment strategy, for forward-looking consumers when $p = 1, X \sim Normal(\mu = 0.5, \sigma^2 = \frac{1}{12})$, and for different values of $v(0) \in [0, 1], v(1) \in [0, v(0)]$. These numerical results are averages of 10,000 simulation runs per set of parameters.

As in Figure 1, the right-most region of the heat map in Figure 5 represents values of $v(0)$ and $v(1)$ where the value of concealment is positive, and the bottom-most region of the heat map represents values of $v(0)$ and $v(1)$ where the value of concealment is negative. The line $v(0) + v(1) = 1$ separates the two regions as it did in Figure 1. Comparing Figures 1 and 5 suggests that our intuition extends beyond the case where $X_i \sim Uniform(\alpha, \beta)$ to the case where $X_i \sim Normal(\mu, \sigma^2)$, and we conjecture that these insights are broadly applicable to many sets of parameters over the forward-looking consumer’s valuations.

B.3. Value of concealment for forward-looking consumers, special case of $N = 3$

Consider the special case where $X_i \sim Uniform(0, 1)$. The following proposition establishes necessary and sufficient conditions on the parameters $(x, y, z) := (p - v(0), p - v(1), p - v(2))$ to establish $\mathbb{E}[\tilde{R}_*^s] \geq \mathbb{E}[R_*^f]$ and vice versa.

PROPOSITION 2. Let $N = 3$ and $X_i \sim \text{Uniform}(0,1)$. For a given trivariate, 7th degree, polynomial, $g(x, y, z)$, the value of concealment is positive if and only if $g(x, y, z) \geq 0$.

Proof of Proposition 2. As in the proof of Theorem 4, let $D^f(D^s)$ represent the number of products the retailer sells when it follows the fixed (sequential) assortment strategy. We calculate $\mathbb{E}[D^f]$ similarly to Theorem 4, by using $\mathbb{E}[D^f] = \sum_{k=1}^3 \mathbb{P}(D^f \geq k)$ and evaluating each term of the sum directly. For $\mathbb{E}[D^s]$, we first calculate the thresholds by applying (29) repeatedly, and subsequently calculate $\mathbb{E}[D^s]$ by recursively evaluating the expected number of items purchased in each period, starting from the last period. We then evaluate g as $\mathbb{E}[\tilde{R}_*^s] - \mathbb{E}[R_*^f] = p(\mathbb{E}[D^s] - \mathbb{E}[D^f]) = Kg(x, y, z)$, where K is a scaling constant. This gives us the following expression for g :

$$\begin{aligned} g(x, y, z) := & x^7 - x^6y + x^6 - 3x^5y^2 + 6x^5y + 2x^4y^3 + x^4y^2z - 10x^4y^2 + x^4yz^2 - 2x^4yz + 8x^4y - x^4z^3 \\ & + 3x^4z^2 - 4x^4z + 2x^4 + 2x^3y^4 - 12x^3y^3 + 2x^3y^2z^2 - 4x^3y^2z + 16x^3y^2 - x^3z^4 \\ & + 4x^3z^3 - 8x^3z^2 + 8x^3z - 16x^3 - 2x^2y^4z + 12x^2y^4 - 4x^2y^3z^2 + 12x^2y^3z - 36x^2y^3 \\ & + 2x^2y^2z^3 - 4x^2y^2z + 20x^2y^2 + x^2yz^4 - 8x^2yz^3 + 20x^2yz^2 - 24x^2yz + 8x^2y - x^2z^4 + \\ & + 4x^2z^3 - 8x^2z^2 + 8x^2z + 4xy^5 - 2xy^4z^2 + 4xy^4z - 16xy^4 + 4xy^3z^2 - 8xy^3z + 16xy^3 + xy^2z^4 \\ & - 4xy^2z^3 + 8xy^2z^2 - 8xy^2z - 2xyz^4 + 8xyz^3 - 16xyz^2 + 16xyz - 4y^5z + 16y^5 + 2y^4z^3 - 16y^4z^2 \\ & + 32y^4z - 48y^4 + 2y^3z^4 - 4y^3z^3 + 12y^3z^2 - 16y^3z + 64y^3 - 3y^2z^5 + 18y^2z^4 - 52y^2z^3 + 84y^2z^2 \\ & - 80y^2z - 24y^2 - yz^6 + 6yz^5 - 20yz^4 + 40yz^3 - 56yz^2 + 48yz + z^7 - 7z^6 + 24z^5 \\ & - 50z^4 + 56z^3 - 24z^2, \end{aligned}$$

which can be seen to be a 7th degree polynomial in the three variables (x, y, z) . Q.E.D.

Figure 6 illustrates the boundary constructed from g where the expected demand under the fixed and sequential assortment strategies are equivalent. Our analysis suggests that it is unlikely that there are simple necessary and sufficient conditions to characterize the sign of the value of concealment.

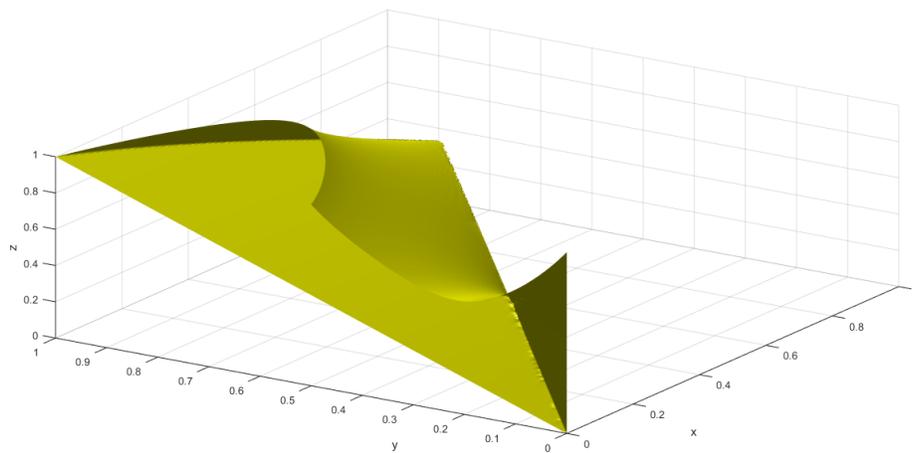


Figure 6 3D plot of boundary on which the value of concealment is zero: $N = 3$ and $X_i \sim Uniform(0,1)$